

Algorithms For Democratic Decision-Making

Jamie Tucker-Foltz • Yale University • Spring 2026

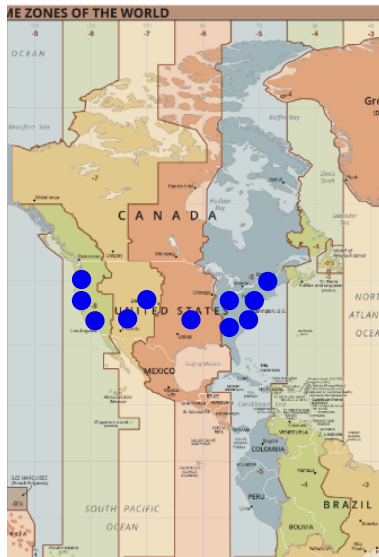
Lecture 5: **Restricted Preferences**

Motivation: time-zone preferences

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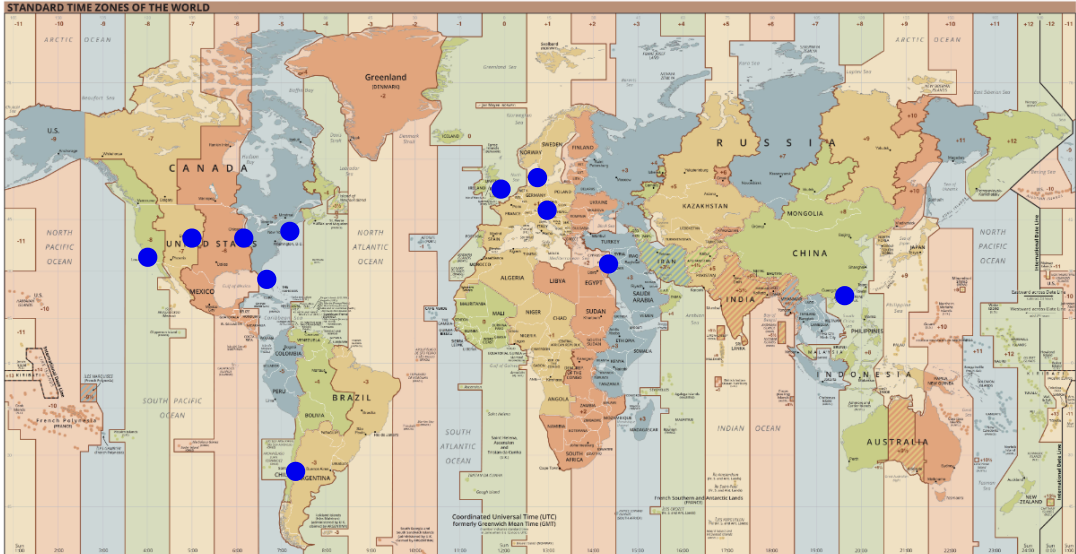
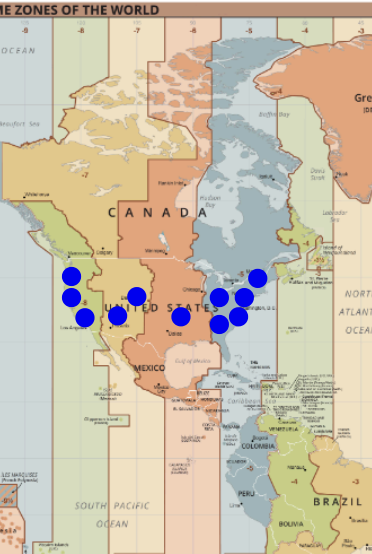


