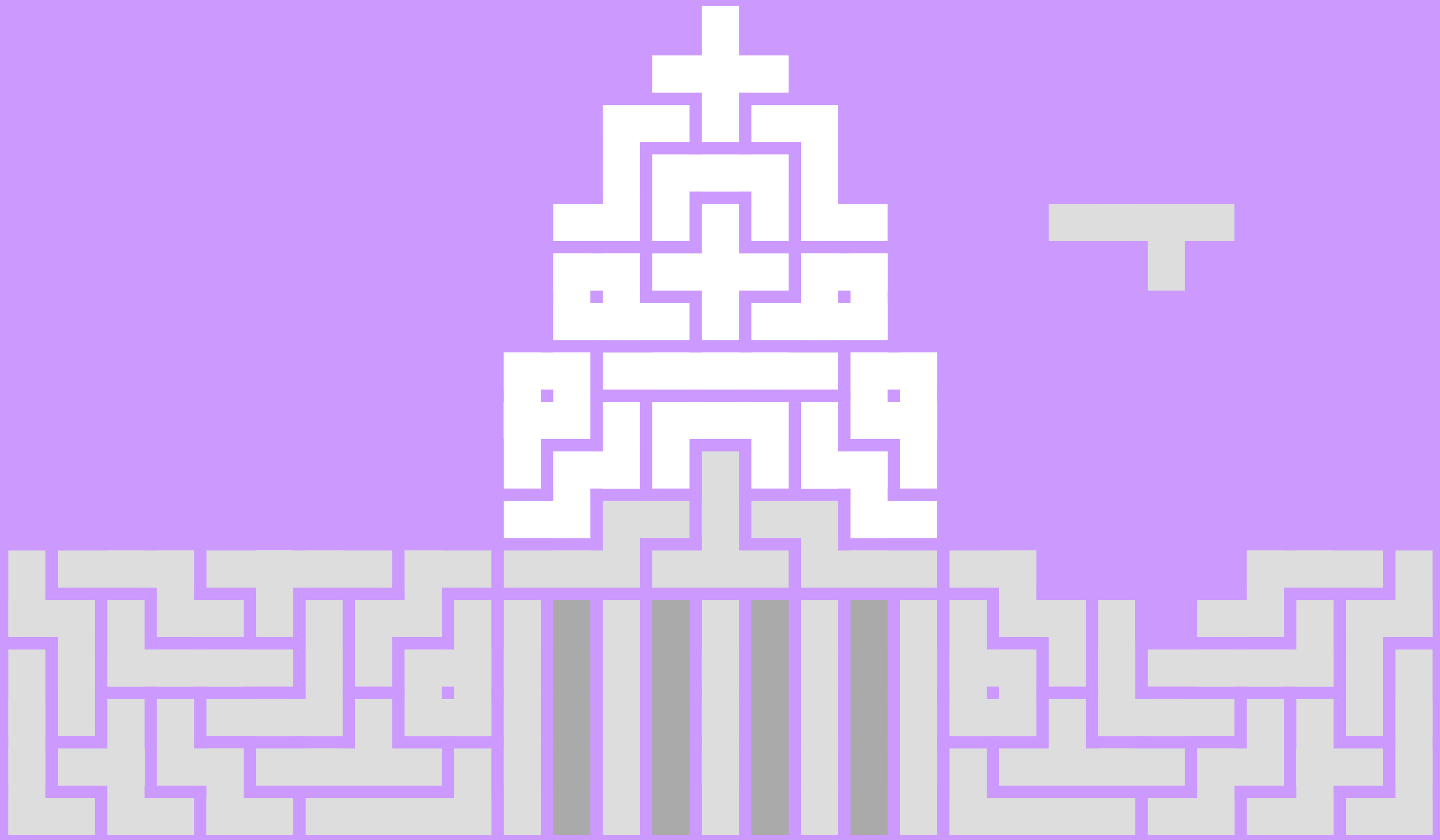
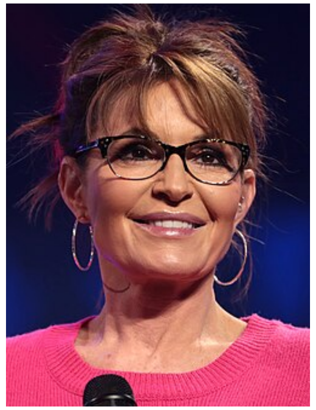


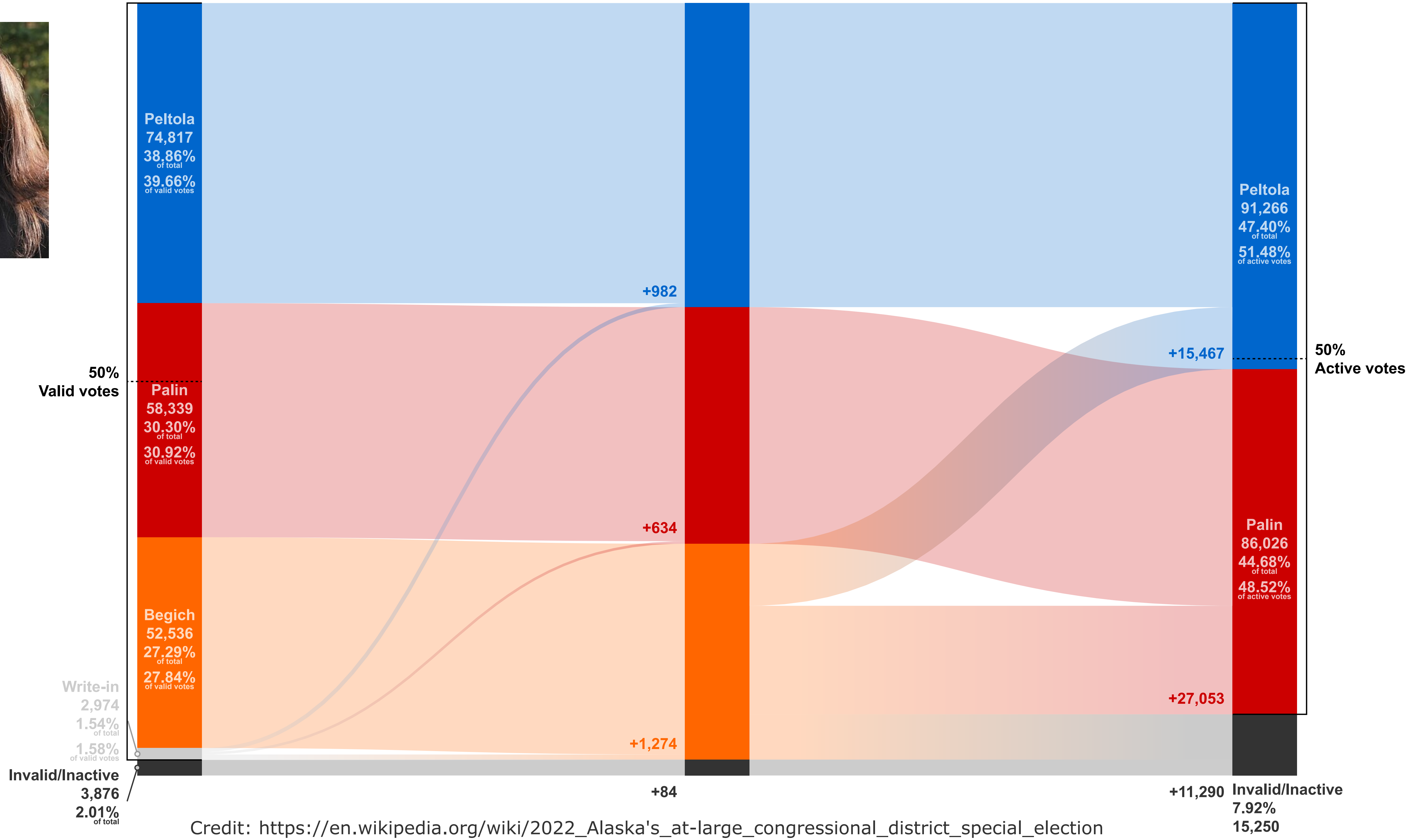
Algorithms For Democratic Decision-Making

Jamie Tucker-Foltz • Yale University • Spring 2026

Lecture 4: **Strategic Voting**

Recall, the 2022 special election in Alaska





Credit: https://en.wikipedia.org/wiki/2022_Alaska's_at-large_congressional_district_special_election

Is strategyproof voting possible?

Theorem (Gibbard-Satterthwaite)

When there are at least 3 alternatives, there is no social choice function satisfying the following three axioms:

- *Onto - Every alternative can win.*
- *Strategyproofness - No voter can ever benefit from misreporting their preferences.*
- *Non-dictatorship - More than one voter's preferences are taken into acount.*

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Limitations:

1. This definition of strategyproofness is very strong: truthtelling must be a *dominant strategy*.

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2. Even if strategic manipulation is possible, it may not be straightforward for voters to discover.

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Limitations:

1. This definition of strategyproofness is very strong: truthtelling must be a *dominant strategy*.
2. Even if strategic manipulation is possible, it may not be straightforward for voters to discover.
3. The theorem only applies to deterministic rules.

Randomized strategyproof rules?

Example 1

Pick a random voter's top choice.

This is called *random dictatorship*.

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Example 2

Pick a random pair of alternatives and select the one with more votes in a head-to-head contest.

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Theorem (Gibbard, 1977)

Any randomized strategyproof voting rule is a lottery over deterministic rules, each of which depends on either:

- *at most one voter, or*
- *at most two alternatives.*

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Corollary 1

Strategyproofness + ex post Pareto Efficiency = random dictatorship.

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Any randomized strategyproof voting rule is a lottery over deterministic rules, each of which depends on either:

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- *at most two alternatives.*

Corollary 1

Strategyproofness + ex post Pareto Efficiency = random dictatorship.

Corollary 2

Strategyproofness + ***ex ante*** *Pareto Efficiency = deterministic dictatorship.*

Strategic voting can be **hard**

Example

46	44	42	40	44	11	8	11	8	20	20
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b
w	z	z	y	y	z	c_1	z	c_2	z	z
c_1	c_1	c_1	z	z	y	z	y	z	y	y
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a

Strategic voting can be **hard**

Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	?
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

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Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	?
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

▸ **You are the last voter. Which candidate(s) can you make win under IRV?**

Respond at:

pollev.com/jtuckerfoltz255 or

bit.ly/jtfpoll or

text jtuckerfoltz255 to 37607

Strategic voting can be **hard**

Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	?
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

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Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	ā	ā	b	b̄	?
y	w	y	w	w	ā	b̄	a	b	b̄	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
ā	ā	ā	ā	ā	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	ā	b̄	a	a	ā	?
b̄	b̄	b̄	b̄	b̄	b̄	b	b	b̄	ā	a	?

z:	46
y:	44
w:	42
c_1:	40
c_2:	44
a:	19
ā:	19
b:	20
b̄:	20

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Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	ā	ā	b	b̄	?
y	w	y	w	w	ā	b̄	a	b	b̄	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
ā	ā	ā	ā	ā	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	ā	b̄	a	a	ā	?
b̄	b̄	b̄	b̄	b̄	b̄	b	b	b̄	ā	a	?

z: 46
y: 44
w: 42
c_1: 40
c_2: 44
a: 19
ā: 19
b: 20
b̄: 20

Symmetry between:

a	ā
b̄	b
c_1	c_2

▸ **You are the last voter. Which candidate(s) can you make win under IRV?** Answer: y and w.

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Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	a
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

z: 46
y: 44
w: 42
c_1: 40
c_2: 44
a: 20
$\bar{a}$: 19
b: 20
$\bar{b}$: 20

Symmetry between:

a	$\bar{a}$
$\bar{b}$	b
c_1	c_2

WLOG, assume that you rank a first.

▸ **You are the last voter. Which candidate(s) can you make win under IRV?** Answer: y and w.

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Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	a
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

z: 46
y: 44
w: 42
c_1: 40
c_2: 44
a: 31
$\bar{a}$: 0
b: 28
$\bar{b}$: 20

Symmetry between:

a	$\bar{a}$
$\bar{b}$	b
c_1	c_2

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46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	a
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

z: 46
y: 44
w: 42
c_1: 40
c_2: 44
a: 31
$\bar{a}$: 0
b: 48
$\bar{b}$: 0

Symmetry between:

a	$\bar{a}$
$\bar{b}$	b
c_1	c_2

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Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	a
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

z:	68
y:	44
w:	42
c_1:	48
c_2:	44
a:	0
$\bar{a}$:	0
b:	48
$\bar{b}$:	0

Symmetry between:

a	$\bar{a}$
$\bar{b}$	b
c_1	c_2

WLOG, assume that you rank a first.

▸ **You are the last voter. Which candidate(s) can you make win under IRV?** Answer: y and w.

Strategic voting can be **hard**

Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	a
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

z:	68	
y:	44	
w:	42	
c_1:	48	40
c_2:	44	52
a:	0	
$\bar{a}$:	0	
b:	48	
$\bar{b}$:	0	

If $\bar{a}$ had instead been ranked first

Symmetry between:

a	$\bar{a}$
$\bar{b}$	b
c_1	c_2

You are the last voter. Which candidate(s) can you make win under IRV? Answer: y and w.

Strategic voting can be **hard**

Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	ā	ā	b	b̄	a
y	w	y	w	w	ā	b̄	a	b	b̄	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
ā	ā	ā	ā	ā	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	ā	b̄	a	a	ā	?
b̄	b̄	b̄	b̄	b̄	b̄	b	b	b̄	ā	a	?

z:	68	
y:	86	
w:	0	
c_1:	48	40
c_2:	44	52
a:	0	
ā:	0	
b:	48	
b̄:	0	

If ā had instead been ranked first

Symmetry between:

a	ā
b̄	b
c_1	c_2

▸ **You are the last voter. Which candidate(s) can you make win under IRV?** Answer: y and w.

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Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	a
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

z:	68	
y:	130	
w:	0	
c_1:	48	40
c_2:	0	52
a:	0	
$\bar{a}$:	0	
b:	48	
$\bar{b}$:	0	

↑ If $\bar{a}$ had instead been ranked first

Symmetry between:

a	$\bar{a}$
$\bar{b}$	b
c_1	c_2

▸ **You are the last voter. Which candidate(s) can you make win under IRV?** Answer: y and w.

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Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	a
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

z:	124	
y:	170	
w:	0	
c_1:	0	40
c_2:	0	52
a:	0	
$\bar{a}$:	0	
b:	0	
$\bar{b}$:	0	

If $\bar{a}$ had instead been ranked first

▸ **You are the last voter. Which candidate(s) can you make win under IRV?** Answer: y and w.

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Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	a
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

z:	124	
y:	170	← Winner
w:	0	
c_1:	0	40
c_2:	0	52
a:	0	
$\bar{a}$:	0	
b:	0	
$\bar{b}$:	0	

↑ If $\bar{a}$ had instead been ranked first

▸ **You are the last voter. Which candidate(s) can you make win under IRV?** Answer: y and w.

Strategic voting can be **hard**

Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	a
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

z:	68	
y:	44	
w:	42	
c_1:	48	40
c_2:	44	52
a:	0	
$\bar{a}$:	0	
b:	48	
$\bar{b}$:	0	

If $\bar{a}$ had instead been ranked first

▸ **You are the last voter. Which candidate(s) can you make win under IRV?** Answer: y and w.

