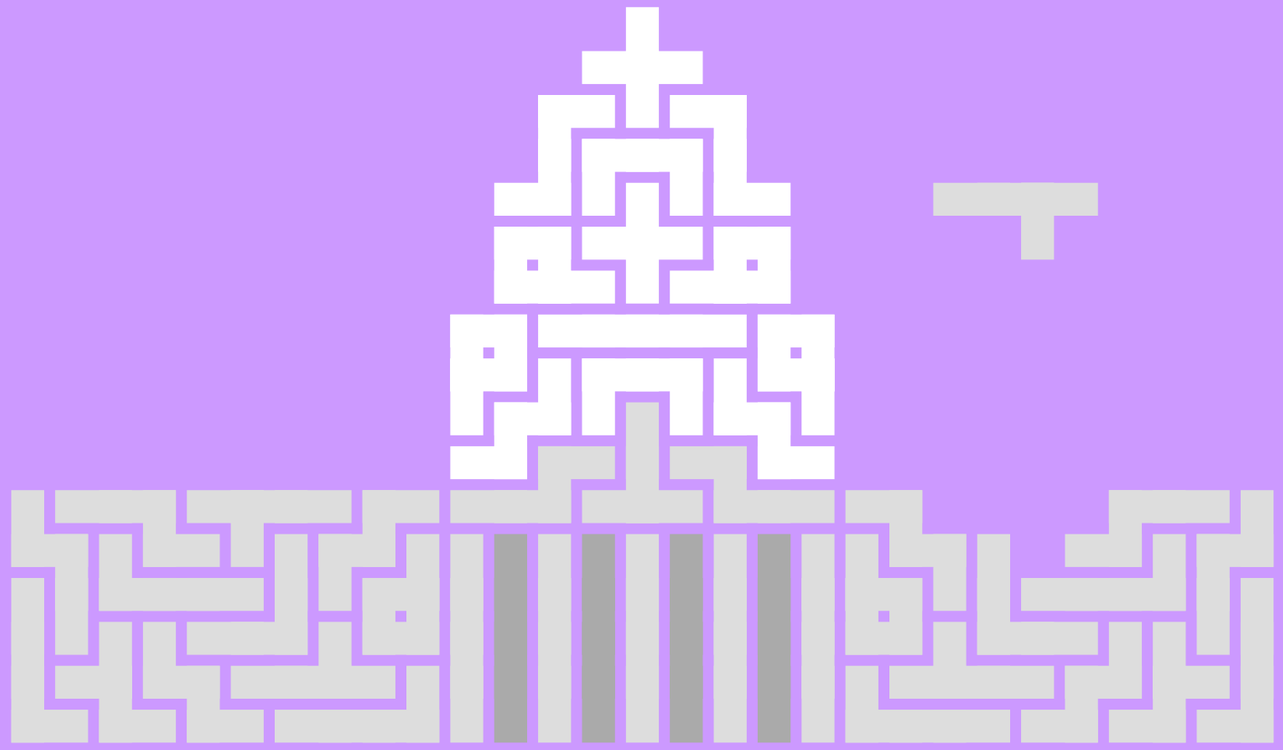

Algorithms For Democratic Decision-Making

Jamie Tucker-Foltz • Yale University • Spring 2026

Lecture 2: **Voting Rules**

Announcements

Please fill out the pre-course survey by TONIGHT if you haven't, even if only auditing!

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Recordings will be posted in the Media Library tab on Canvas

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Studies methods for making societal decisions from agents with heterogeneous preferences

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There are *m* alternatives (AKA candidates)

There are *n* voters

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- A subset of "approved" alternatives
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Most widely-studied setting: picking a single candidate from ranked-choice ballots:

$$f : \Sigma([m])^n \to [m]$$

An interesting preference profile

Example

m = 5 candidates, n = 100 voters, preferences as follows:

Num voters:	33	16	3	8	18	22
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Last choice:	e	a	e	a	a	a

▸ **Which candidate should win? (No right answer!)**

Respond at:

pollev.com/jtuckerfoltz255 or

bit.ly/jtfpoll or

text jtuckerfoltz255 to 37607

Instant Runoff Voting (IRV)

AKA:

- Elimination voting
- Single Transferrable Vote (STV)
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The last candidate remaining wins.