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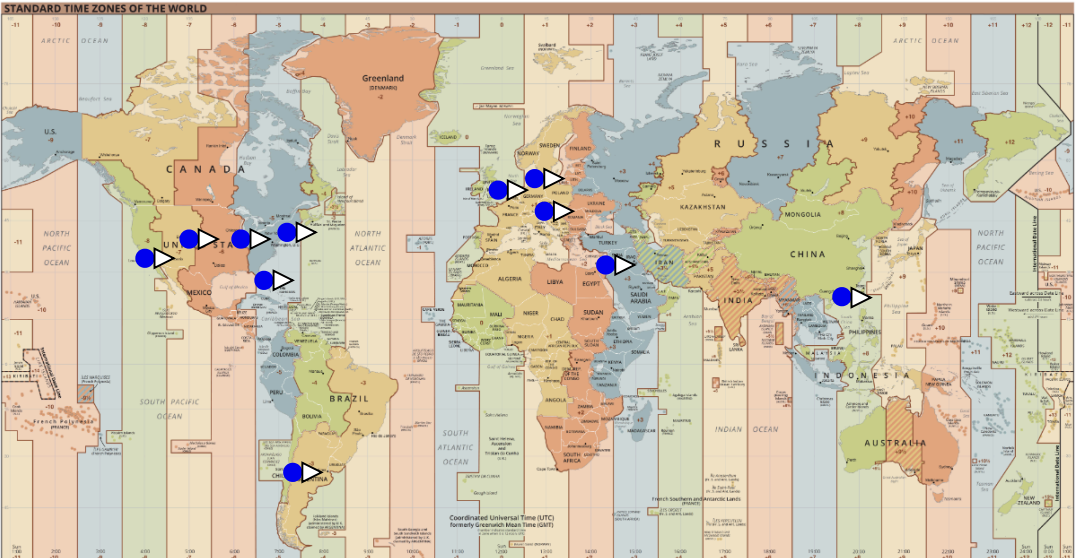
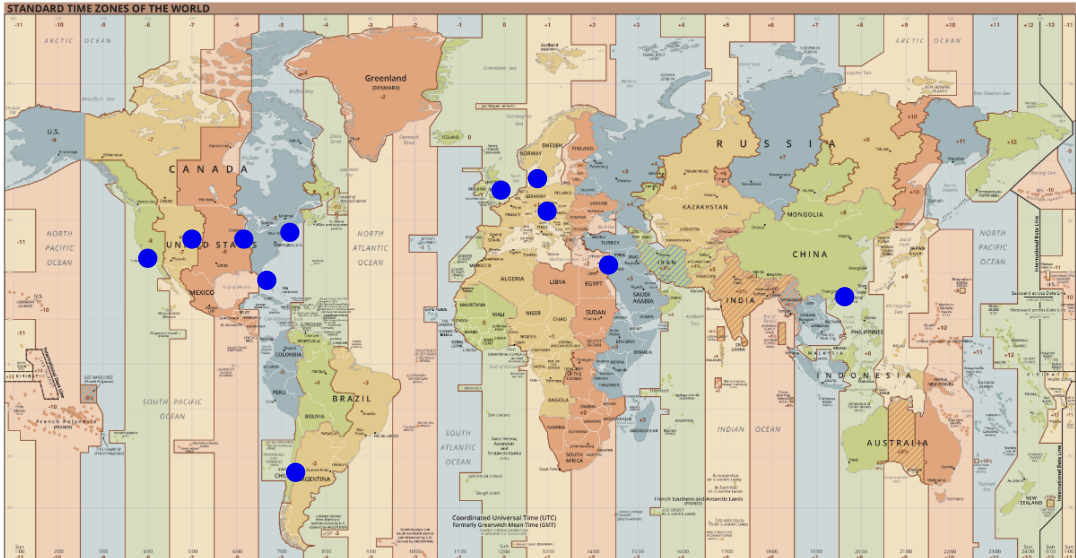
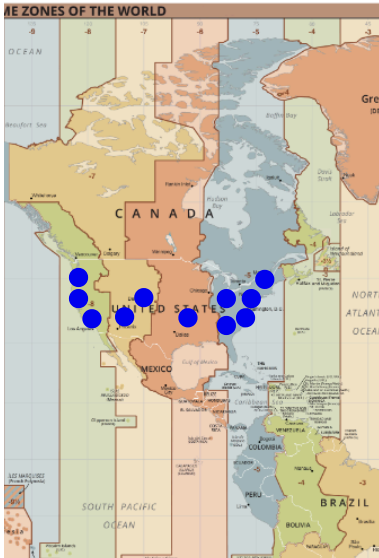
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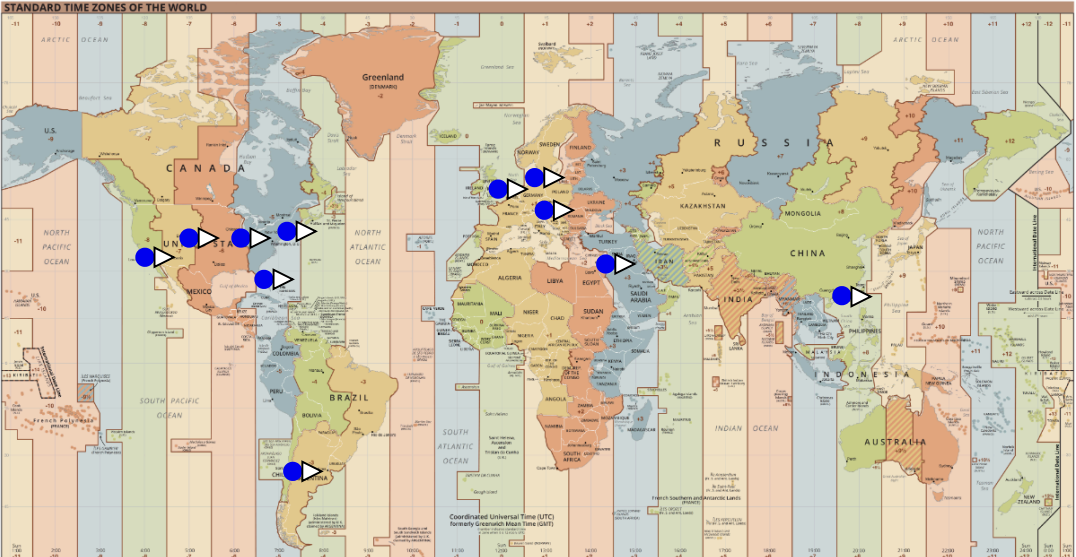
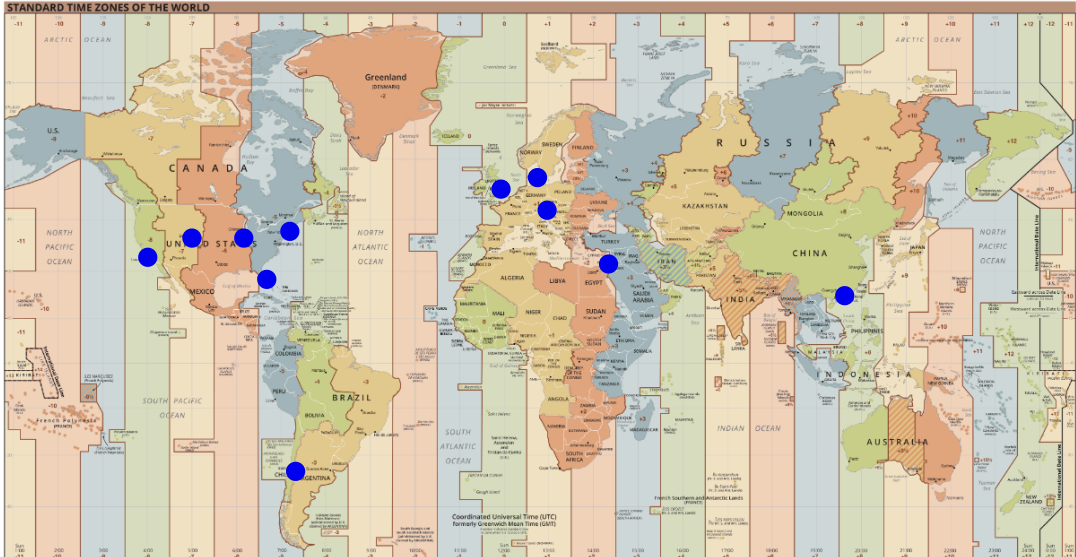
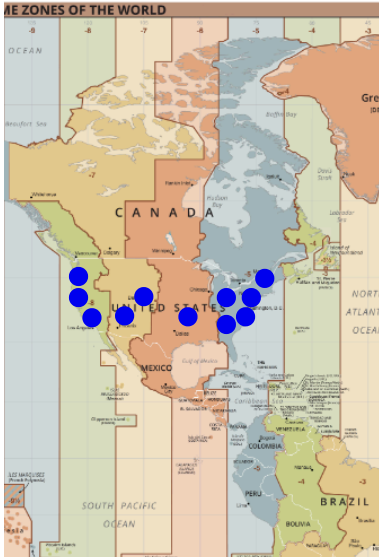
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- *Pareto efficient* - If all candidates rank a before b , b should not win.
- *Group strategyproof (SP)* - No group of voters can ever benefit from misreporting their preferences (to other orderings in D).