Strategic voting can be **hard**

Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	$\bar{a}$
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

z:	68
y:	44
w:	42
c_1:	40
c_2:	52
a:	0
$\bar{a}$:	0
b:	0
$\bar{b}$:	48

▸ **You are the last voter. Which candidate(s) can you make win under IRV?** Answer: y and w.

Strategic voting can be **hard**

Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	$\bar{a}$
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

z:	68
y:	44
w:	82
c_1:	0
c_2:	52
a:	0
$\bar{a}$:	0
b:	0
$\bar{b}$:	48

▸ **You are the last voter. Which candidate(s) can you make win under IRV?** Answer: y and w.

Strategic voting can be **hard**

Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	ā	ā	b	b̄	ā
y	w	y	w	w	ā	b̄	a	b	b̄	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
ā	ā	ā	ā	ā	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	ā	b̄	a	a	ā	?
b̄	b̄	b̄	b̄	b̄	b̄	b	b	b̄	ā	a	?

z:	68
y:	0
w:	126
c_1:	0
c_2:	52
a:	0
ā:	0
b:	0
b̄:	48

▸ **You are the last voter. Which candidate(s) can you make win under IRV?** Answer: y and w.

Strategic voting can be **hard**

Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	$\bar{a}$
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

z:	116
y:	0
w:	126
c_1:	0
c_2:	52
a:	0
$\bar{a}$:	0
b:	0
$\bar{b}$:	0

▸ **You are the last voter. Which candidate(s) can you make win under IRV?** Answer: y and w.

Strategic voting can be **hard**

Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$	$\bar{a}$
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$	?
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a	?

z:	124
y:	0
w:	170
c_1:	0
c_2:	0
a:	0
$\bar{a}$:	0
b:	0
$\bar{b}$:	0

▸ **You are the last voter. Which candidate(s) can you make win under IRV?** Answer: y and w.

Strategic voting can be **hard**

Example

46	44	42	40	44	11	8	11	8	20	20	1
z	y	w	c_1	c_2	a	a	ā	ā	b	b̄	ā
y	w	y	w	w	ā	b̄	a	b	b̄	b	?
w	z	z	y	y	z	c_1	z	c_2	z	z	?
c_1	c_1	c_1	z	z	y	z	y	z	y	y	?
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w	?
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1	?
ā	ā	ā	ā	ā	c_2	c_2	c_1	c_1	c_2	c_2	?
b	b	b	b	b	b	ā	b̄	a	a	ā	?
b̄	b̄	b̄	b̄	b̄	b̄	b	b	b̄	ā	a	?

z:	124
y:	0
w:	170
c_1:	0
c_2:	0
a:	0
ā:	0
b:	0
b̄:	0

← Winner (w)

You are the last voter. Which candidate(s) can you make win under IRV? Answer: y and w.

Polynomial time

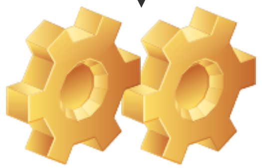
46	44	42	40	44	11	8	11	8	20	20
z	y	w	c_1	c_2	a	a	ā	ā	b	b̄
y	w	y	w	w	ā	b̄	a	b	b̄	b
w	z	z	y	y	z	c_1	z	c_2	z	z
c_1	c_1	c_1	z	z	y	z	y	z	y	y
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1
ā	ā	ā	ā	ā	c_2	c_2	c_1	c_1	c_2	c_2
b	b	b	b	b	b	ā	b̄	a	a	ā
b̄	b̄	b̄	b̄	b̄	b̄	b	b	b̄	ā	a

Polynomial time

46	44	42	40	44	11	8	11	8	20	20
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b
w	z	z	y	y	z	c_1	z	c_2	z	z
c_1	c_1	c_1	z	z	y	z	y	z	y	y
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a

, y

1001101110101000010...



Yes, you can make y win.

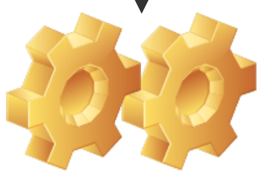
Polynomial time

46	44	42	40	44	11	8	11	8	20	20
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b
w	z	z	y	y	z	c_1	z	c_2	z	z
c_1	c_1	c_1	z	z	y	z	y	z	y	y
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a

, y

Length n

10011011101010000010...



f(n) steps

Yes, you can make y win.

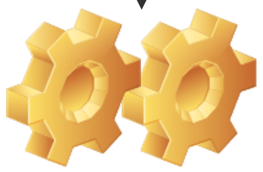
Polynomial time

46	44	42	40	44	11	8	11	8	20	20
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b
w	z	z	y	y	z	c_1	z	c_2	z	z
c_1	c_1	c_1	z	z	y	z	y	z	y	y
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a

, y

Length *n*

10011011101010000010...



f(*n*) steps

Yes, you can make y win.

An algorithm runs in *polynomial time* if the number of steps *f*(*n*) is bounded by some polynomial function *p*(*n*).

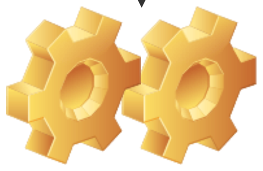
Polynomial time

46	44	42	40	44	11	8	11	8	20	20
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b
w	z	z	y	y	z	c_1	z	c_2	z	z
c_1	c_1	c_1	z	z	y	z	y	z	y	y
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a

, y

Length *n*

1001101110101000010...



f(*n*) steps

Yes, you can make y win.

An algorithm runs in *polynomial time* if the number of steps *f*(*n*) is bounded by some polynomial function *p*(*n*). This definition remains the same even if you change:

- The computational model

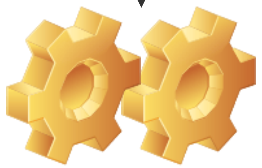
Polynomial time

46	44	42	40	44	11	8	11	8	20	20
z	y	w	c_1	c_2	a	a	$\bar{a}$	$\bar{a}$	b	$\bar{b}$
y	w	y	w	w	$\bar{a}$	$\bar{b}$	a	b	$\bar{b}$	b
w	z	z	y	y	z	c_1	z	c_2	z	z
c_1	c_1	c_1	z	z	y	z	y	z	y	y
c_2	c_2	c_2	c_2	c_1	w	y	w	y	w	w
a	a	a	a	a	c_1	w	c_2	w	c_1	c_1
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	c_2	c_2	c_1	c_1	c_2	c_2
b	b	b	b	b	b	$\bar{a}$	$\bar{b}$	a	a	$\bar{a}$
$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	b	b	$\bar{b}$	$\bar{a}$	a

, y

Length *n*

100110111010100001 0...



f(*n*) steps

Yes, you can make y win.

An algorithm runs in *polynomial time* if the number of steps *f*(*n*) is bounded by some polynomial function *p*(*n*). This definition remains the same even if you change:

- The computational model
- The definition of *n* (e.g., we can take *n* to be the number of voters + the number of candidates, or the size of the table of preferences (writing out duplicate columns))

Boolean satisfiability

The *boolean satisfiability problem* (or *SAT* for short) is the problem of determining whether a given boolean formula is satisfiable. For example,

$$(x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_1} \lor \overline{x_2}) \land (x_2 \lor x_3 \lor x_3) \land (\overline{x_2} \lor \overline{x_3} \lor \overline{x_3})$$

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$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3})$$

requires that

- at least one of x_1 and x_2 is **TRUE** and the other is FALSE, and
- at least one of x_2 and x_3 is **TRUE** and the other is FALSE

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requires that

- at least one of x_1 and x_2 is **TRUE** and the other is FALSE, and
- at least one of x_2 and x_3 is **TRUE** and the other is FALSE

so we can satisfy the formula by setting

$$x_1 = x_3 = \mathtt{TRUE},\ x_2 = \mathtt{FALSE}, \qquad \text{or} \qquad x_1 = x_3 = \mathtt{TRUE},\ x_2 = \mathtt{FALSE}.$$

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The *boolean satisfiability problem* (or *SAT* for short) is the problem of determining whether a given boolean formula is satisfiable. For example,

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3})$$

requires that

- at least one of x_1 and x_2 is **`TRUE`** and the other is FALSE, and
- at least one of x_2 and x_3 is **`TRUE`** and the other is FALSE

so we can satisfy the formula by setting

$$x_1 = x_3 = \mathtt{TRUE},\ x_2 = \mathtt{FALSE}, \qquad \text{or} \qquad x_1 = x_3 = \mathtt{TRUE},\ x_2 = \mathtt{FALSE}.$$

The special case of SAT where the formula is a conjunction of clauses, each with 3 literals, each clause is all positive or all negative, and each literal appears exactly twice, is called *Monotone 3-SAT-(2, 2)*.

Strategic voting can require solving Monotone 3-SAT-(2, 2)

Proposition (adapted from Bartholdi and Orlin, 2003)

For any Monotone 3-SAT-(2, 2) formula f with n variables, there is an associated preference profile (with one undetermined ballot) where f is satisfiable if and only if candidate y can be made the IRV winner:

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For any Monotone 3-SAT-(2, 2) formula f with n variables, there is an associated preference profile (with one undetermined ballot) where f is satisfiable if and only if candidate y can be made the IRV winner:

$200n + 6$	$200n + 4$	$200n + 2$
z	y	w
y		y

For each clause j

$200n$
c_j
w
y

For each variable i that appears positively in clauses p1(i) and p2(i) and negatively in clauses n1(i) and n2(i)

11	4	4	11	4	4	20	20
a_i	a_i	a_i	$\bar{a}_i$	$\bar{a}_i$	$\bar{a}_i$	b_i	$\bar{b}_i$
$\bar{a}_i$	$\bar{b}_i$	$\bar{b}_i$	a_i	b_i	b_i	z	z
z	$c_{p1(i)}$	$c_{p2(i)}$	z	$c_{n1(i)}$	$c_{n2(i)}$	y	y
y	z	z	y	z	z		
	y	y		y	y		

$100(n + i - 1)$			
a_i	$\bar{a}_i$	b_i	$\bar{b}_i$
z	z	$\bar{b}_i$	b_i
y	y	y	y

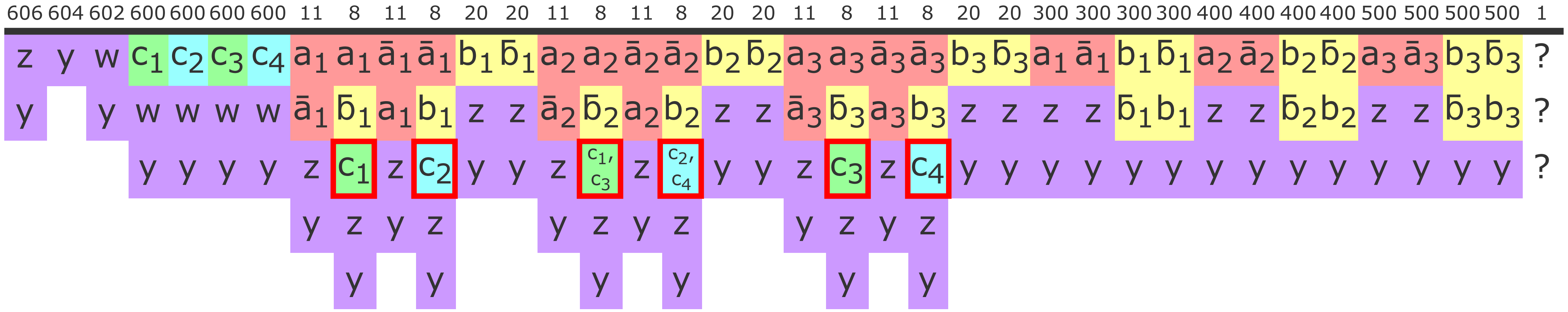
Proof illustration

$$(x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_1} \lor \overline{x_2}) \land (x_2 \lor x_3 \lor x_3) \land (\overline{x_2} \lor \overline{x_3} \lor \overline{x_3}) \mapsto$$

606	604	602	600	600	600	600	11	8	11	8	20	20	11	8	11	8	20	20	11	8	11	8	20	20	300	300	300	300	400	400	400	400	500	500	500	500	1
z	y	w	c_1	c_2	c_3	c_4	a_1	a_1	$\bar{a}_1$	$\bar{a}_1$	b_1	$\bar{b}_1$	a_2	a_2	$\bar{a}_2$	$\bar{a}_2$	b_2	$\bar{b}_2$	a_3	a_3	$\bar{a}_3$	$\bar{a}_3$	b_3	$\bar{b}_3$	a_1	$\bar{a}_1$	b_1	$\bar{b}_1$	a_2	$\bar{a}_2$	b_2	$\bar{b}_2$	a_3	$\bar{a}_3$	b_3	$\bar{b}_3$	?
y		y	w	w	w	w	$\bar{a}_1$	$\bar{b}_1$	a_1	b_1	z	z	$\bar{a}_2$	$\bar{b}_2$	a_2	b_2	z	z	$\bar{a}_3$	$\bar{b}_3$	a_3	b_3	z	z	z	z	$\bar{b}_1$	b_1	z	z	$\bar{b}_2$	b_2	z	z	$\bar{b}_3$	b_3	?
			y	y	y	y	z	c_1	z	c_2	y	y	z	c_1, c_3	z	c_2, c_4	y	y	z	c_3	z	c_4	y	y	y	y	y	y	y	y	y	y	y	y	y	y	?
							y	z	y	z			y	z	y	z			y	z	y	z															
								y		y				y		y				y		y															