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a:	33
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a:	33
b:	16
c:	0
d:	21
e:	30

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a:	33
b:	0
c:	0
d:	37
e:	30

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a:	33
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c:	0
d:	67
e:	0

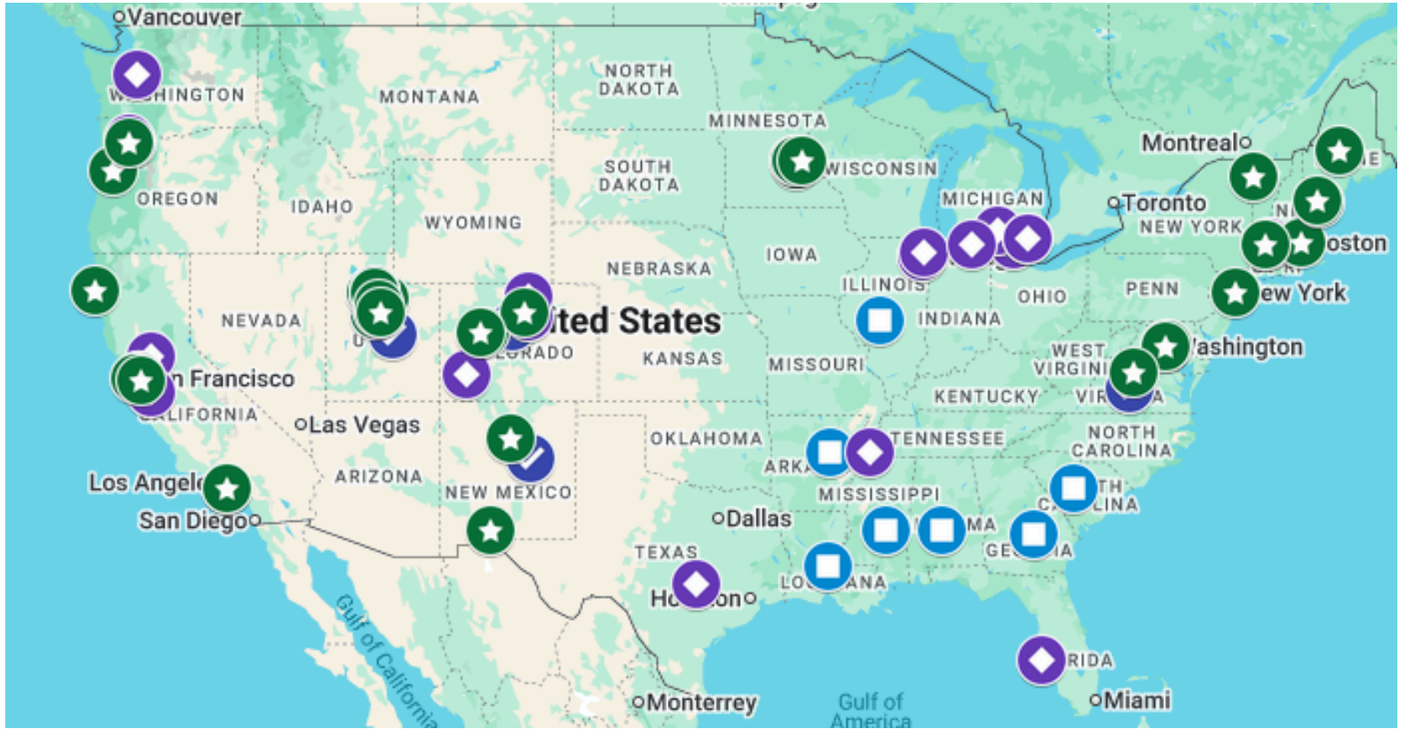
Winner →

Note: We can stop when some candidate has a majority!