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Characterization of CD's in terms of "never" conditions:

Theorem (Sen, 1966)

A domain D is a Condorcet domain if and only if, for every triple of alternatives $a_1, a_2, a_3 \in [m]$, there exist $i, j \in \{1, 2, 3\}$ such that alternative a_i never occurs in position j when restricting any ranking in D to the subset $\{a_1, a_2, a_3\}$.

Proof of Sen's characterization

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A domain D is a **not** a Condorcet domain if and only if, for some triple of alternatives $a_1, a_2, a_3 \in [m]$, **for all** $i, j \in \{1, 2, 3\}$, alternative a_i **can occur** in position j when restricting **some** ranking in D to the subset $\{a_1, a_2, a_3\}$.

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Proof. Backward direction: Suppose that (a, b, c) is a triple where every candidate occurs in every position.

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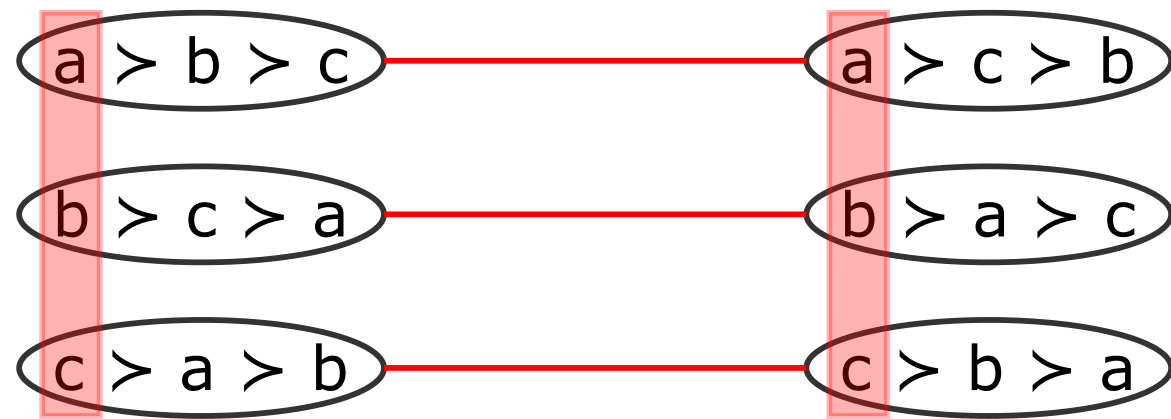
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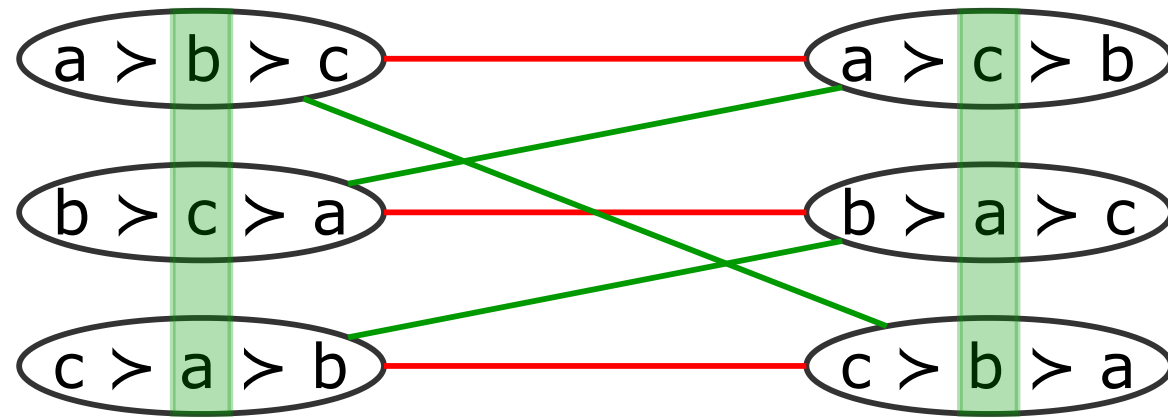


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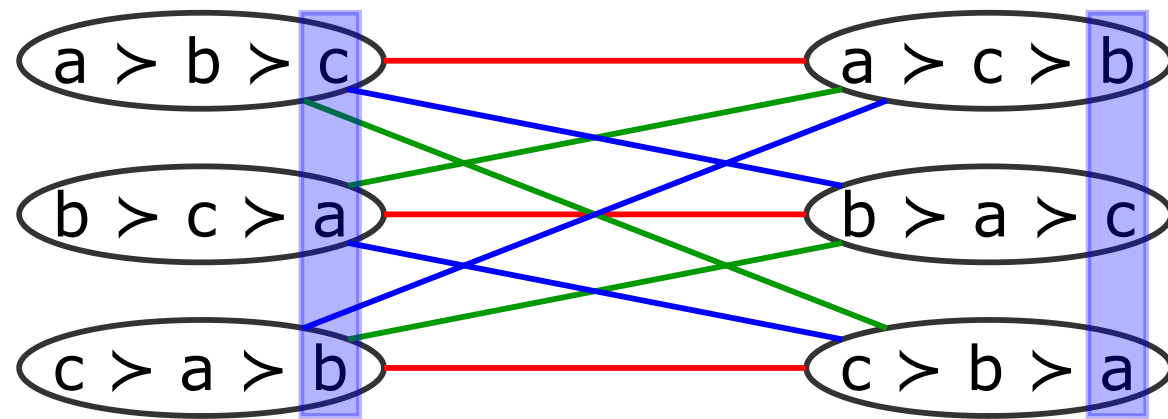