Proof illustration

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$



Proof illustration

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$

606	604	602	600	600	600	600	11	8	11	8	20	20	11	8	11	8	20	20	11	8	11	8	20	20	300	300	300	300	400	400	400	400	500	500	500	500	1
z	y	w	c_1	c_2	c_3	c_4	a_1	a_1	$\bar{a}_1$	$\bar{a}_1$	b_1	$\bar{b}_1$	a_2	a_2	$\bar{a}_2$	$\bar{a}_2$	b_2	$\bar{b}_2$	a_3	a_3	$\bar{a}_3$	$\bar{a}_3$	b_3	$\bar{b}_3$	a_1	$\bar{a}_1$	b_1	$\bar{b}_1$	a_2	$\bar{a}_2$	b_2	$\bar{b}_2$	a_3	$\bar{a}_3$	b_3	$\bar{b}_3$	?
y		y	w	w	w	w	$\bar{a}_1$	$\bar{b}_1$	a_1	b_1	z	z	$\bar{a}_2$	$\bar{b}_2$	a_2	b_2	z	z	$\bar{a}_3$	$\bar{b}_3$	a_3	b_3	z	z	z	z	$\bar{b}_1$	b_1	z	z	$\bar{b}_2$	b_2	z	z	$\bar{b}_3$	b_3	?
			y	y	y	y	z	c_1	z	c_2	y	y	z	c_1, c_3	z	c_2, c_4	y	y	z	c_3	z	c_4	y	y	y	y	y	y	y	y	y	y	y	y	y	y	?
							y	z	y	z			y	z	y	z			y	z	y	z															
								y		y				y		y				y		y															

a_1:	319
$\bar{a}_1$:	319
a_2:	419
$\bar{a}_2$:	419
a_3:	519
$\bar{a}_3$:	519

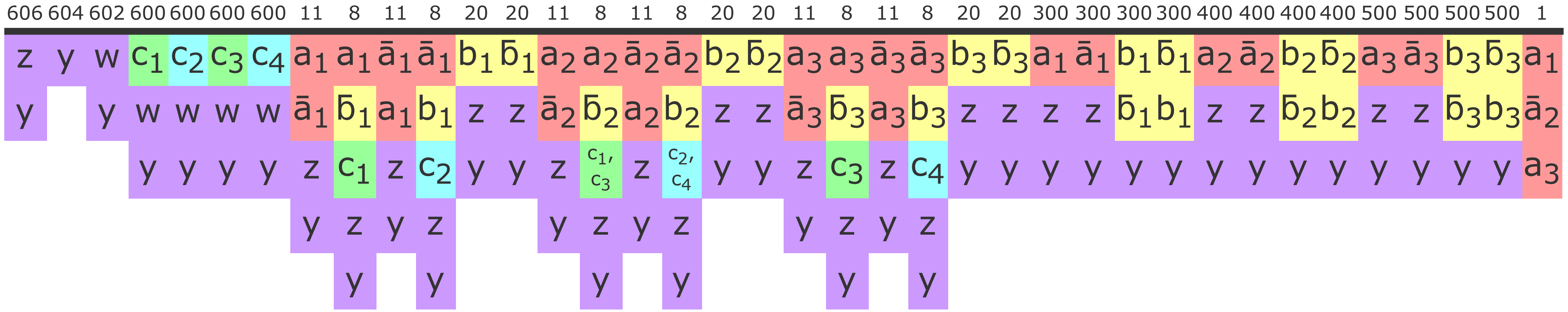
b_1:	320
$\bar{b}_1$:	320
b_2:	420
$\bar{b}_2$:	420
b_3:	520
$\bar{b}_3$:	520

c_1:	600
c_2:	600
c_3:	600
c_4:	600

w:	602
y:	604
z:	606

Proof illustration

$$(x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_1} \lor \overline{x_2}) \land (x_2 \lor x_3 \lor x_3) \land (\overline{x_2} \lor \overline{x_3} \lor \overline{x_3}) \mapsto$$



a_1:	320
$\bar{a}_1$:	319
a_2:	419
$\bar{a}_2$:	419
a_3:	519
$\bar{a}_3$:	519

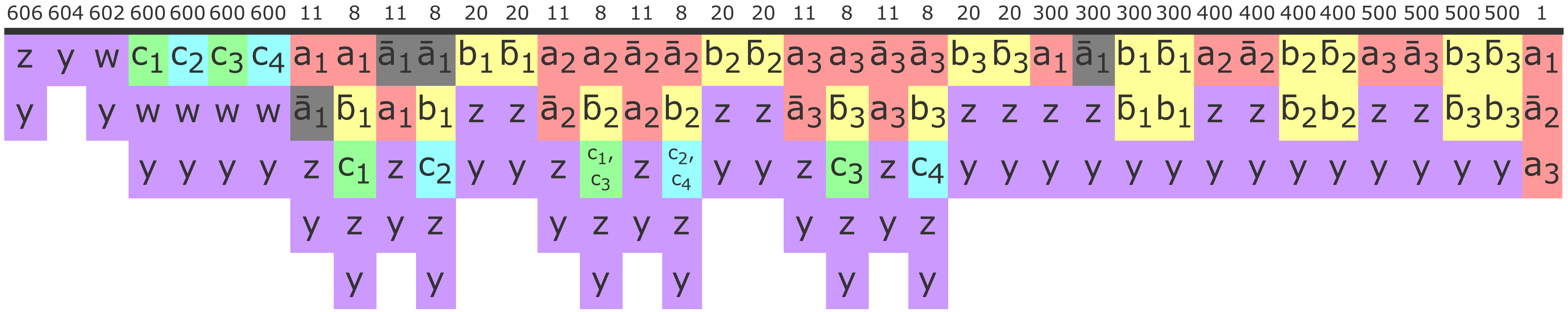
b_1:	320
$\bar{b}_1$:	320
b_2:	420
$\bar{b}_2$:	420
b_3:	520
$\bar{b}_3$:	520

c_1:	600
c_2:	600
c_3:	600
c_4:	600

w:	602
y:	604
z:	606

Proof illustration

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$



a_1: 331
$\bar{a}_1$: 0
a_2: 419
$\bar{a}_2$: 419
a_3: 519
$\bar{a}_3$: 519

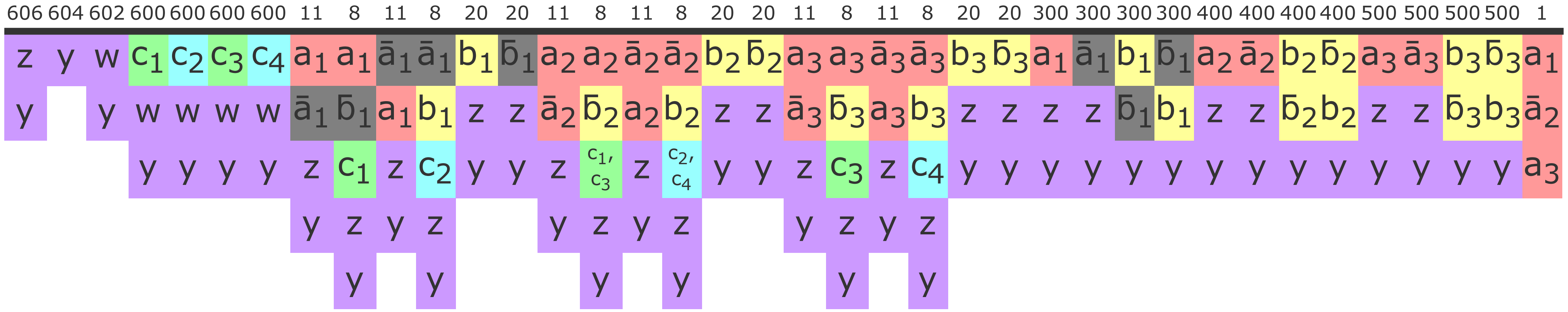
b_1: 328
$\bar{b}_1$: 320
b_2: 420
$\bar{b}_2$: 420
b_3: 520
$\bar{b}_3$: 520

c_1: 600
c_2: 600
c_3: 600
c_4: 600

w: 602
y: 604
z: 906

Proof illustration

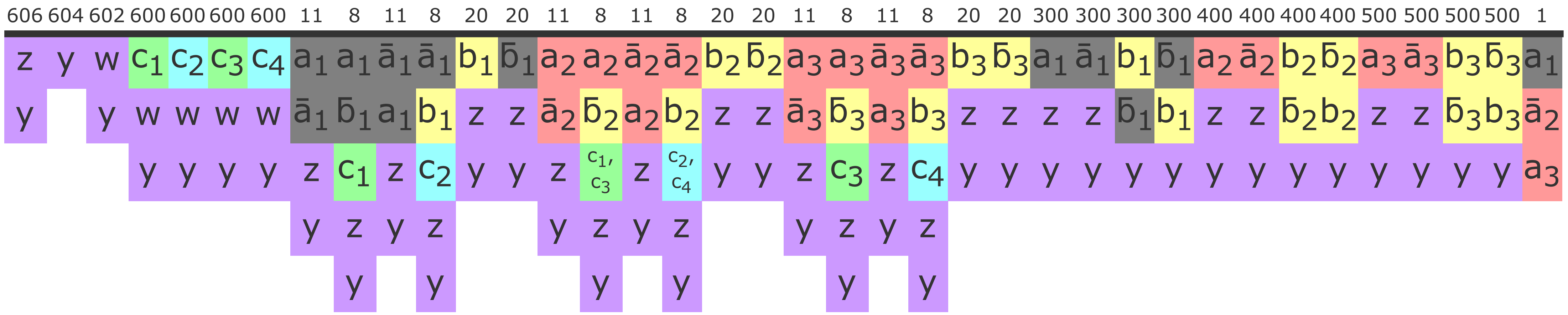
$$(x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_1} \lor \overline{x_2}) \land (x_2 \lor x_3 \lor x_3) \land (\overline{x_2} \lor \overline{x_3} \lor \overline{x_3}) \mapsto$$



a_1: 331	b_1: 628	c_1: 600	w: 602
$\bar{a}_1$: 0	$\bar{b}_1$: 0	c_2: 600	y: 604
a_2: 419	b_2: 420	c_3: 600	z: 926
$\bar{a}_2$: 419	$\bar{b}_2$: 420	c_4: 600	
a_3: 519	b_3: 520		
$\bar{a}_3$: 519	$\bar{b}_3$: 520		

Proof illustration

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$



a_1: 0
$\bar{a}_1$: 0
a_2: 419
$\bar{a}_2$: 420
a_3: 519
$\bar{a}_3$: 519

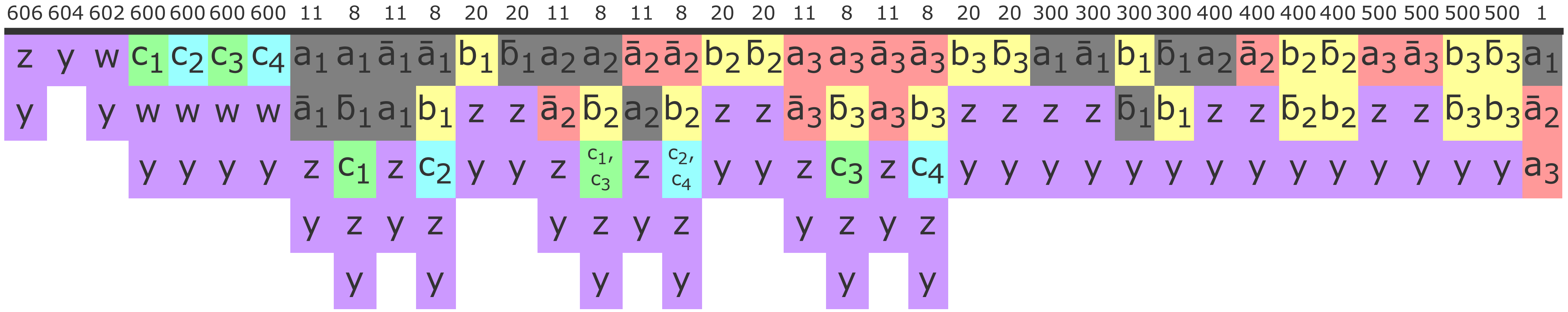
b_1: 628
$\bar{b}_1$: 0
b_2: 420
$\bar{b}_2$: 420
b_3: 520
$\bar{b}_3$: 520

c_1: 608
c_2: 600
c_3: 600
c_4: 600

w: 602
y: 604
z: 1248

Proof illustration

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$



a_1: 0
$\bar{a}_1$: 0
a_2: 0
$\bar{a}_2$: 431
a_3: 519
$\bar{a}_3$: 519

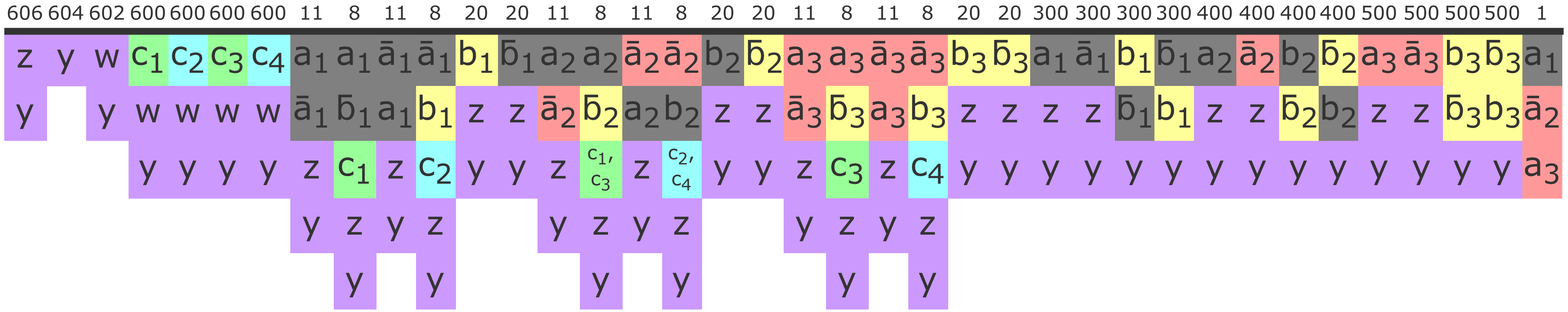
b_1: 628
$\bar{b}_1$: 0
b_2: 420
$\bar{b}_2$: 428
b_3: 520
$\bar{b}_3$: 520

c_1: 608
c_2: 600
c_3: 600
c_4: 600

w: 602
y: 604
z: 1648

Proof illustration

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$



a_1: 0
$\bar{a}_1$: 0
a_2: 0
$\bar{a}_2$: 431
a_3: 519
$\bar{a}_3$: 519

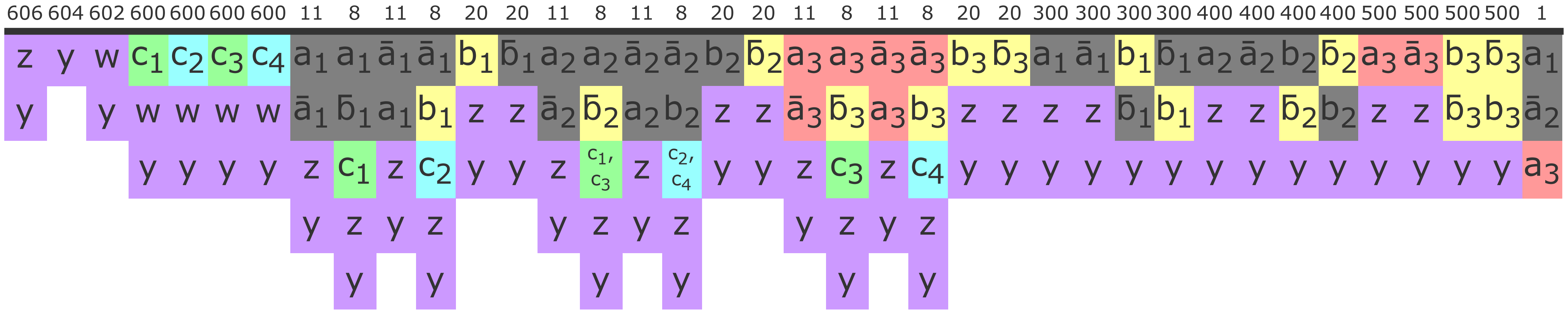
b_1: 628
$\bar{b}_1$: 0
b_2: 0
$\bar{b}_2$: 828
b_3: 520
$\bar{b}_3$: 520