IRV use

Was or is currently used in:

- Australia
- Canada
- Estonia
- Hong Kong
- Hungary
- India
- Ireland
- Malta
- New Zealand
- United Kingdom



Credit: https://www.rcvresources.org/where-is-rcv-used/

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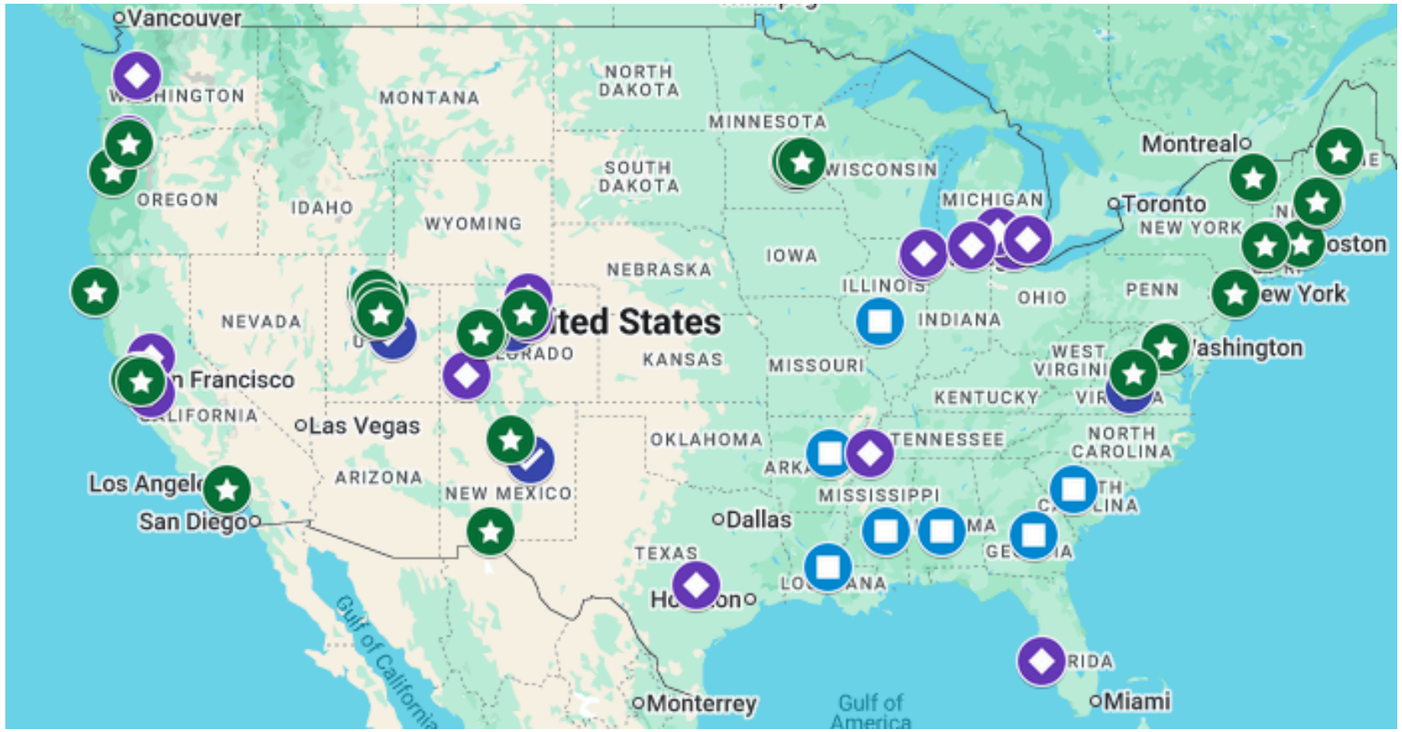