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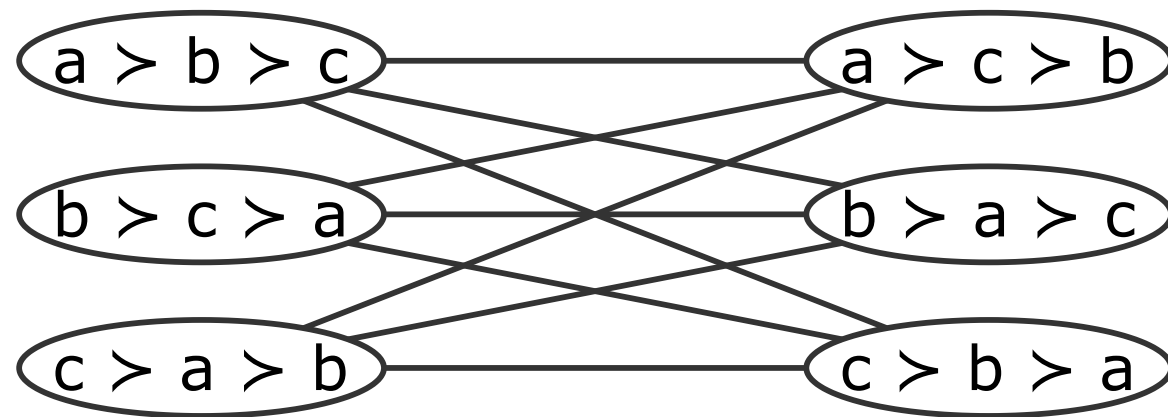


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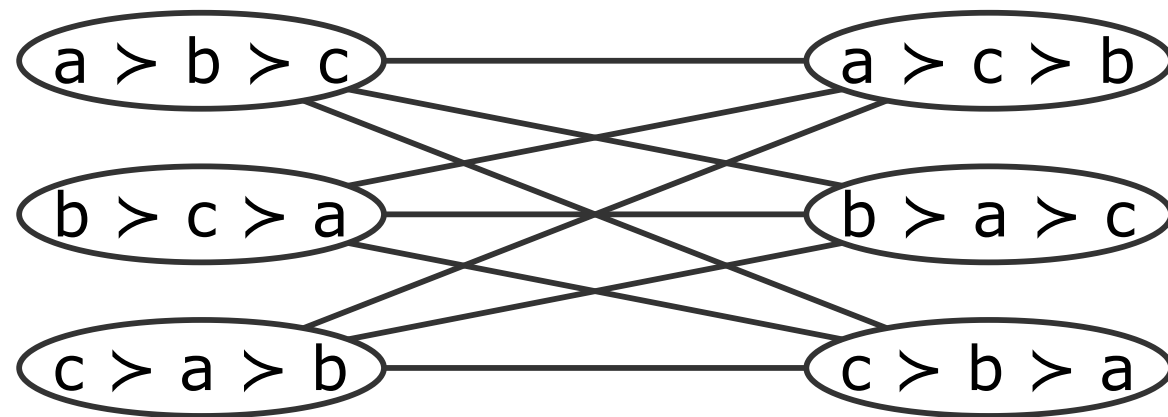


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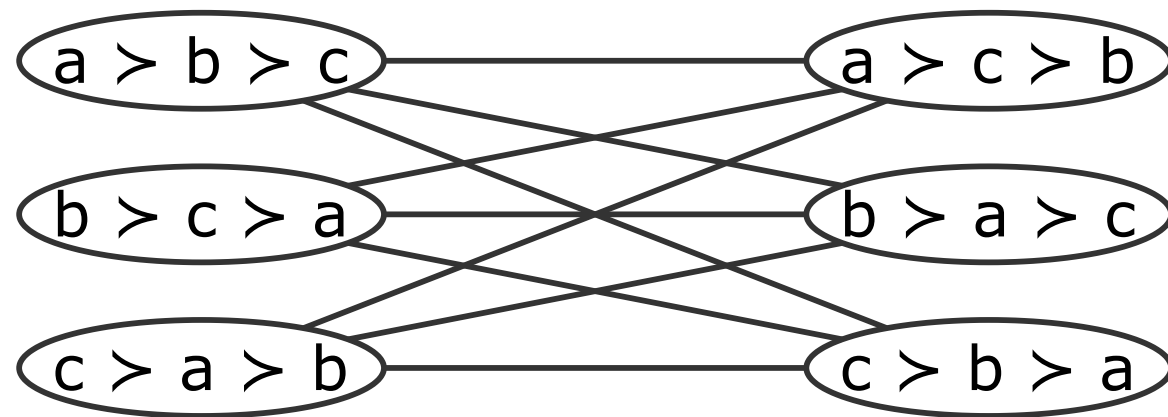
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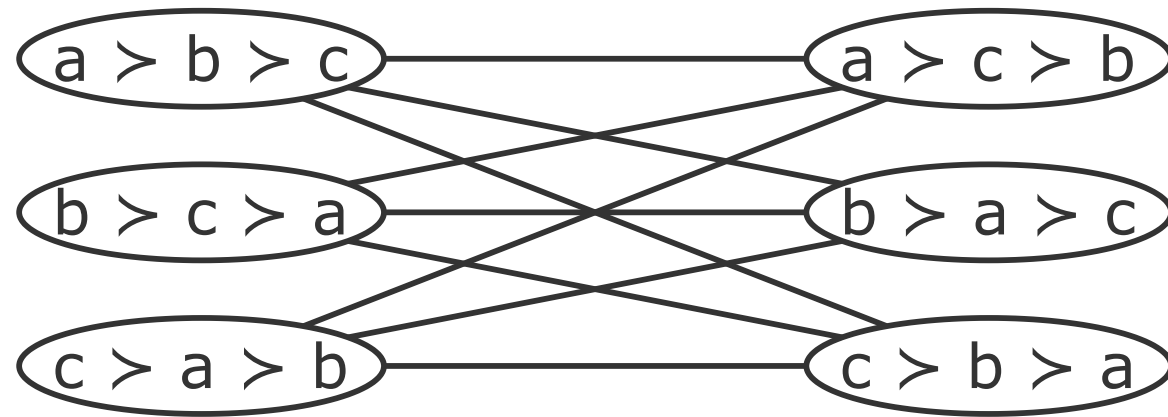
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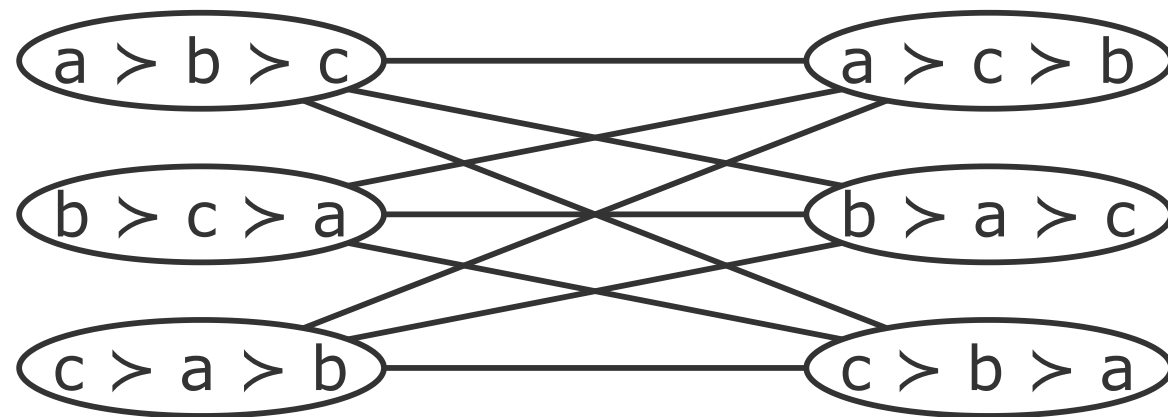
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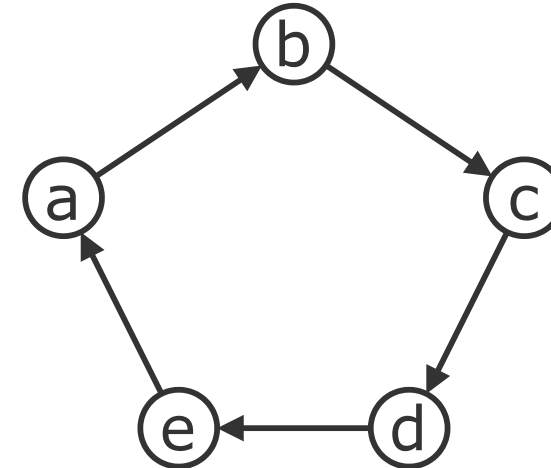
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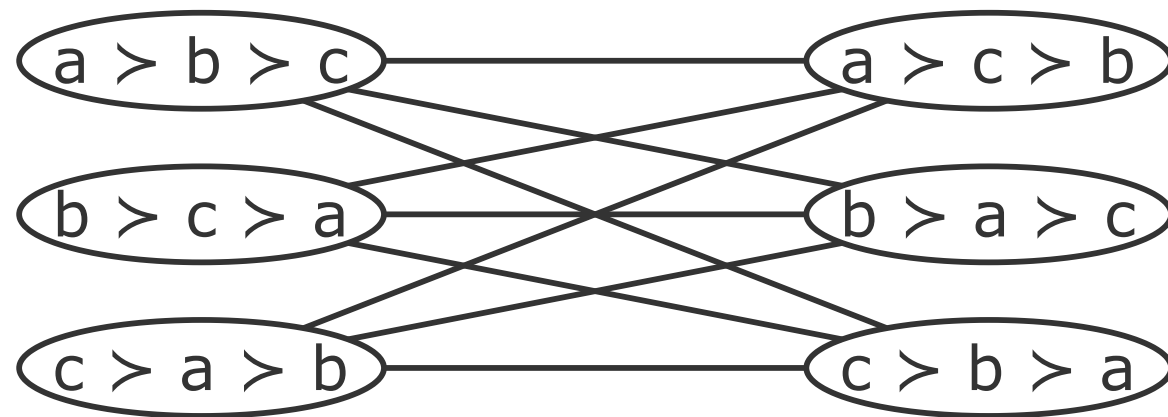