c_1: 608
c_2: 600
c_3: 600
c_4: 600

w: 602
y: 604
z: 1668

Proof illustration

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$



a_1: 0
$\bar{a}_1$: 0
a_2: 0
$\bar{a}_2$: 0
a_3: 520
$\bar{a}_3$: 519

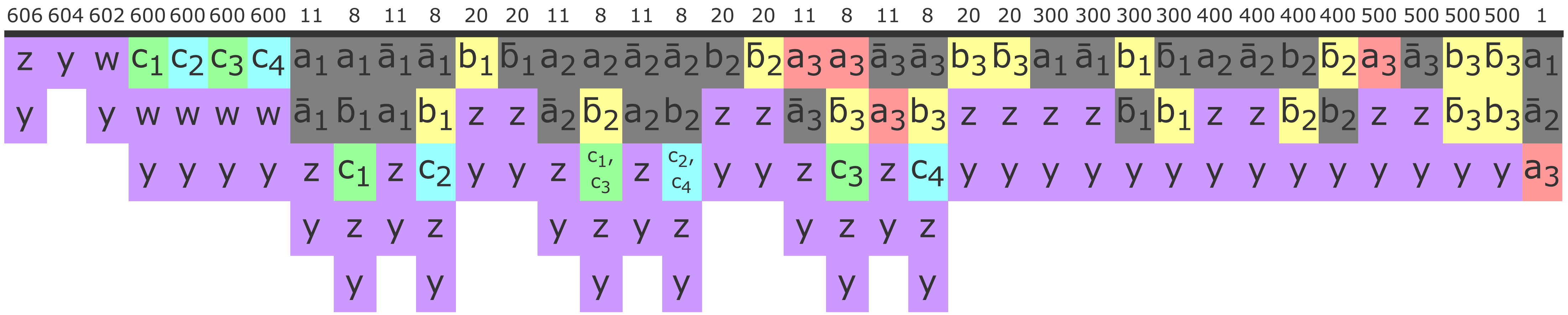
b_1: 628
$\bar{b}_1$: 0
b_2: 0
$\bar{b}_2$: 828
b_3: 520
$\bar{b}_3$: 520

c_1: 608
c_2: 604
c_3: 600
c_4: 604

w: 602
y: 604
z: 2090

Proof illustration

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$



a_1: 0
$\bar{a}_1$: 0
a_2: 0
$\bar{a}_2$: 0
a_3: 531
$\bar{a}_3$: 0

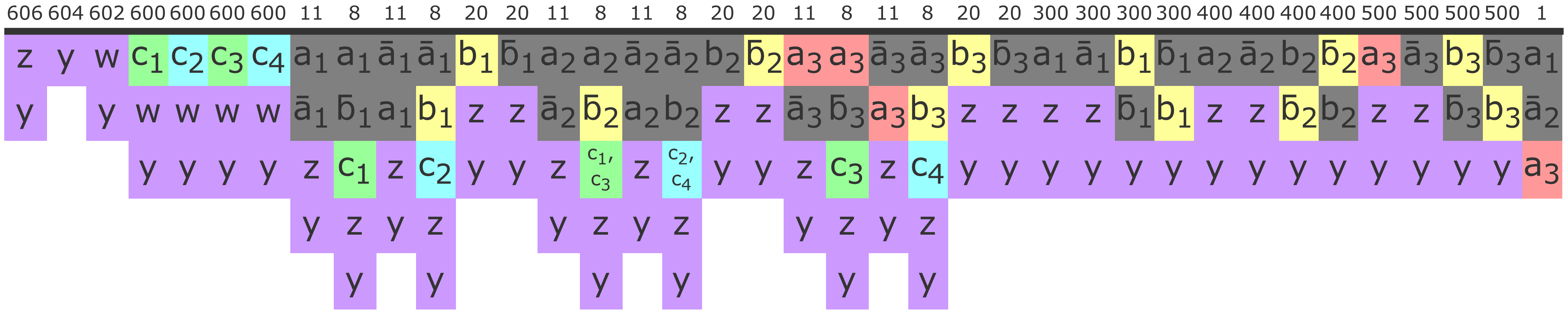
b_1: 628
$\bar{b}_1$: 0
b_2: 0
$\bar{b}_2$: 828
b_3: 528
$\bar{b}_3$: 520

c_1: 608
c_2: 604
c_3: 600
c_4: 604

w: 602
y: 604
z: 2590

Proof illustration

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$



a_1: 0
$\bar{a}_1$: 0
a_2: 0
$\bar{a}_2$: 0
a_3: 531
$\bar{a}_3$: 0

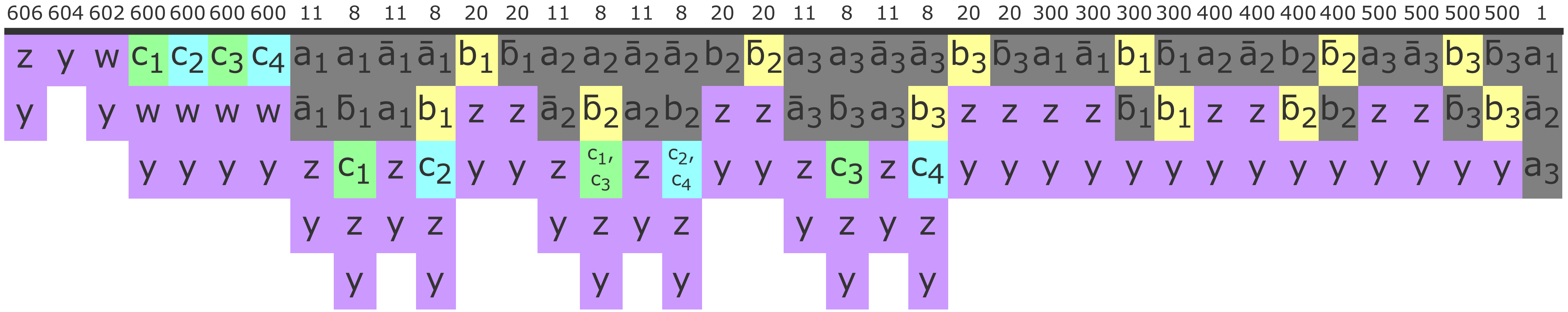
b_1: 628
$\bar{b}_1$: 0
b_2: 0
$\bar{b}_2$: 828
b_3: 1028
$\bar{b}_3$: 0

c_1: 608
c_2: 604
c_3: 600
c_4: 604

w: 602
y: 604
z: 2610

Proof illustration

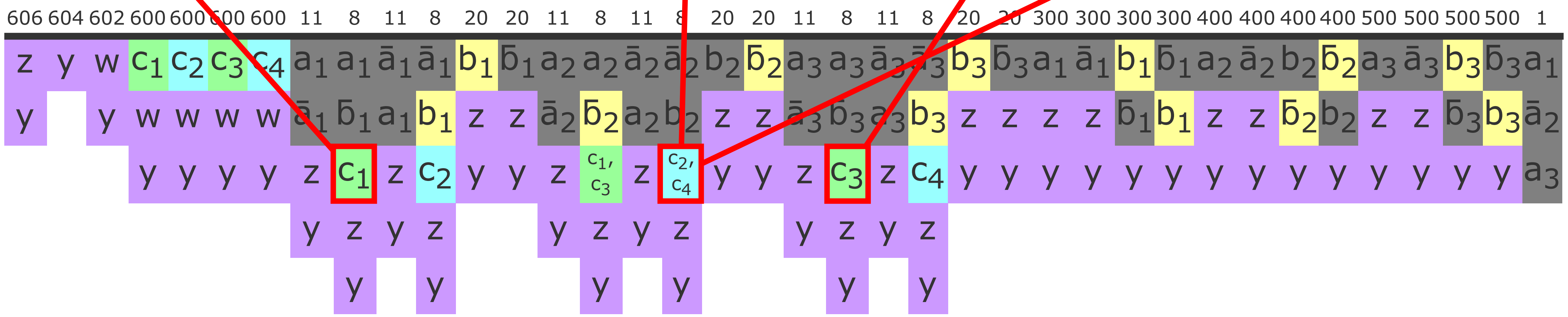
$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$



a_1: 0	b_1: 628	c_1: 608	w: 602
$\bar{a}_1$: 0	$\bar{b}_1$: 0	c_2: 604	y: 604
a_2: 0	b_2: 0	c_3: 608	z: 3132
$\bar{a}_2$: 0	$\bar{b}_2$: 828	c_4: 604	
a_3: 0	b_3: 1028		
$\bar{a}_3$: 0	$\bar{b}_3$: 0		

Proof illustration

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$



a_1:	0
$\bar{a}_1$:	0
a_2:	0
$\bar{a}_2$:	0
a_3:	0
$\bar{a}_3$:	0

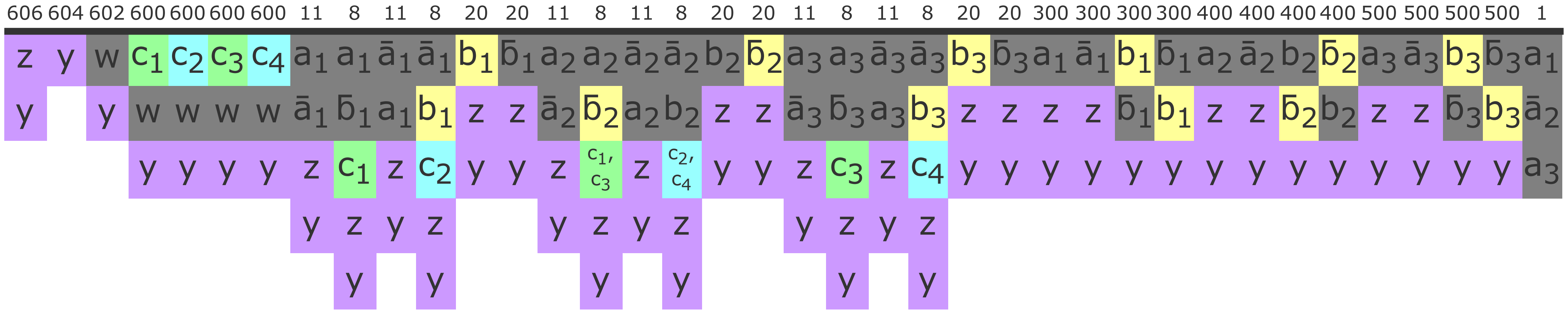
b_1:	628
$\bar{b}_1$:	0
b_2:	0
$\bar{b}_2$:	828
b_3:	1028
$\bar{b}_3$:	0

c_1:	608
c_2:	604
c_3:	608
c_4:	604

w:	602
y:	604
z:	3132

Proof illustration

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$



a_1: 0
$\bar{a}_1$: 0
a_2: 0
$\bar{a}_2$: 0
a_3: 0
$\bar{a}_3$: 0

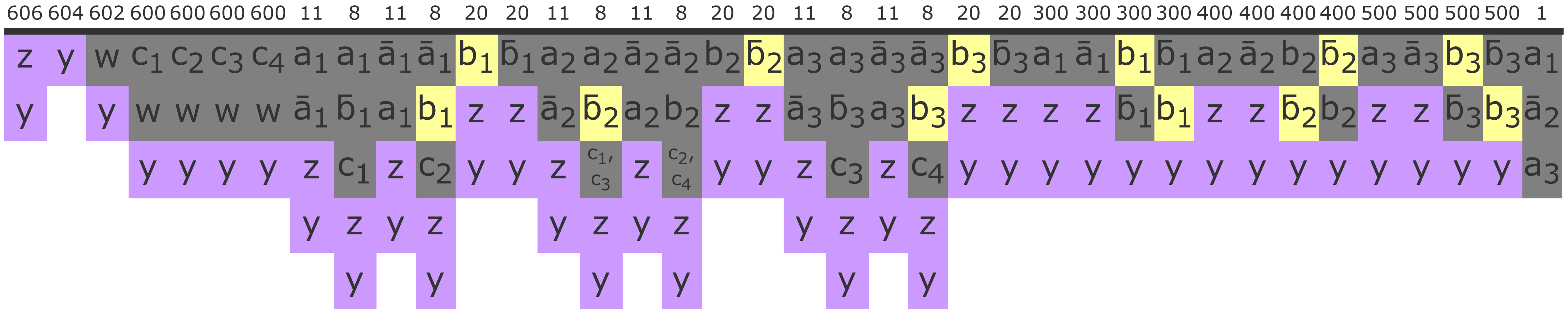
b_1: 628
$\bar{b}_1$: 0
b_2: 0
$\bar{b}_2$: 828
b_3: 1028
$\bar{b}_3$: 0

c_1: 608
c_2: 604
c_3: 608
c_4: 604

w: 0
y: 1206
z: 3132

Proof illustration

$$(x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_1} \lor \overline{x_2}) \land (x_2 \lor x_3 \lor x_3) \land (\overline{x_2} \lor \overline{x_3} \lor \overline{x_3}) \mapsto$$



a_1: 0
$\bar{a}_1$: 0
a_2: 0
$\bar{a}_2$: 0
a_3: 0
$\bar{a}_3$: 0

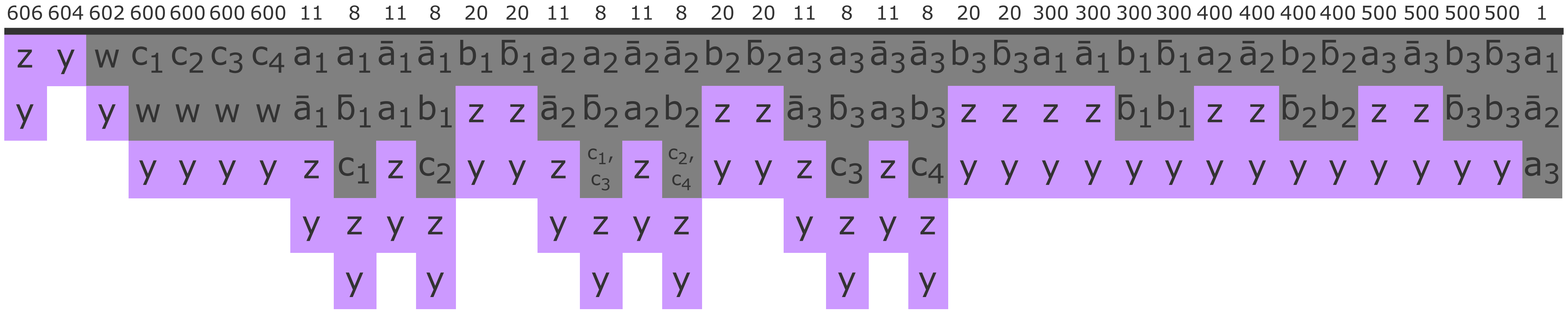
b_1: 628
$\bar{b}_1$: 0
b_2: 0
$\bar{b}_2$: 828
b_3: 1028
$\bar{b}_3$: 0