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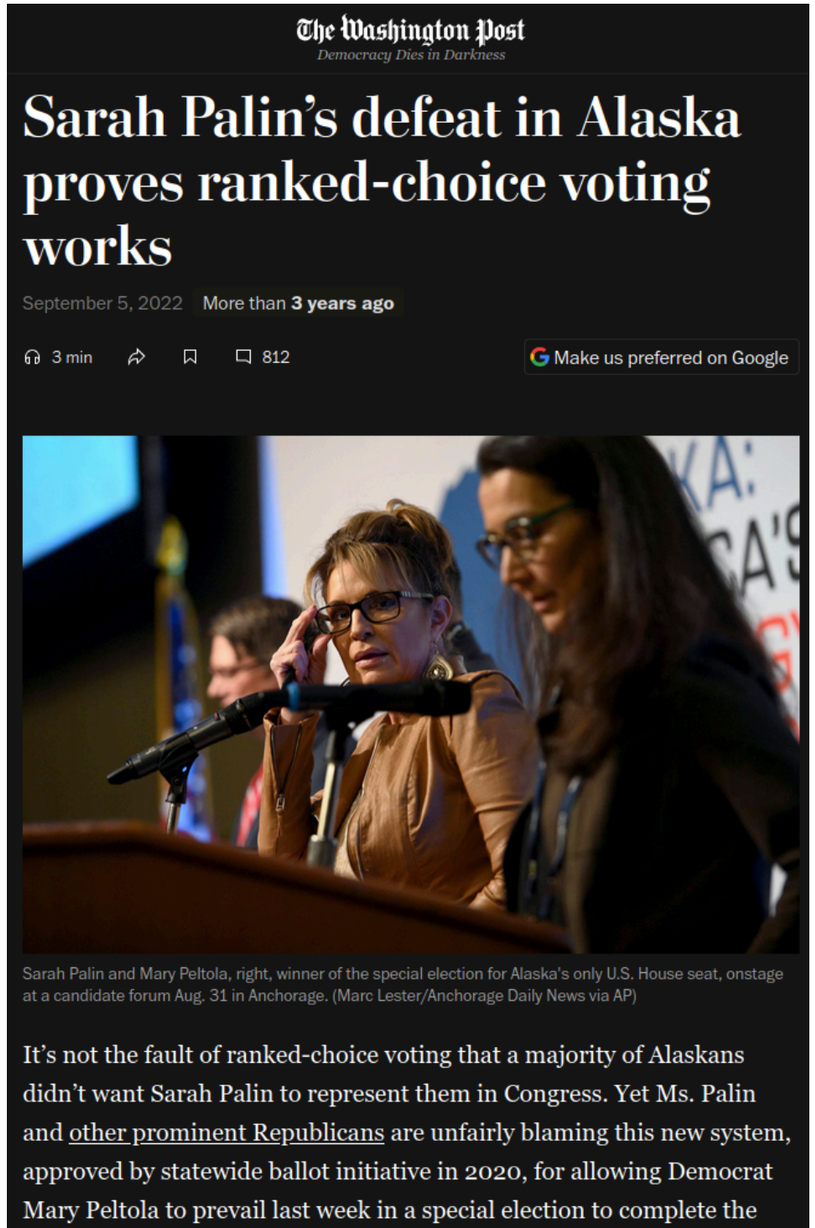


The Washington Post
Democracy Dies in Darkness

Sarah Palin's defeat in Alaska proves ranked-choice voting works

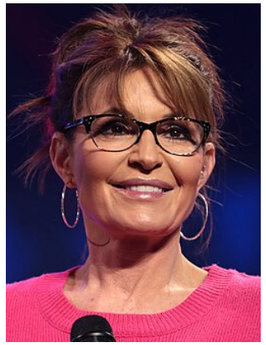
September 5, 2022 More than **3 years ago**