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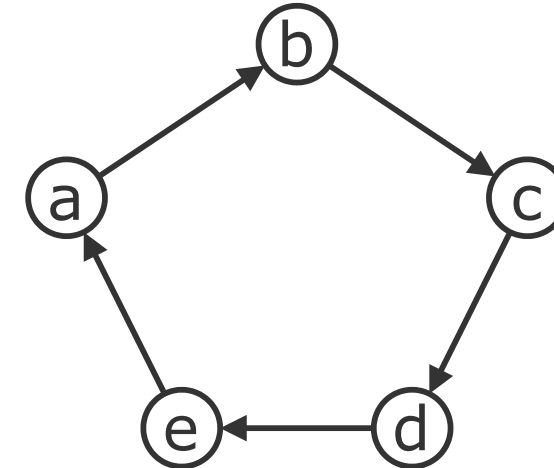
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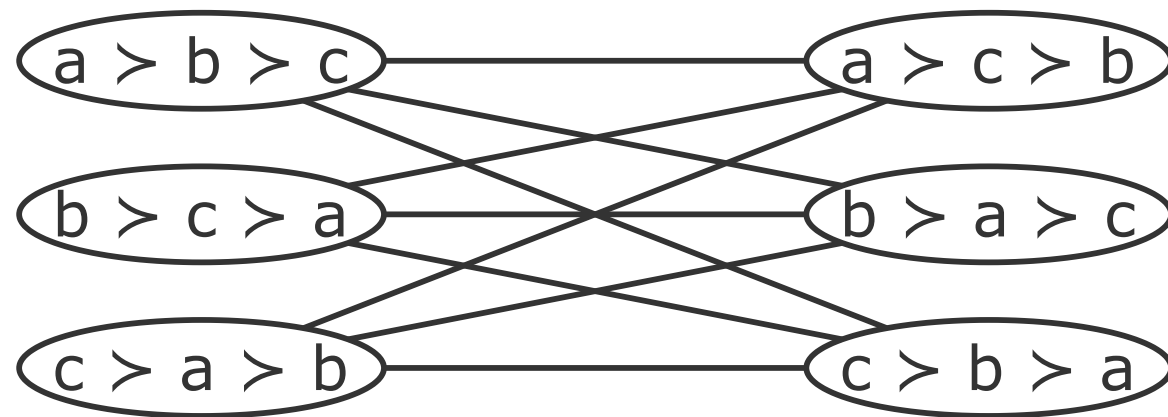