c_1: 0
c_2: 0
c_3: 0
c_4: 0

w: 0
y: 3606
z: 3156

Proof illustration

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$



a_1: 0
$\bar{a}_1$: 0
a_2: 0
$\bar{a}_2$: 0
a_3: 0
$\bar{a}_3$: 0

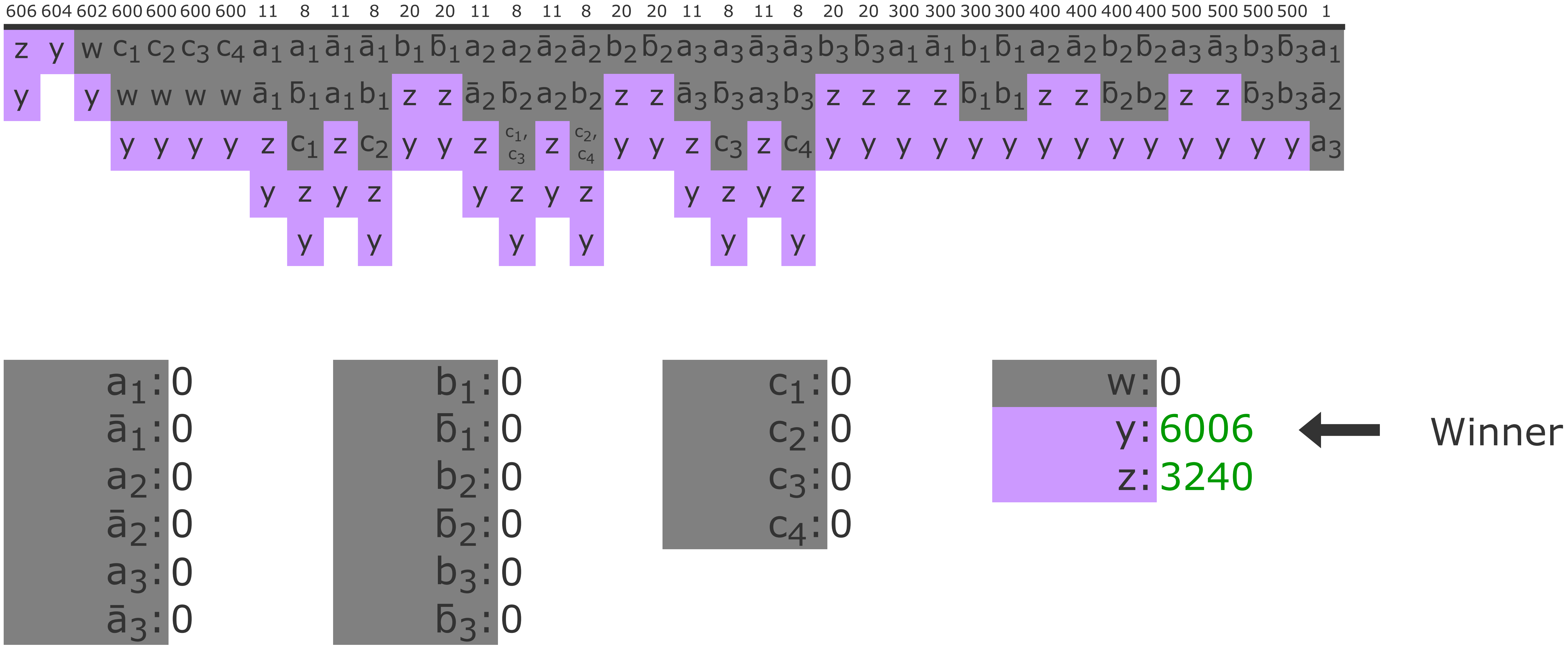
b_1: 0
$\bar{b}_1$: 0
b_2: 0
$\bar{b}_2$: 0
b_3: 0
$\bar{b}_3$: 0

c_1: 0
c_2: 0
c_3: 0
c_4: 0

w: 0
y: 6006
z: 3240

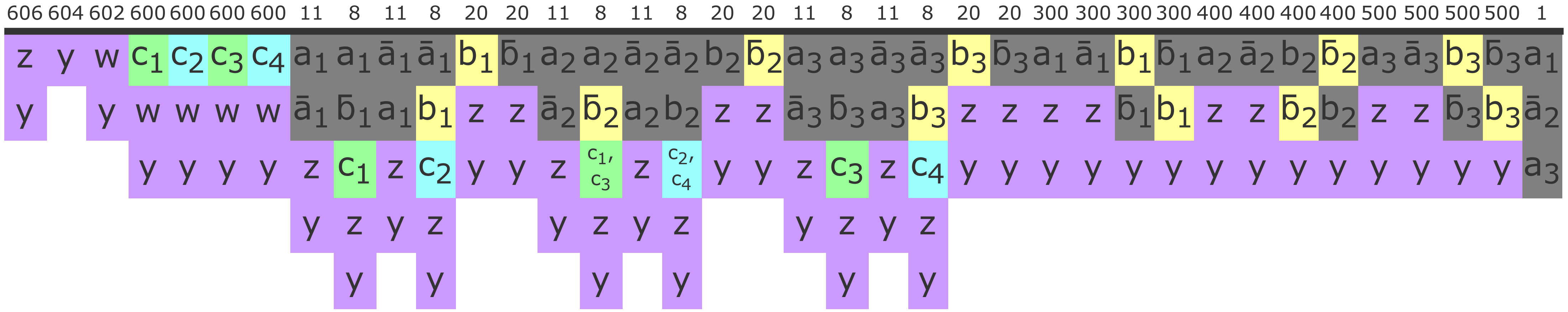
Proof illustration

$$(x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_1} \lor \overline{x_2}) \land (x_2 \lor x_3 \lor x_3) \land (\overline{x_2} \lor \overline{x_3} \lor \overline{x_3}) \mapsto$$



Proof illustration

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$



a_1: 0
$\bar{a}_1$: 0
a_2: 0
$\bar{a}_2$: 0
a_3: 0
$\bar{a}_3$: 0

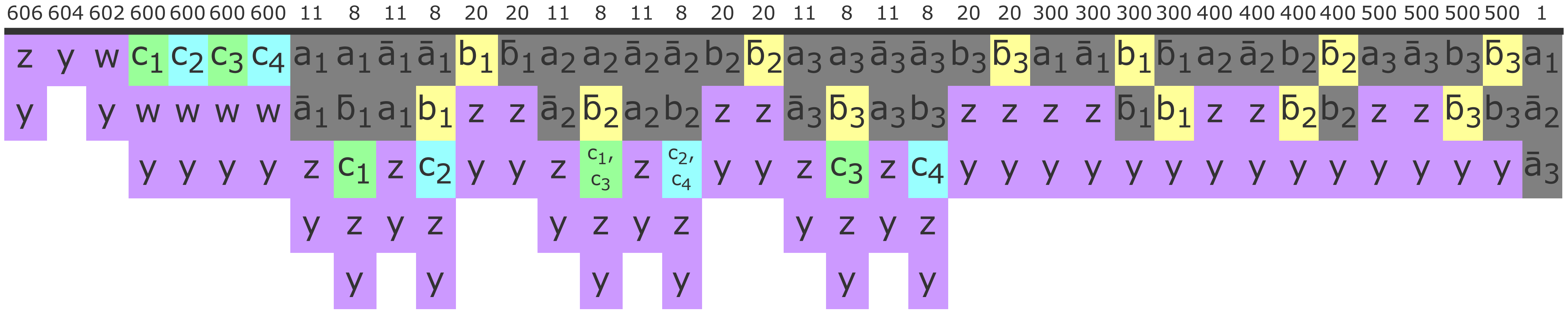
b_1: 628
$\bar{b}_1$: 0
b_2: 0
$\bar{b}_2$: 828
b_3: 1028
$\bar{b}_3$: 0

c_1: 608
c_2: 604
c_3: 608
c_4: 604

w: 602
y: 604
z: 3132

Proof illustration

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$



a_1: 0
$\bar{a}_1$: 0
a_2: 0
$\bar{a}_2$: 0
a_3: 0
$\bar{a}_3$: 0

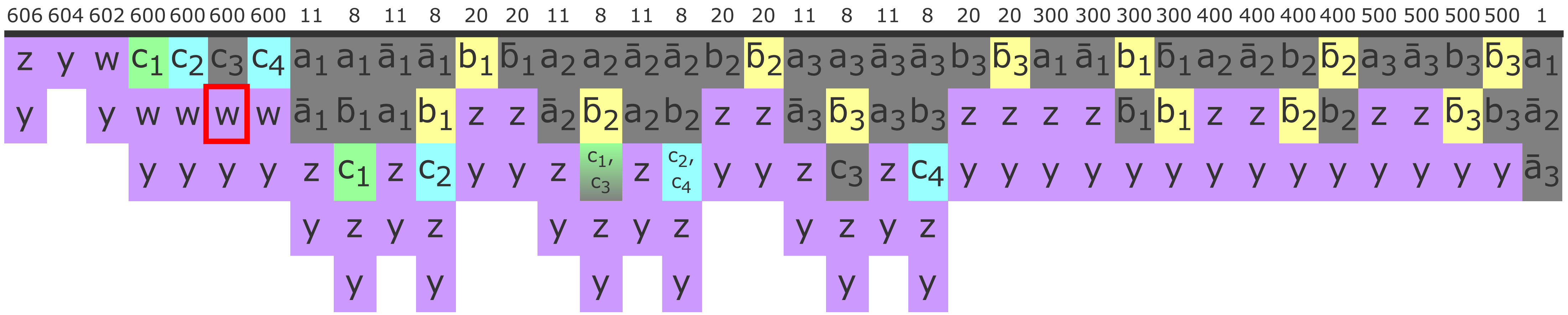
b_1: 628
$\bar{b}_1$: 0
b_2: 0
$\bar{b}_2$: 828
b_3: 0
$\bar{b}_3$: 1028

c_1: 608
c_2: 604
c_3: 600
c_4: 612

w: 602
y: 604
z: 3132

Proof illustration

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_3}) \mapsto$$



a_1: 0
$\bar{a}_1$: 0
a_2: 0
$\bar{a}_2$: 0
a_3: 0
$\bar{a}_3$: 0

b_1: 628
$\bar{b}_1$: 0
b_2: 0
$\bar{b}_2$: 828
b_3: 0
$\bar{b}_3$: 1028

c_1: 608
c_2: 604
c_3: 0
c_4: 612

w: 1202
y: 604
z: 3132

Will be eliminated!

Complexity implication

Proposition (adapted from Bartholdi and Orlin, 2003)

For any Monotone 3-SAT-(2, 2) formula f with n variables, there is an associated preference profile (with one undetermined ballot) where f is satisfiable iff candidate y can be made the IRV winner.

Complexity implication

Proposition (adapted from Bartholdi and Orlin, 2003)

For any Monotone 3-SAT-(2, 2) formula f with n variables, there is an associated preference profile (with one undetermined ballot) where f is satisfiable iff candidate y can be made the IRV winner. ***This profile can be computed in polynomial time.***

Complexity implication

Proposition (adapted from Bartholdi and Orlin, 2003)

For any Monotone 3-SAT-(2, 2) formula f with n variables, there is an associated preference profile (with one undetermined ballot) where f is satisfiable iff candidate y can be made the IRV winner. ***This profile can be computed in polynomial time.***

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If there is a polynomial-time algorithm to compute whether a given candidate can be made the IRV winner, then there is also a polynomial-time algorithm to solve the Monotone 3-SAT-(2, 2) problem.

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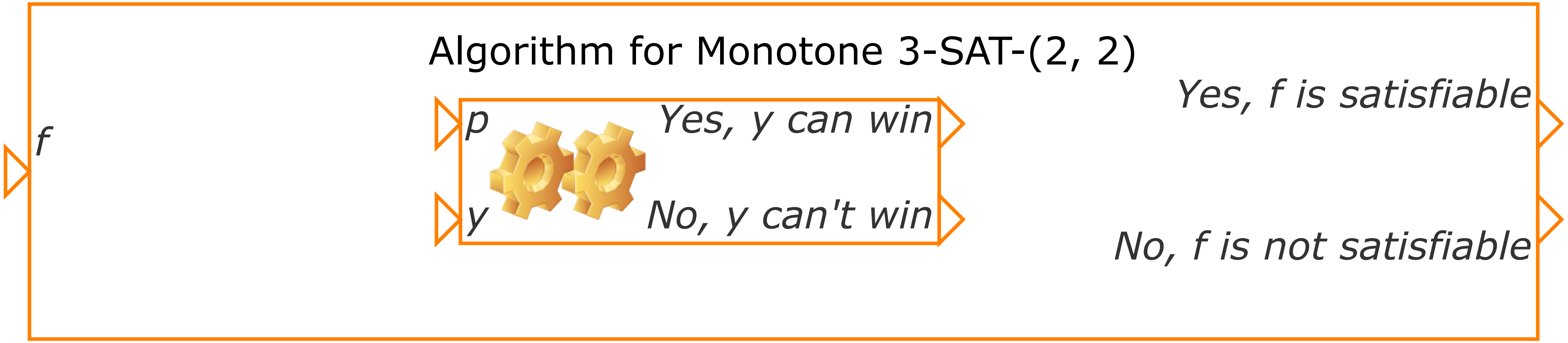

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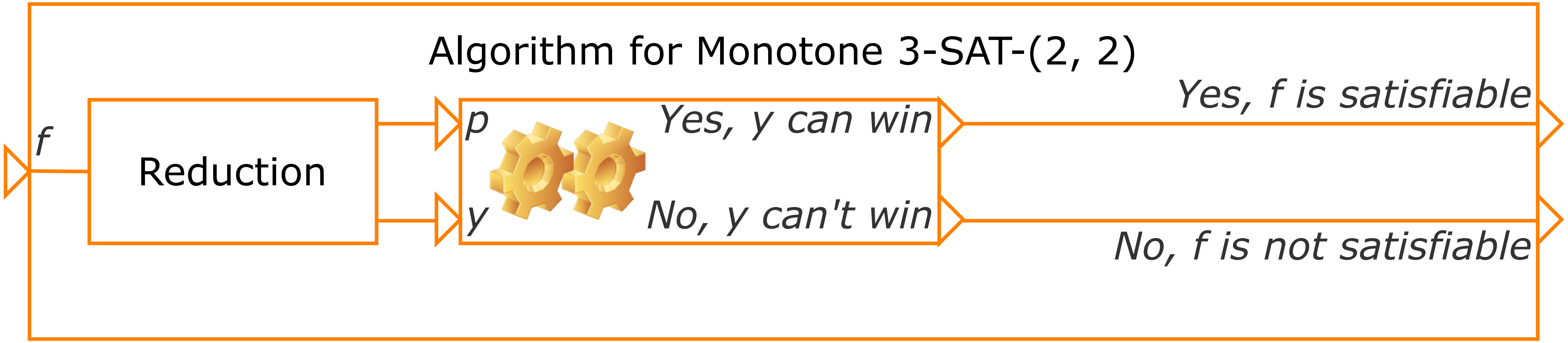


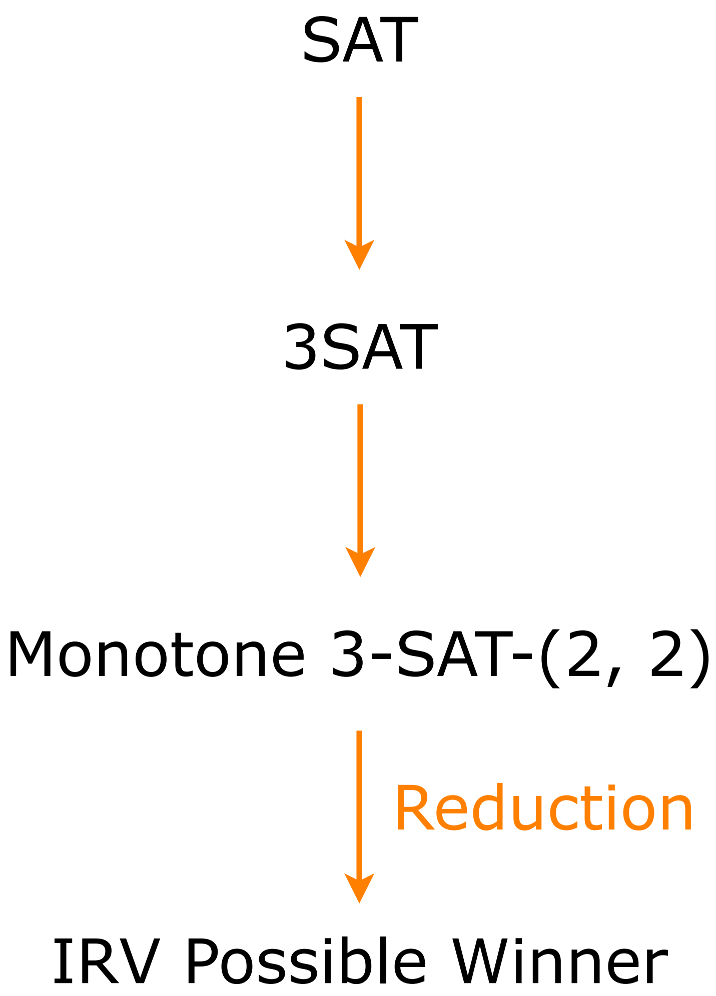
NP-completeness

Monotone 3-SAT-(2, 2)