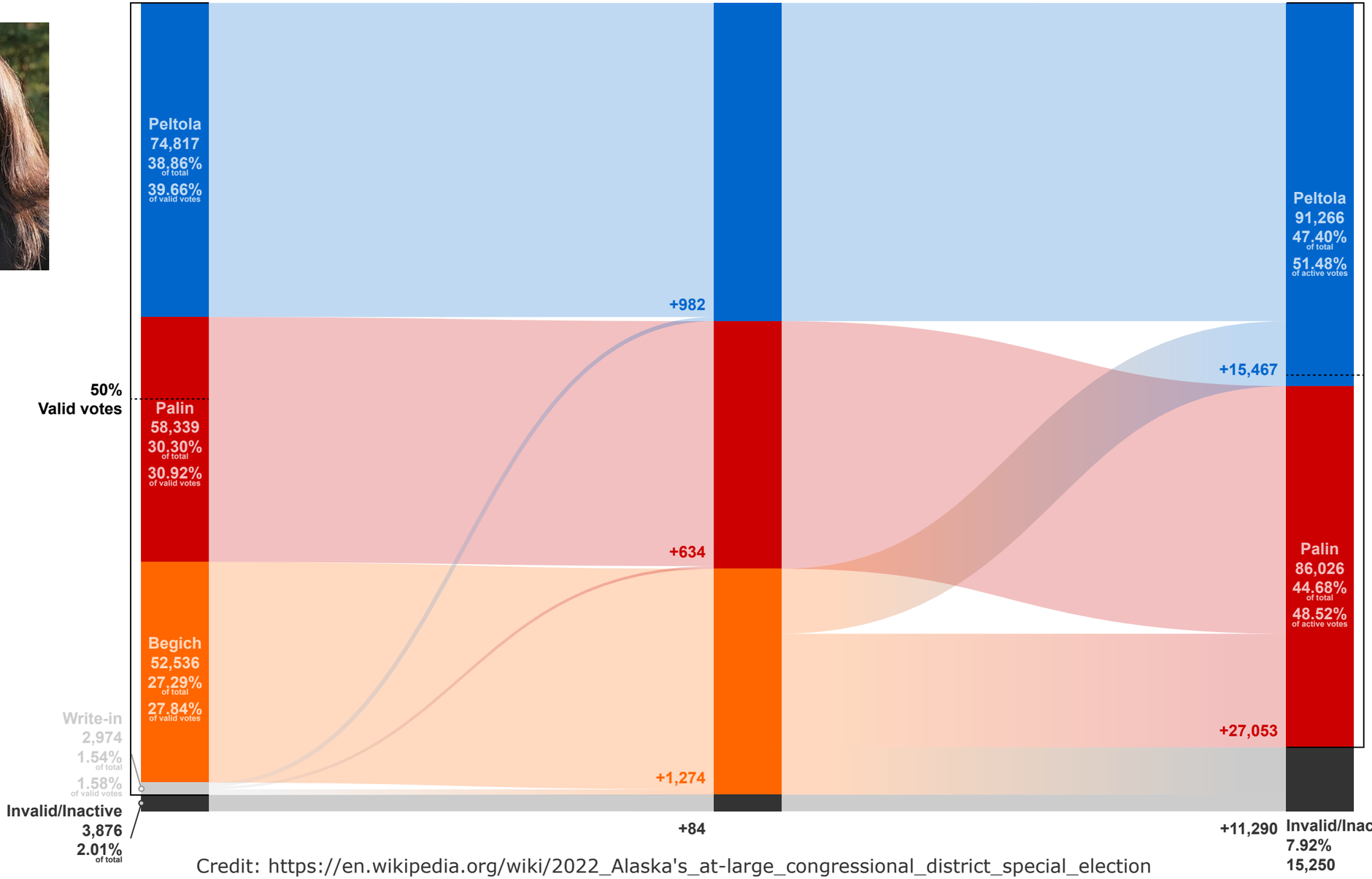
3 min 812 Make us preferred on Google

Sarah Palin and Mary Peltola, right, winner of the special election for Alaska's only U.S. House seat, onstage at a candidate forum Aug. 31 in Anchorage. (Marc Lester/Anchorage Daily News via AP)

It's not the fault of ranked-choice voting that a majority of Alaskans didn't want Sarah Palin to represent them in Congress. Yet Ms. Palin and other prominent Republicans are unfairly blaming this new system, approved by statewide ballot initiative in 2020, for allowing Democrat Mary Peltola to prevail last week in a special election to complete the

Case study: 2022 special election in Alaska





Credit: https://en.wikipedia.org/wiki/2022_Alaska's_at-large_congressional_district_special_election

Borda count

How it works:

(1) For each ballot, assign $m - j$ points to the candidate ranked in position j

(2) Add scores across ballots

(3) Select the candidate with the largest score

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Last choice:	e	a	e	a	a	a

a: $4 \times 33 + 1 \times 3 = 135$

b: $4 \times 16 + 3 \times 33 + 2 \times (3 + 8 + 22) + 1 \times 18 = 247$

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a:	4×33 + 1×3 = 135
b:	4×16 + 3×33 + 2×(3 + 8 + 22) + 1×18 = 247
c:	4×11 + 3×22 + 2×(33 + 16 + 18) = 244
d:	4×18 + 3×(16 + 3) + 1×(33 + 8 + 22) = 192
e:	4×22 + 3×(8 + 18) + 1×16 = 182

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Condorcet vs Borda

Marquis de Condorcet
(1743-1794)

Jean-Charles de Borda
(1733-1799)

Condorcet vs Borda

Marquis de Condorcet
(1743-1794)

b should win

c should win

Jean-Charles de Borda
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c>b: 3 + 8 + 18 + 22 = 51

Condorcet winners

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c>a: 16 + 3 + 8 + 18 + 22 = 67

c>b: 3 + 8 + 18 + 22 = 51

c>d: 33 + 3 + 8 + 22 = 66

c>e: 33 + 16 + 3 + 8 = 60

Condorcet winners

A *Condorcet winner* is an alternative that beats every other alternative in a head-to-head contest.

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✅ Candidate c has > 50/100 votes in any contest

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▸ **Does a condorcet winner always exist?**

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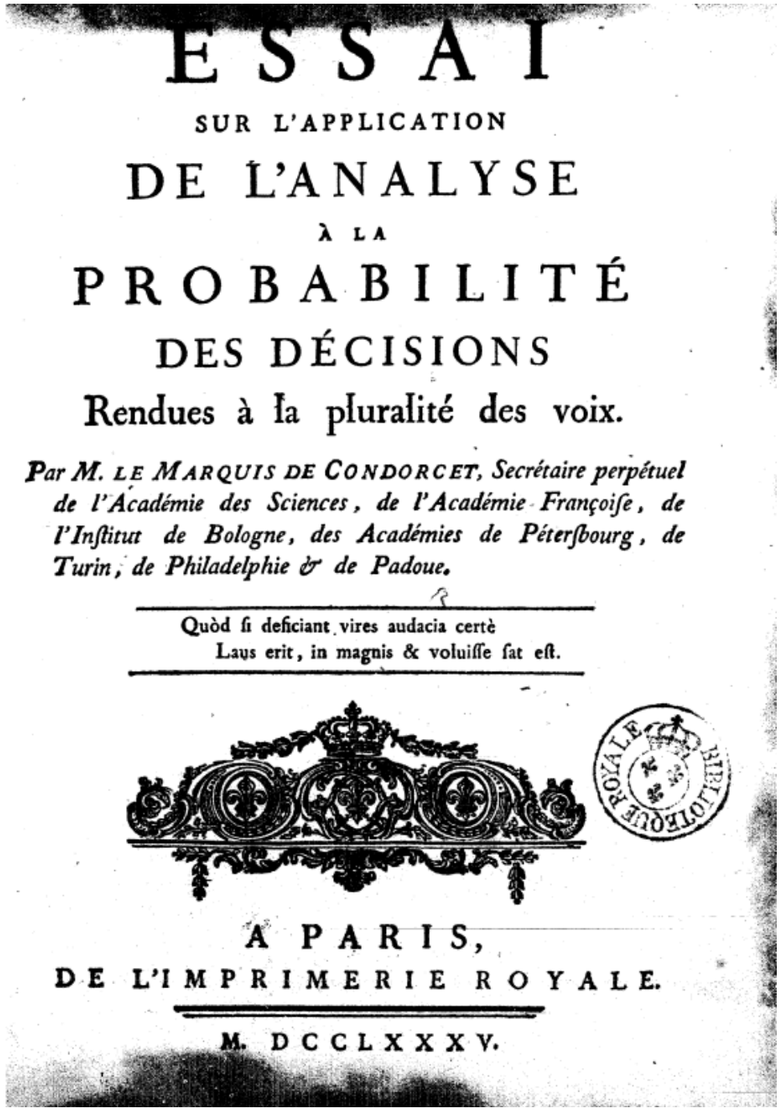
Condorcet's paradox



ESSAI

SUR L'APPLICATION

DE L'ANALYSE

À LA

PROBABILITÉ

DES DÉCISIONS

Rendues à la pluralité des voix.