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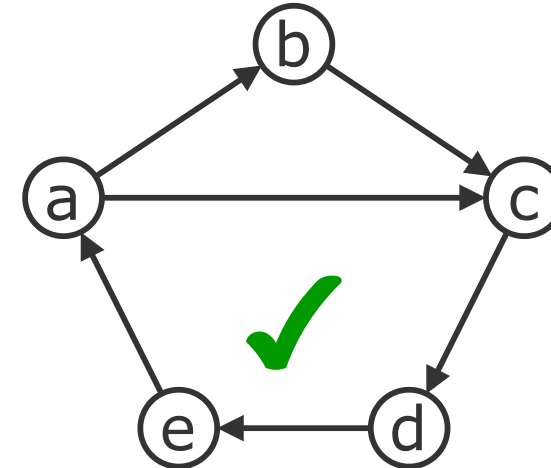
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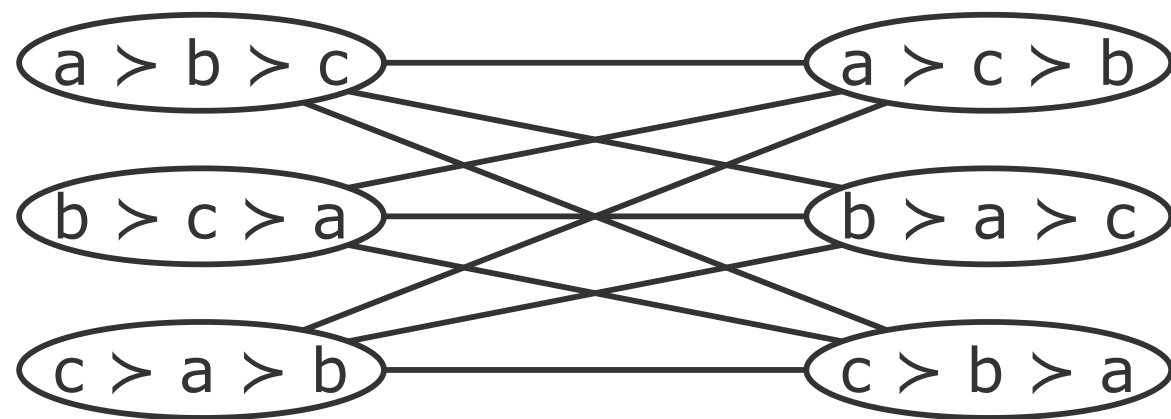


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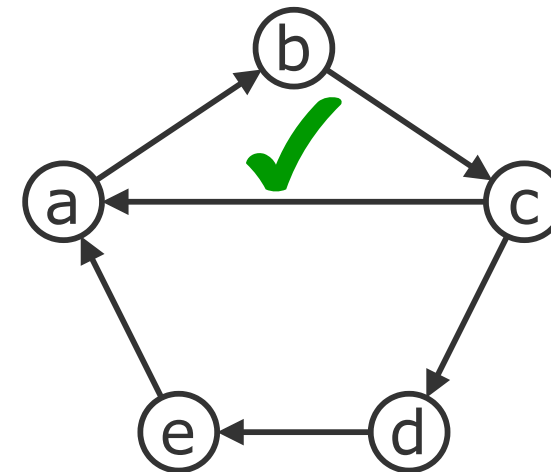
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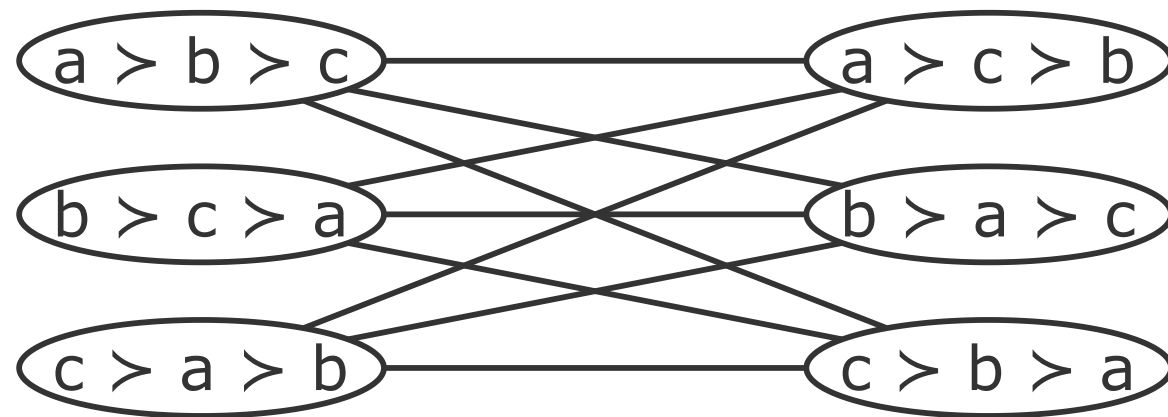


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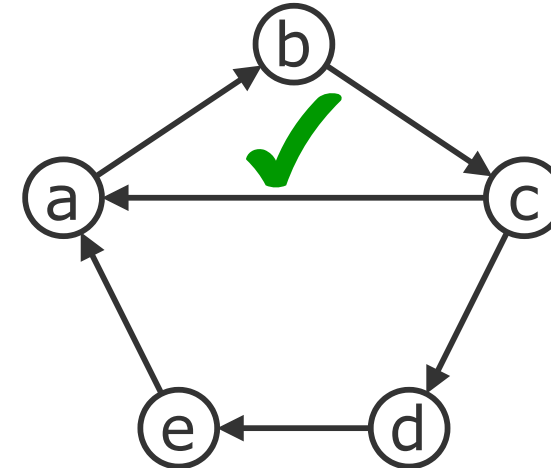


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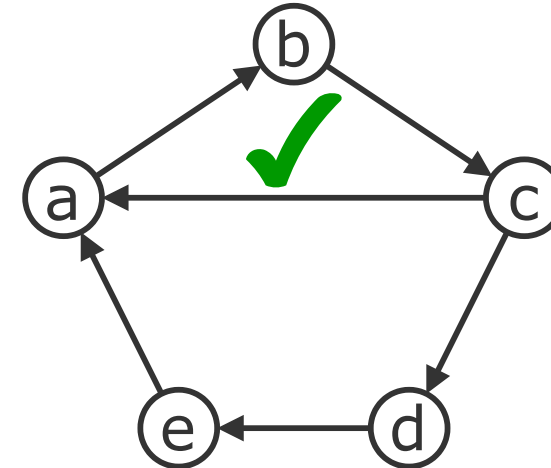


So p has a Condorcet cycle of size 3, say abc .

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We need to show each alternative (WLOG alternative a) can occur in each position.