↓ Reduction

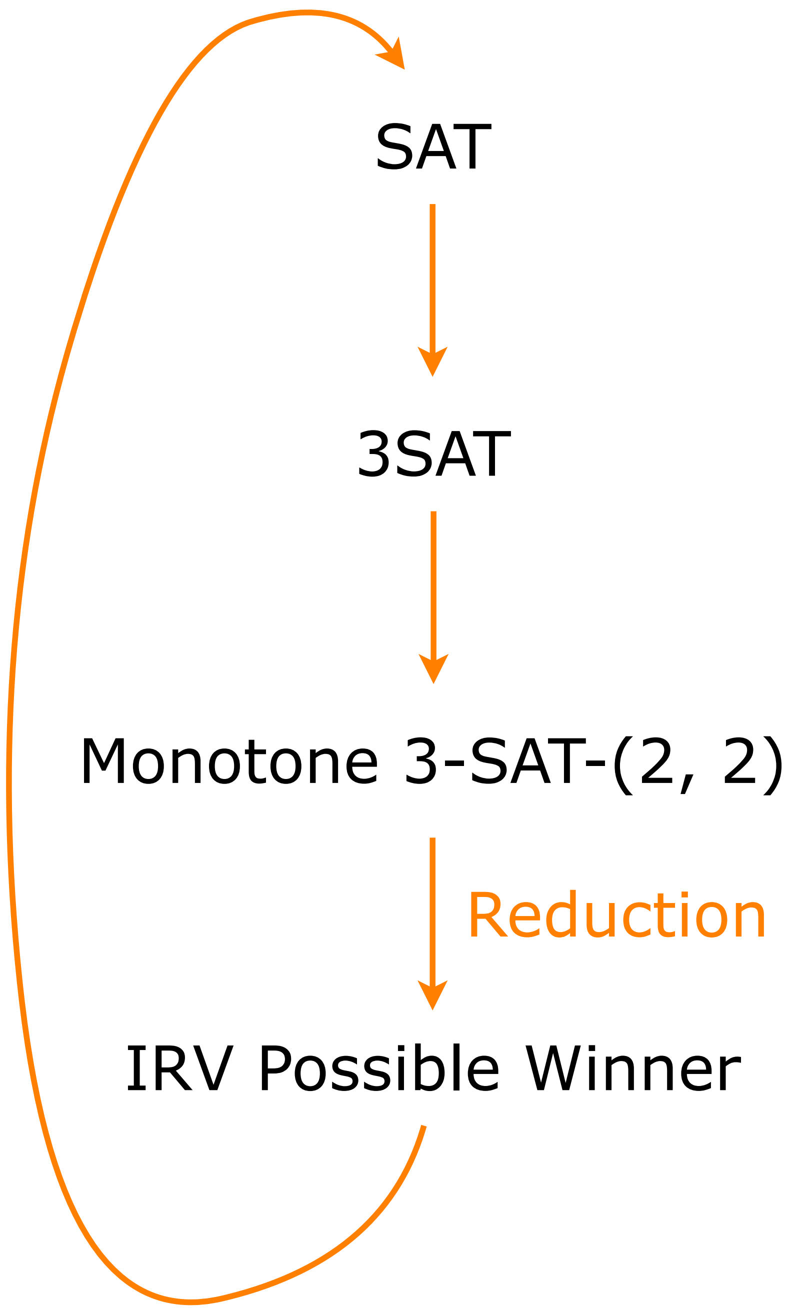
IRV Possible Winner

NP-completeness



Since any boolean formula can be efficiently converted into a Monotone 3-SAT-(2, 2) formula, the IRV Possible Winner problem is at least as hard as SAT as well!

NP-completeness



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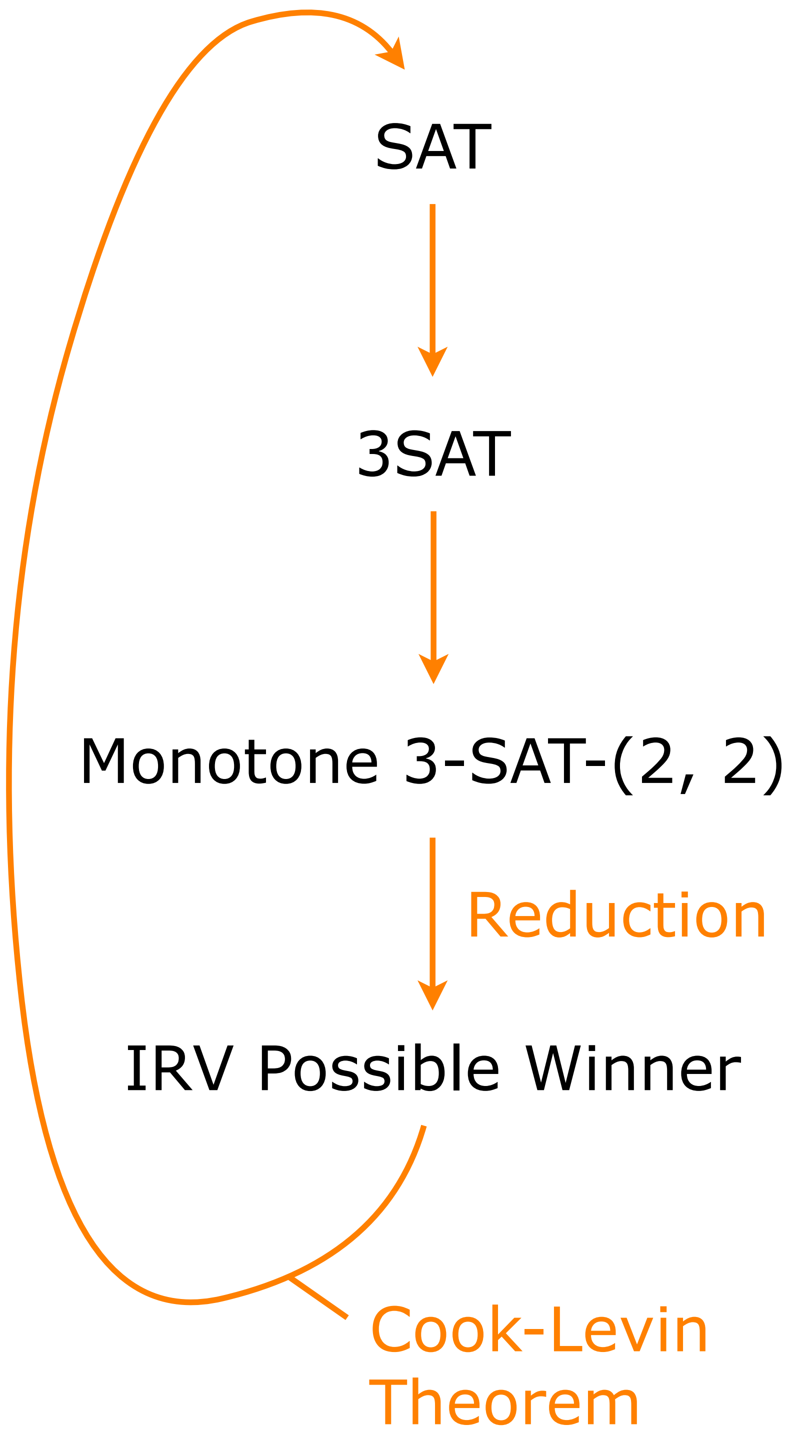

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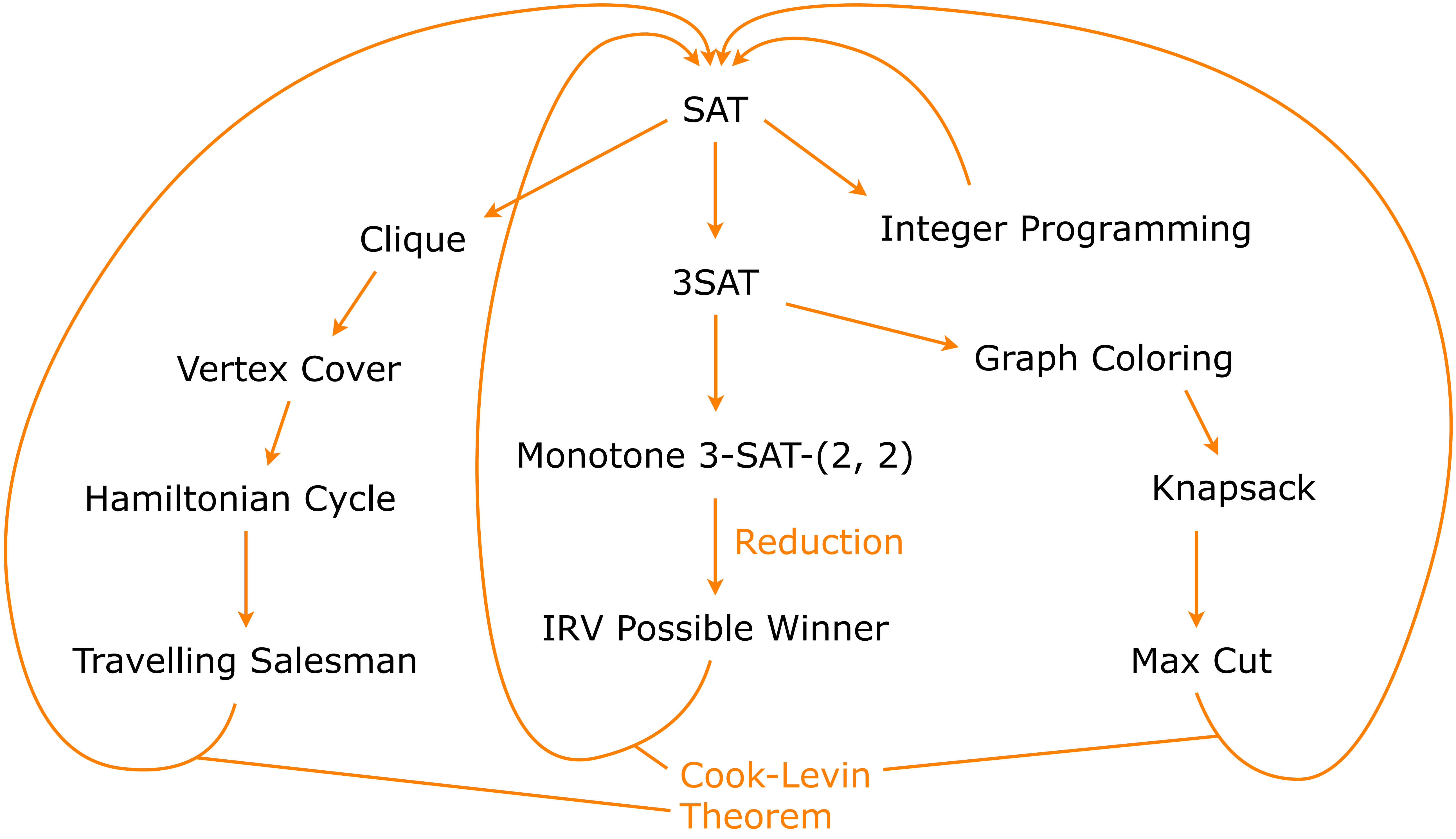
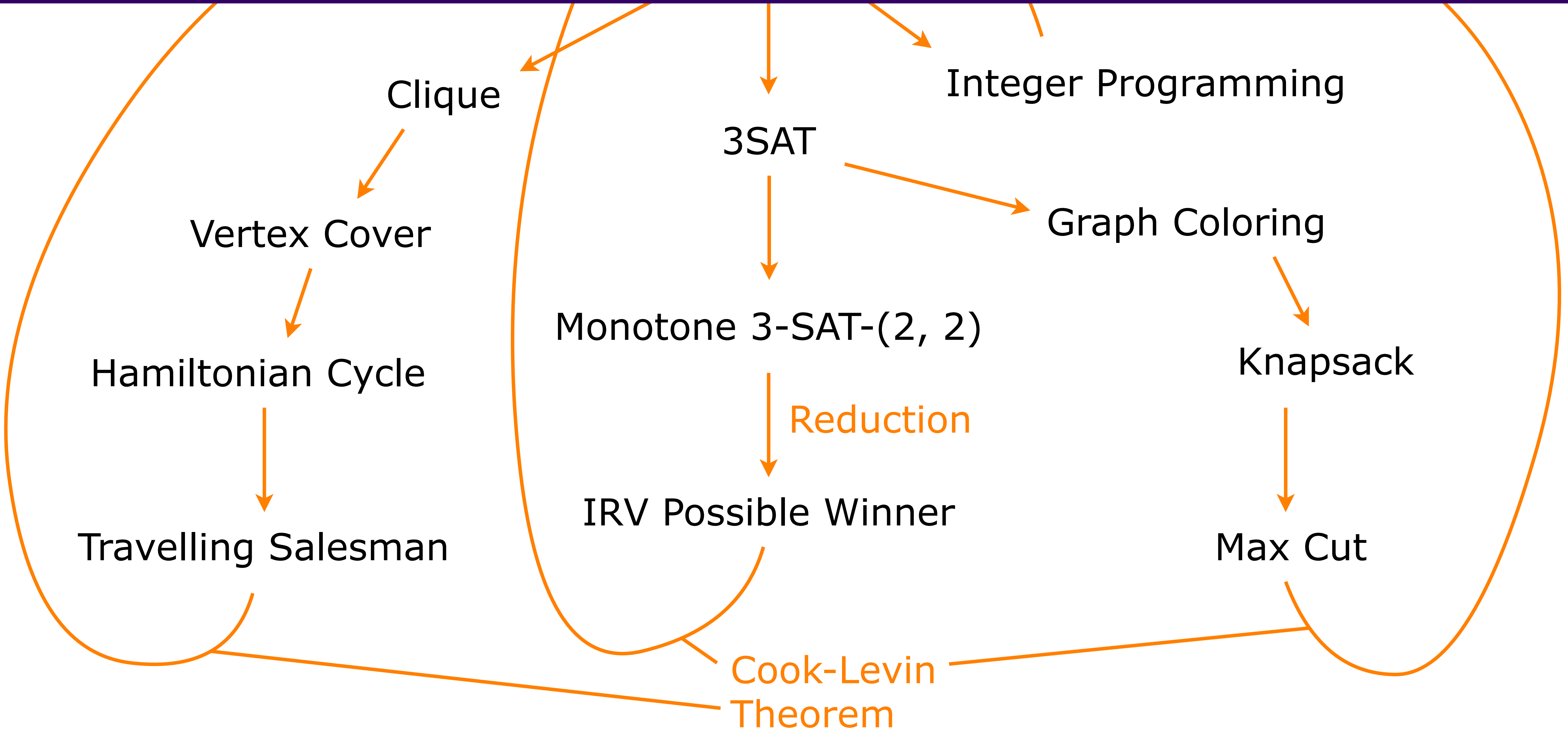


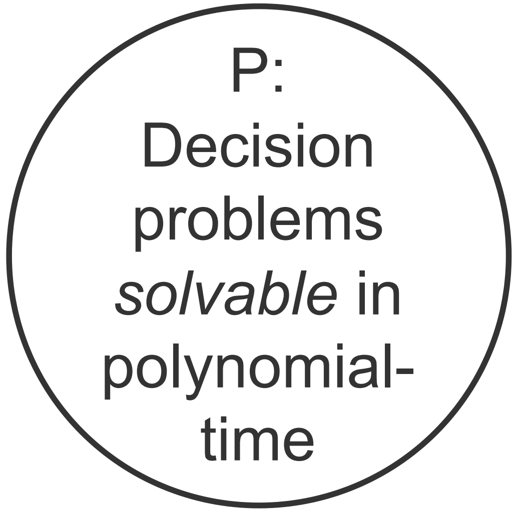
NP-completeness

Theorem

Either all of these problems have a polynomial-time algorithm, or none of them do.