Par M. LE MARQUIS DE CONDORCET, Secrétaire perpétuel de l'Académie des Sciences, de l'Académie Françoiſe, de l'Inſtitut de Bologne, des Académies de Péterſbourg, de Turin, de Philadelphie & de Padoue.

Quòd ſi deficiant vires audacia certè
Laus erit, in magnis & voluiſſe ſat eſt.

A PARIS,

DE L'IMPRIMERIE ROYALE.

M. DCCLXXXV.

Suppoſons en effet que dans l'exemple déjà choiſi, où l'on a 23 voix pour *A*, 19 pour *B*, 18 pour *C*, les 23 voix pour *A* ſoient pour la propoſition *B* vaut mieux que *C*; cette propoſition aura une pluralité de 42 voix contre 18.

Suppoſons enſuite que des 19 voix en faveur de *B*, il y en ait 17 pour *C* vaut mieux que *A*, & 2 pour la propoſition contradictoire; cette propoſition *C* vaut mieux que *A* aura une pluralité de 35 voix contre 25. Suppoſons enfin que des 18 voix pour *C*, 10 ſoient pour la propoſition *A* vaut mieux que *B*, & 8 pour la propoſition contradictoire, nous aurons une pluralité de 33 voix contre 27 en faveur de la propoſition *A* vaut mieux que *B*. Le ſyſtème qui obtient la pluralité ſera donc compoſé des trois propoſitions,

A vaut mieux que *B*,
C vaut mieux que *A*,
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Ce ſyſtème eſt le troiſième, & un de ceux qui impliquent contradiction.

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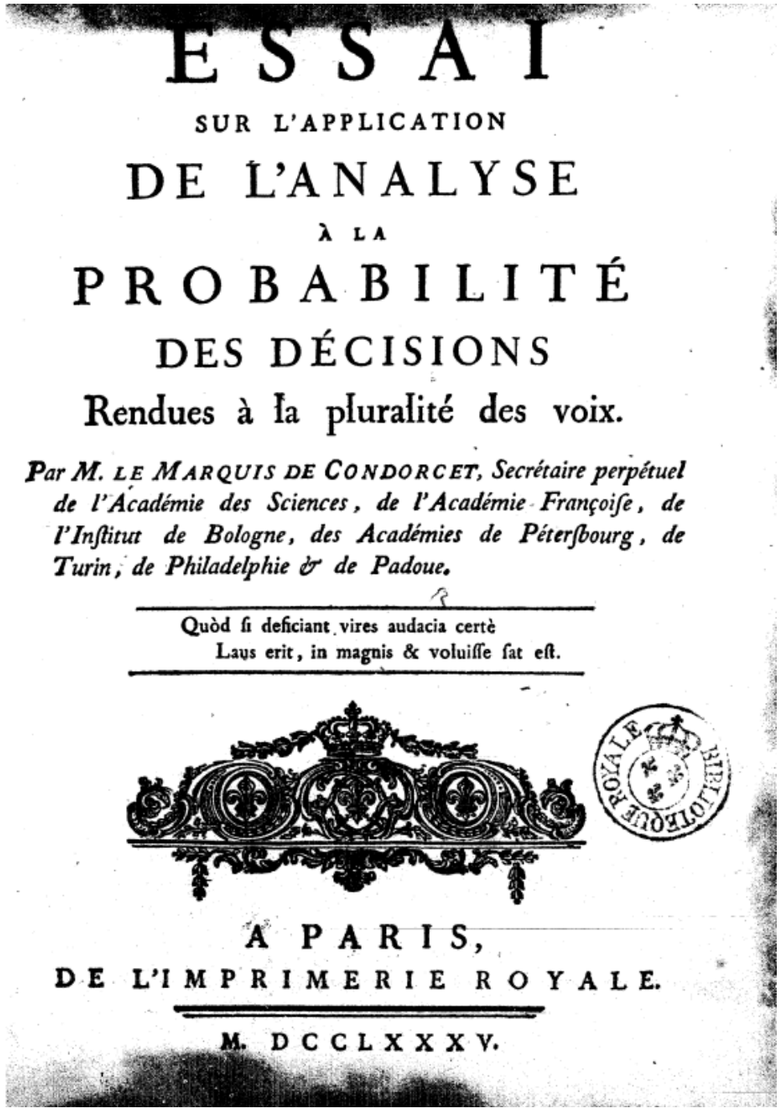