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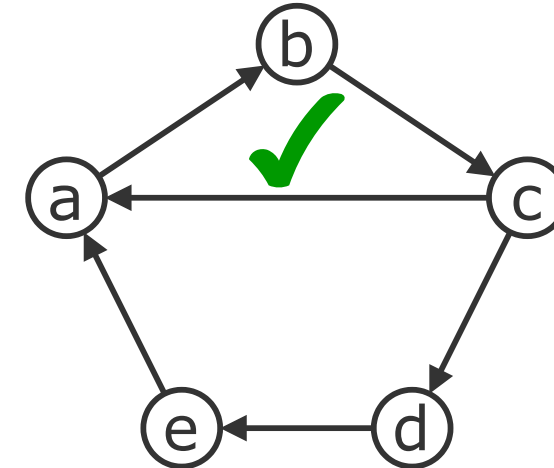
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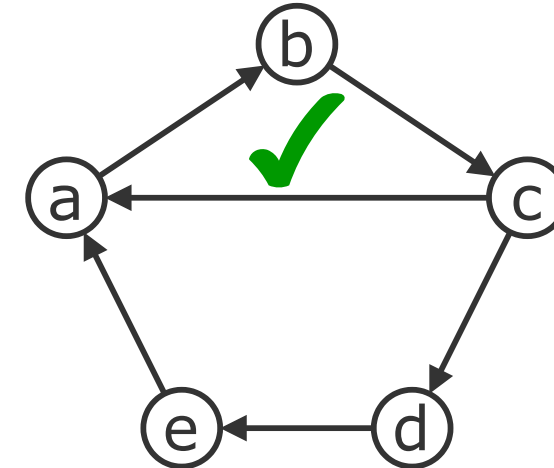
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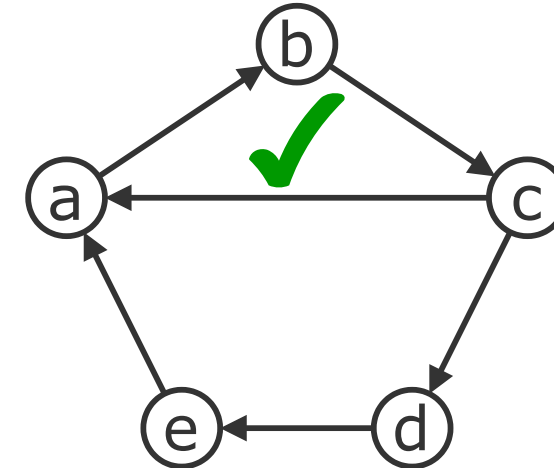
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We need to show each alternative (WLOG alternative a) can occur in each position.

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Forward direction: Suppose a Condorcet cycle can occur for some $p \in \Delta(D)$. Let C be a minimal cycle in the tournament graph of p . Claim: $|C| = 3$.



So p has a Condorcet cycle of size 3, say abc .

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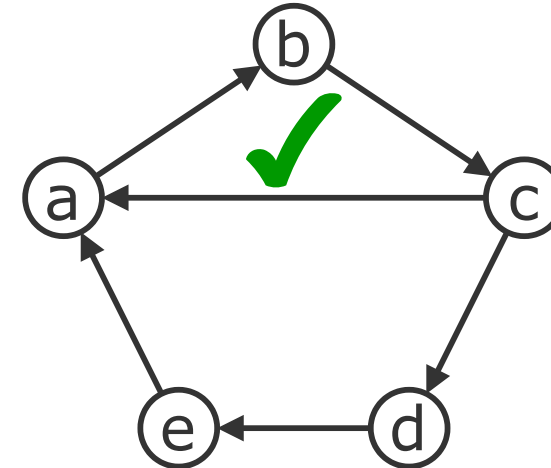
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Analogously, if a doesn't occur 3rd, then c must defeat it in the 2nd position. So a majority have preference $c \succ a \succ b$. Same contradiction as before.

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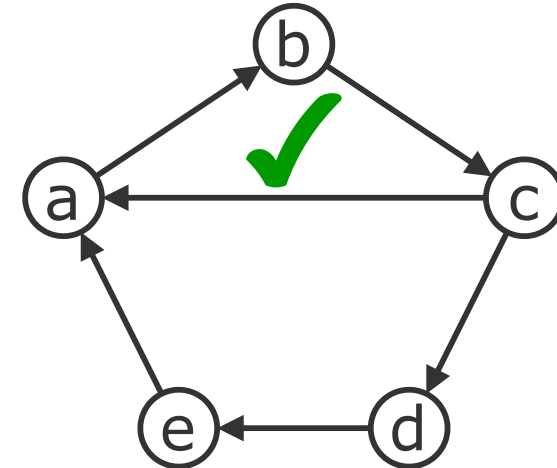
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Finally, if a doesn't occur 2nd, it defeats b from the 1st position, so a majority rank a first. So c can't defeat a , contradiction. ■



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Enumerating Condorcet domains

Open Question: What is the size of the largest Condorcet domain on m alternatives?