P versus NP



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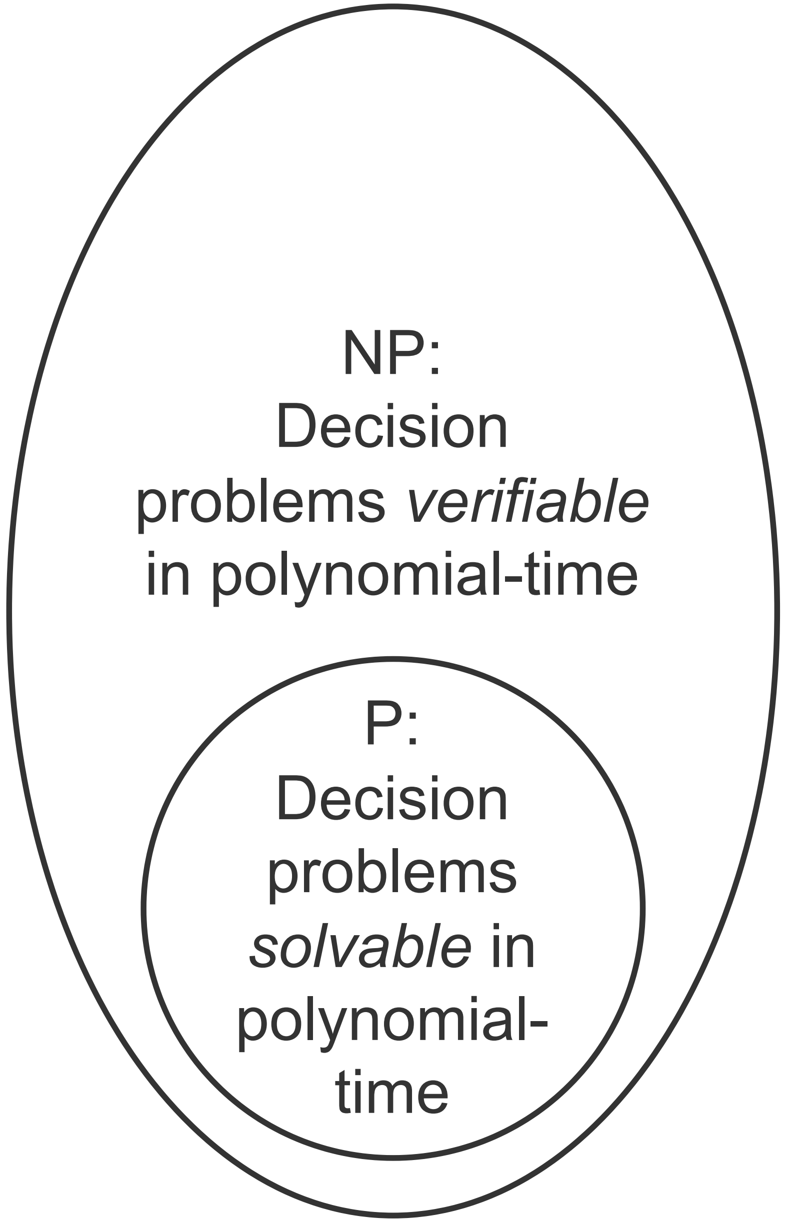
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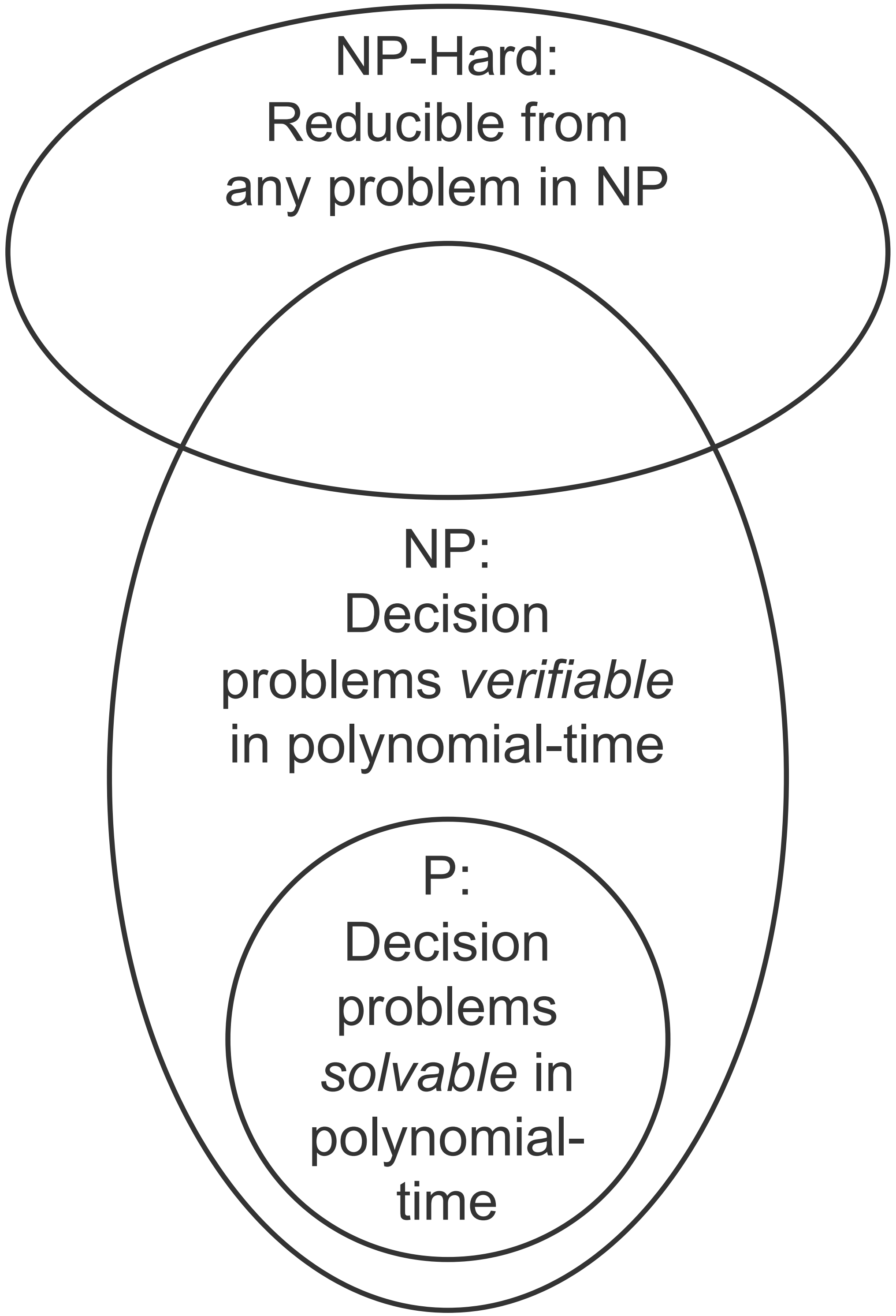

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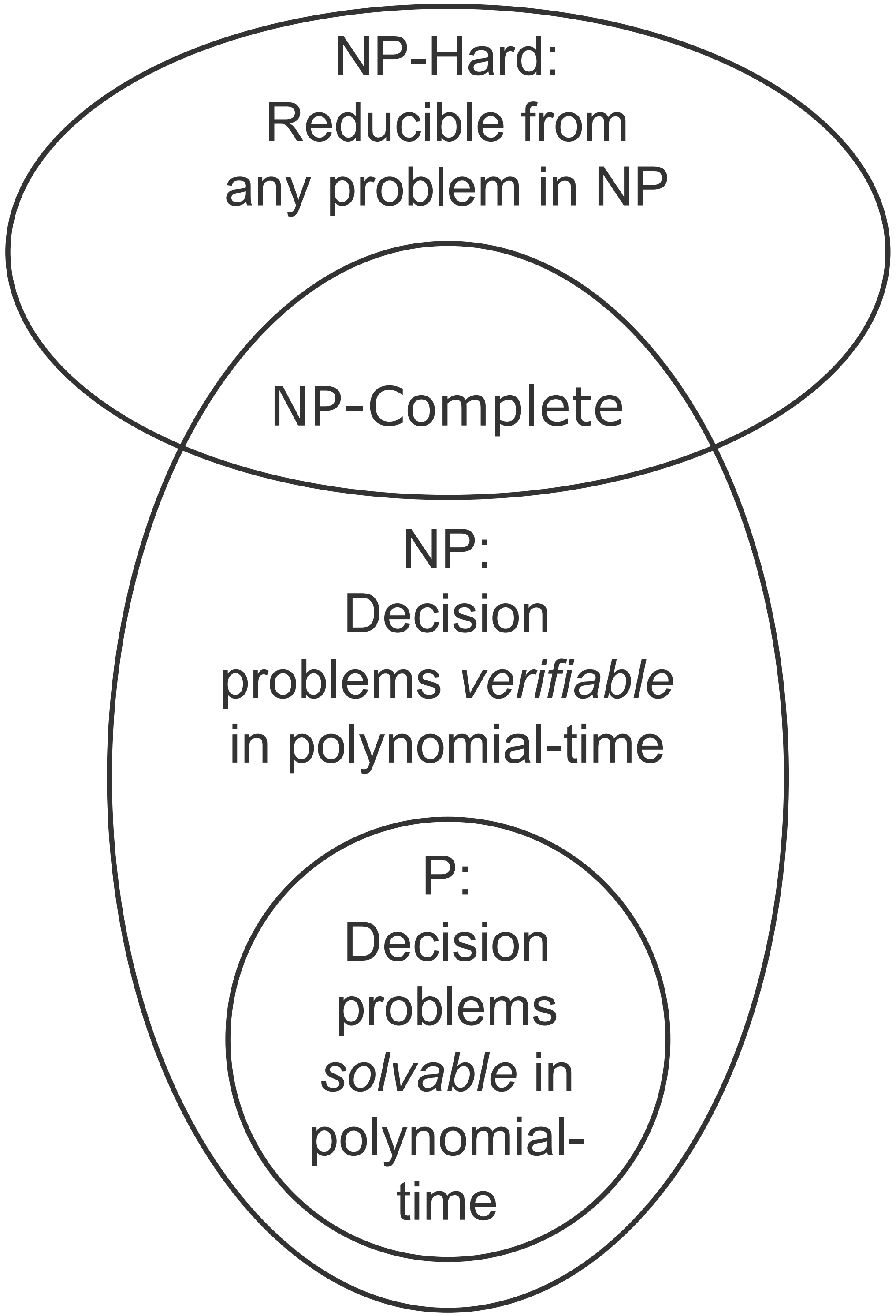
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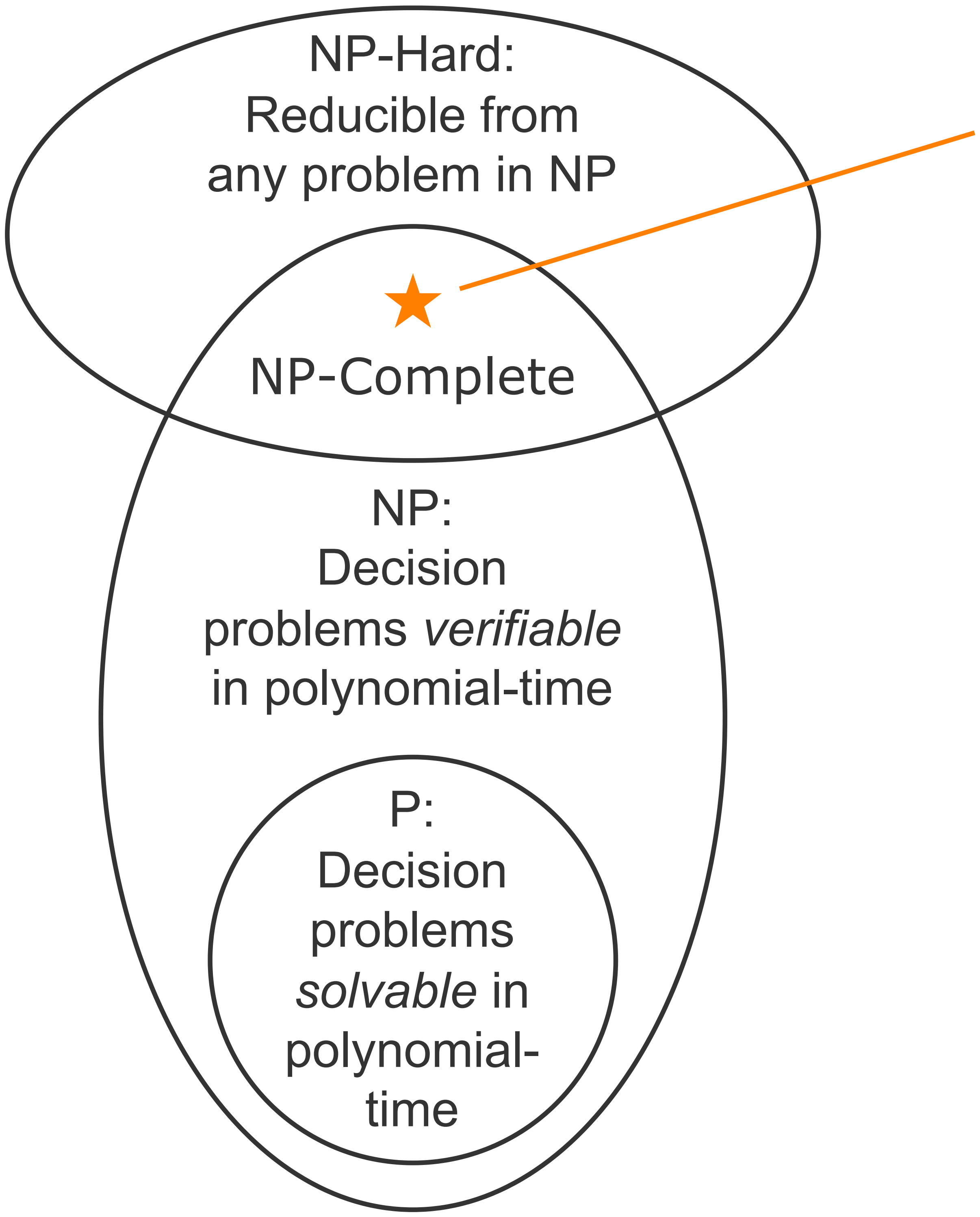