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Example

1	1	1
a	b	c
b	c	a
c	a	b

Condorcet consistency and tournament graphs

A voting rule is *Condorcet consistent* if it selects a condorcet winner when it exists

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Proposition

The Borda count and IRV are not Condorcet consistent.

Proof. See previous example. ■

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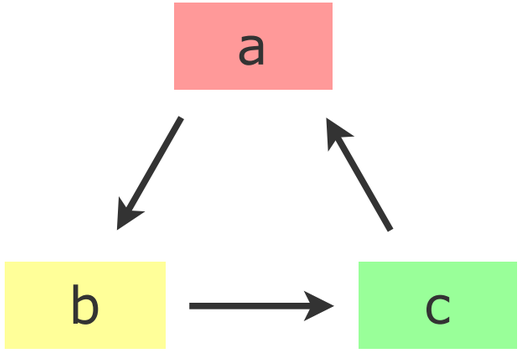

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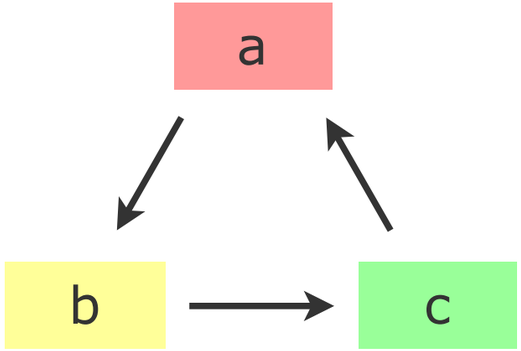
A Condorcet winner is a source in the tournament graph



A *Condorcet cycle* is a cycle in the tournament graph

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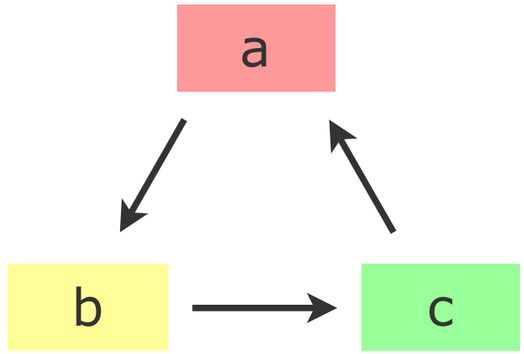
Proposition

The Borda count and IRV are not Condorcet consistent.

Proof. See previous example. ■

Example

1	1	1
a	b	c
b	c	a
c	a	b



A Condorcet winner is a source in the tournament graph

A *Condorcet cycle* is a cycle in the tournament graph

The *[weighted] tournament graph* of a preference profile has a vertex for each candidate and an edge from a to b if a beats b in a head-to-head contest. The weight is the fraction of voters preferring a to b.

Voting rule classification [Fishburn, 1977]:

(C1) Depends only on the tournament graph
(C2) Depends on the weighted tournament graph
(C3) All other rules

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  - The amount by which a beats b is the weight of the weakest link in the best path
  - There always exists a candidate x that beats any other candidate y by more than y beats x. That x is the winner.

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  - The Smith set is the "top cycle of the tournament graph" - a set of candidates that beats every other candidate outside the set in a head-to-head contest
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- Dodgson's rule (C3): Select the candidate minimizing the number of swaps to become a Condorcet winner