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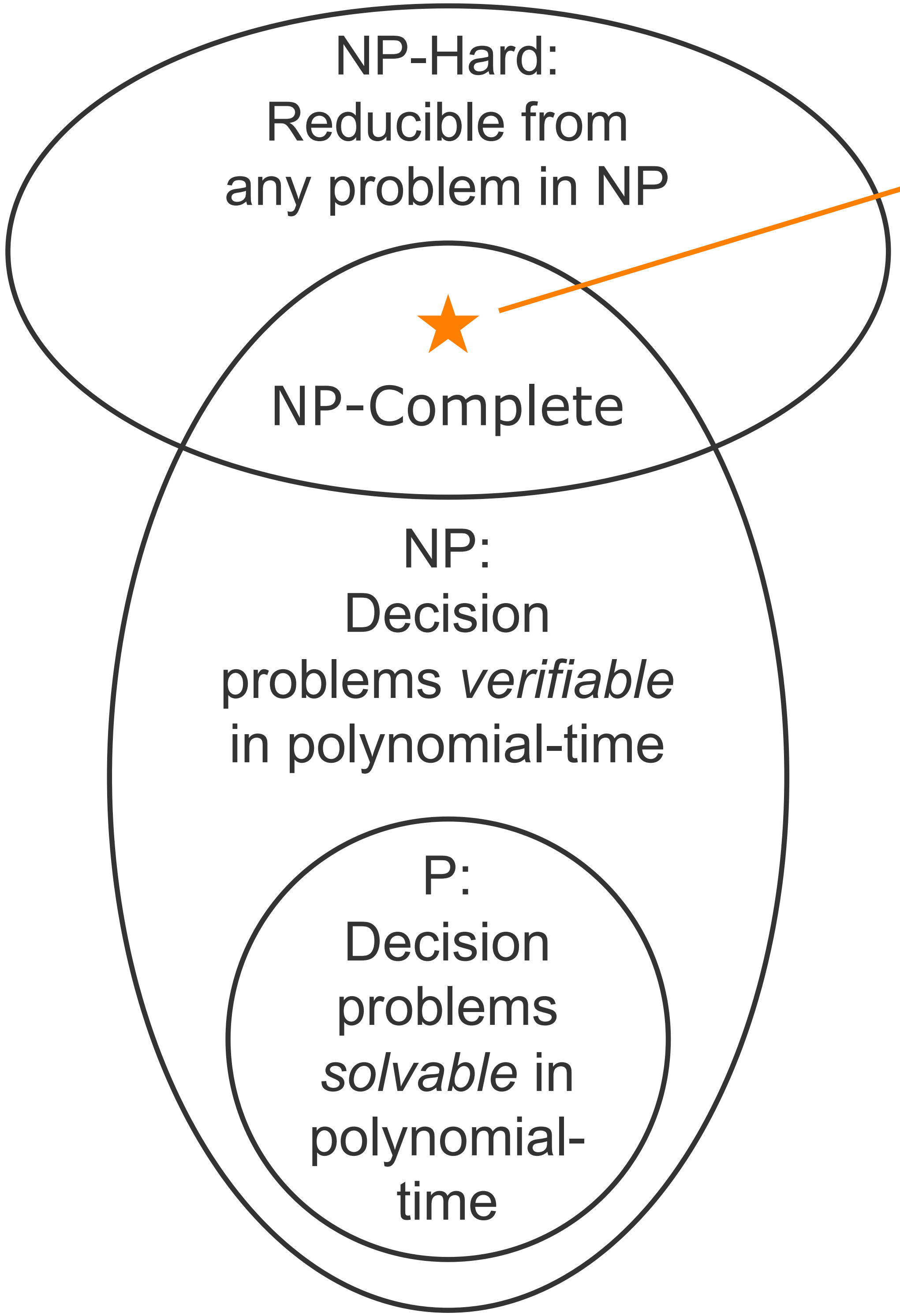
P versus NP



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$$\iff$$

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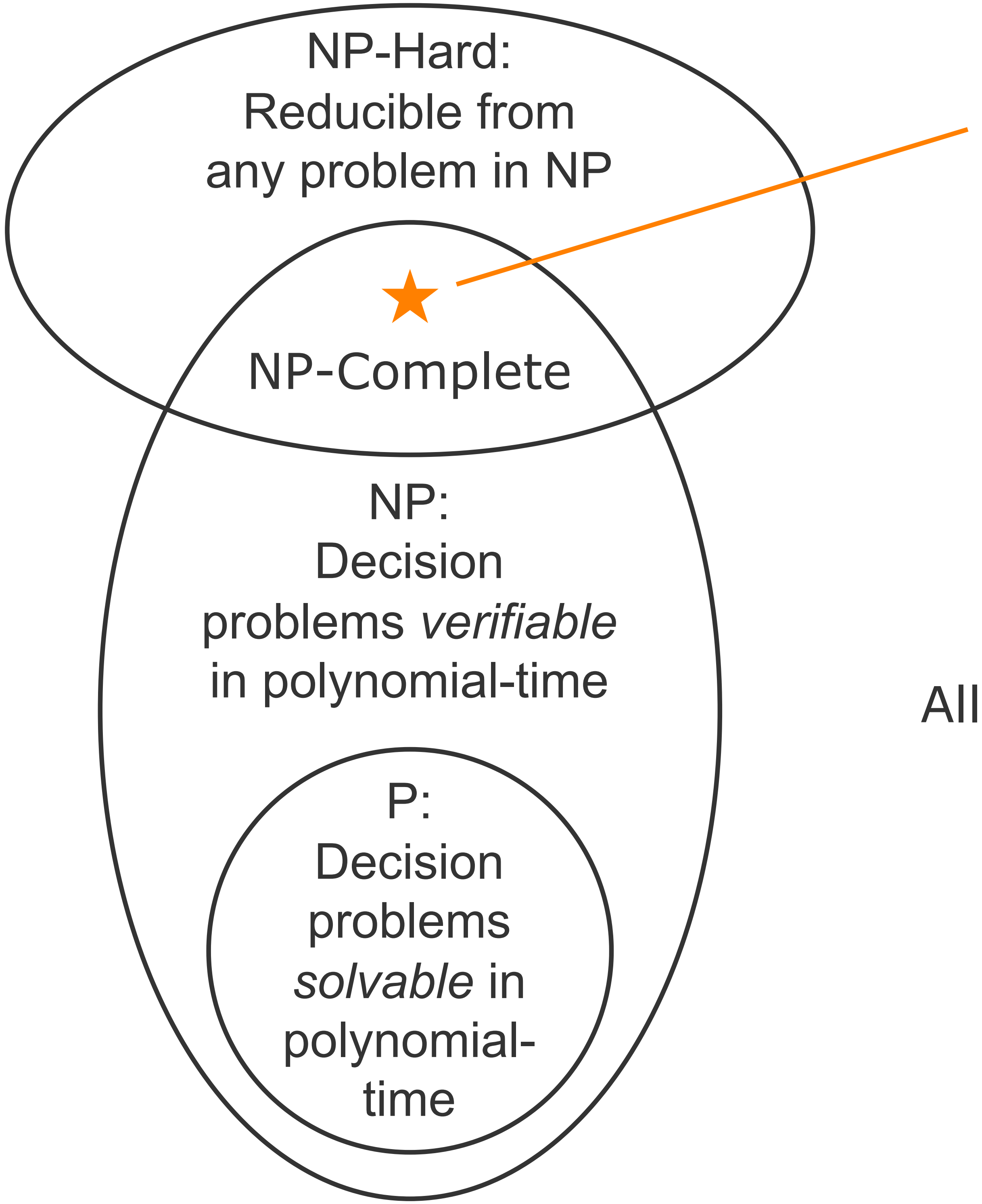

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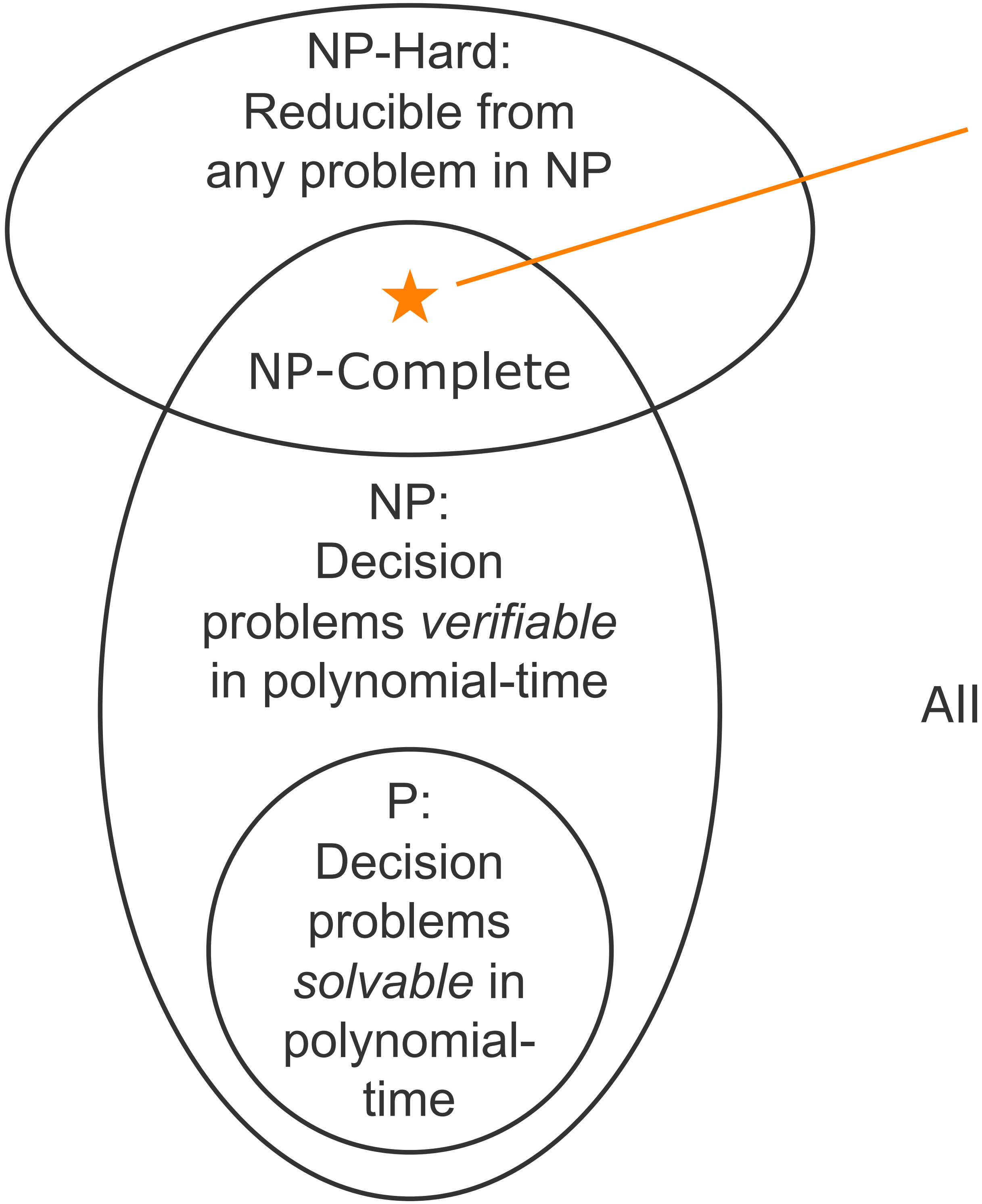

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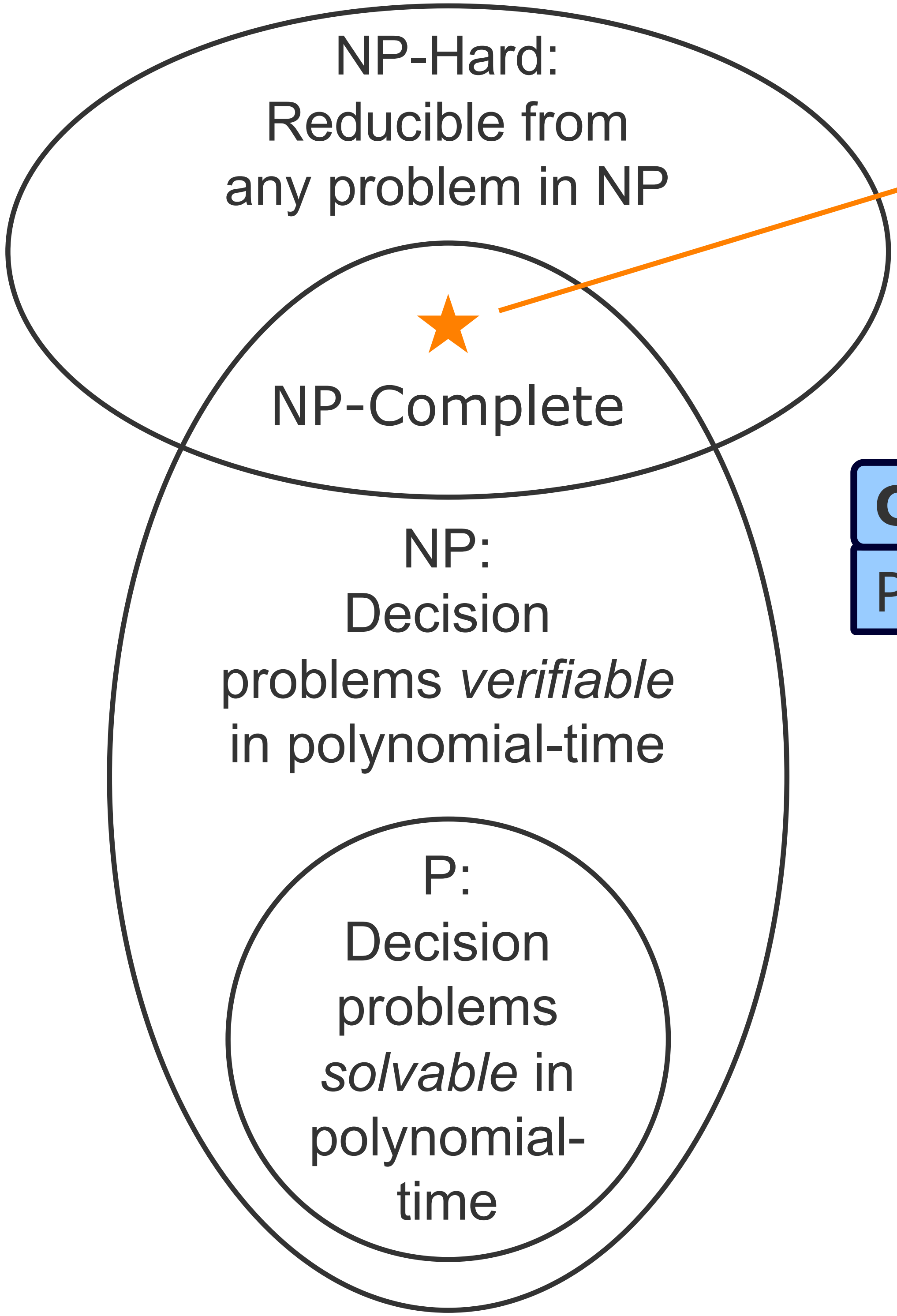

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Conjecture
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"Weakly monotone" means if the manipulators rank a first and b last yet b still wins, then there's no way that they can make a win.