m	$f(m)$
1	1
2	2
3	4
4	9
5	20
6	45
7	100
8	224
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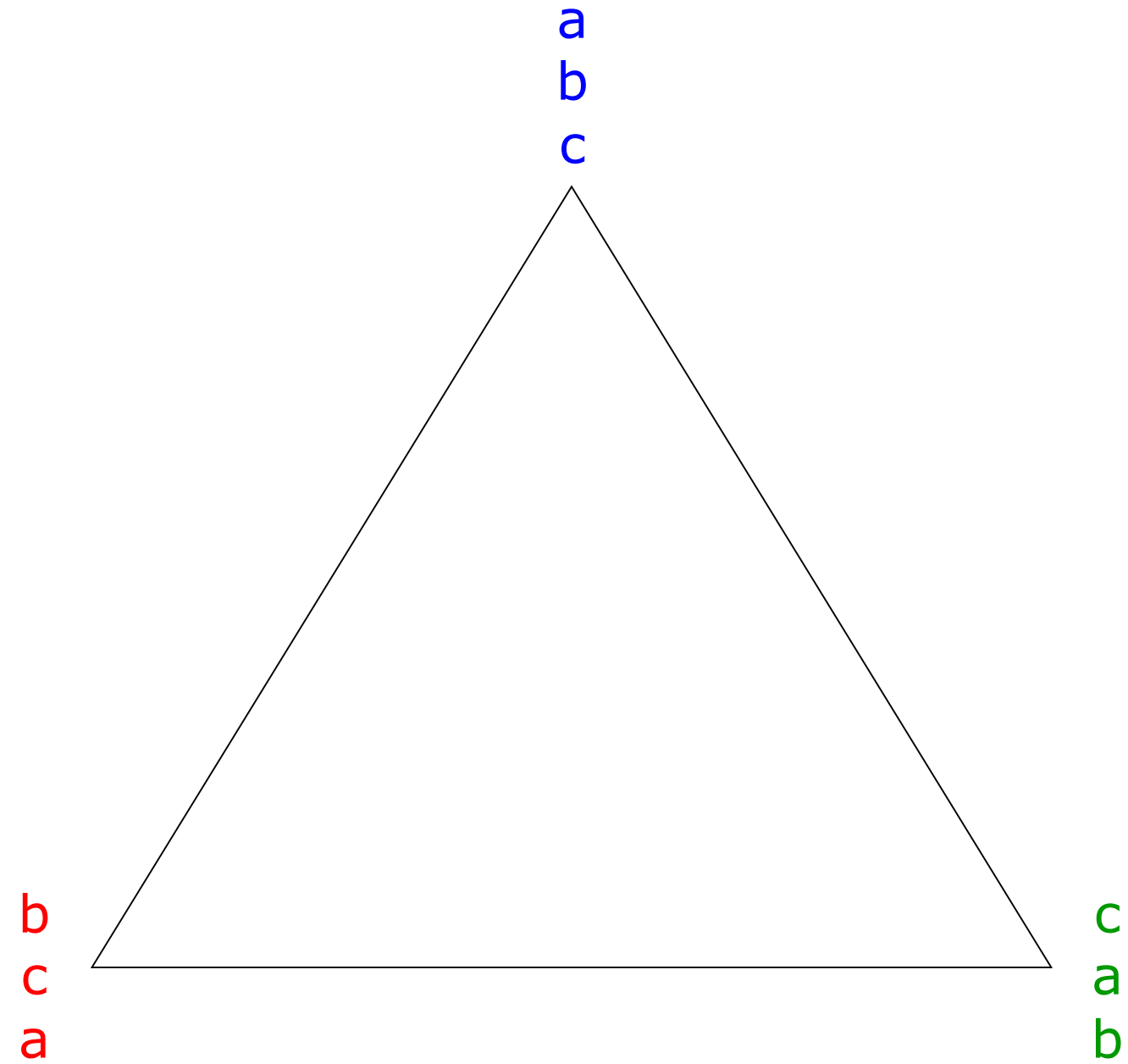
$$2.198139 = (3155366)^{1/19}$$


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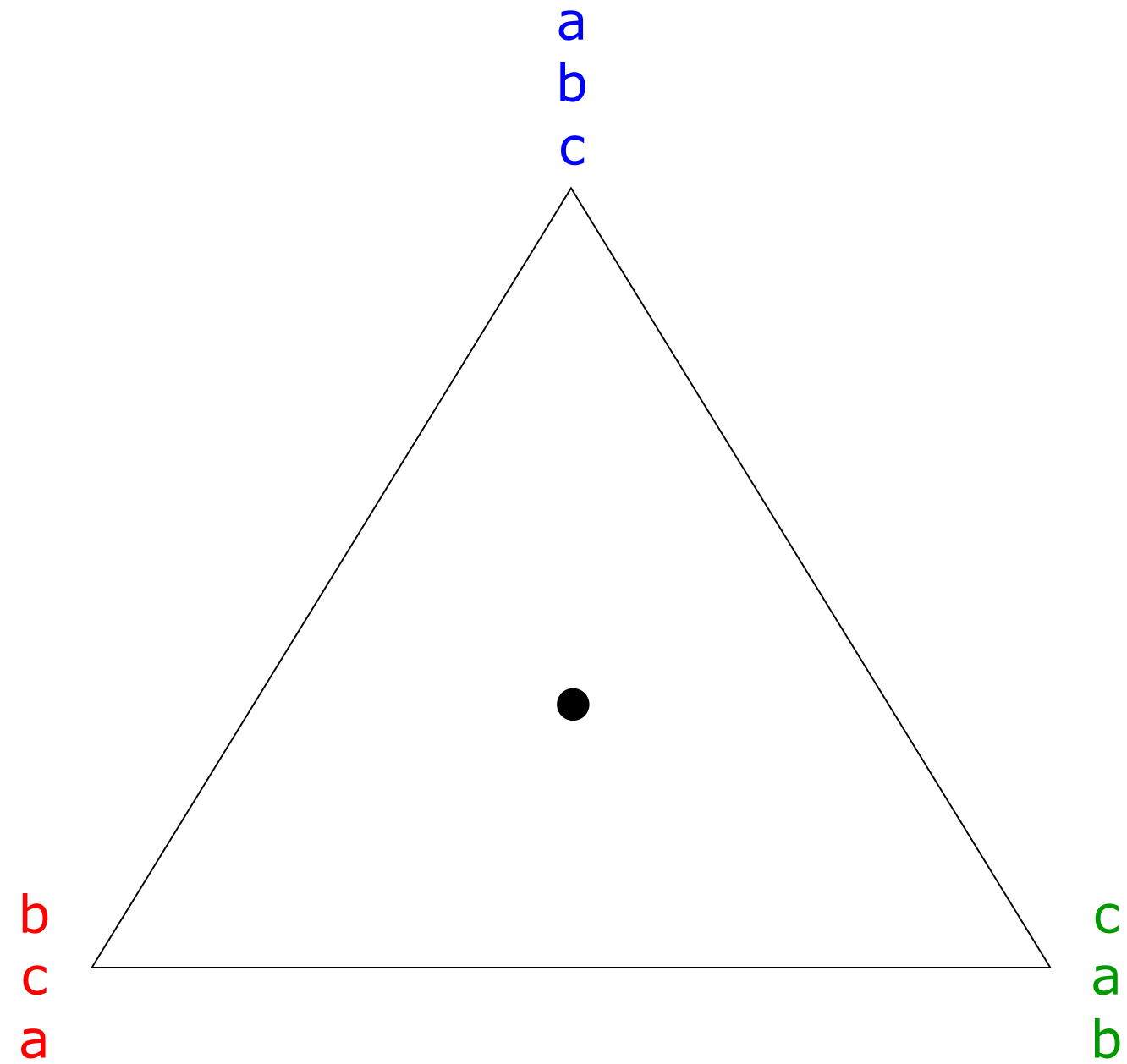
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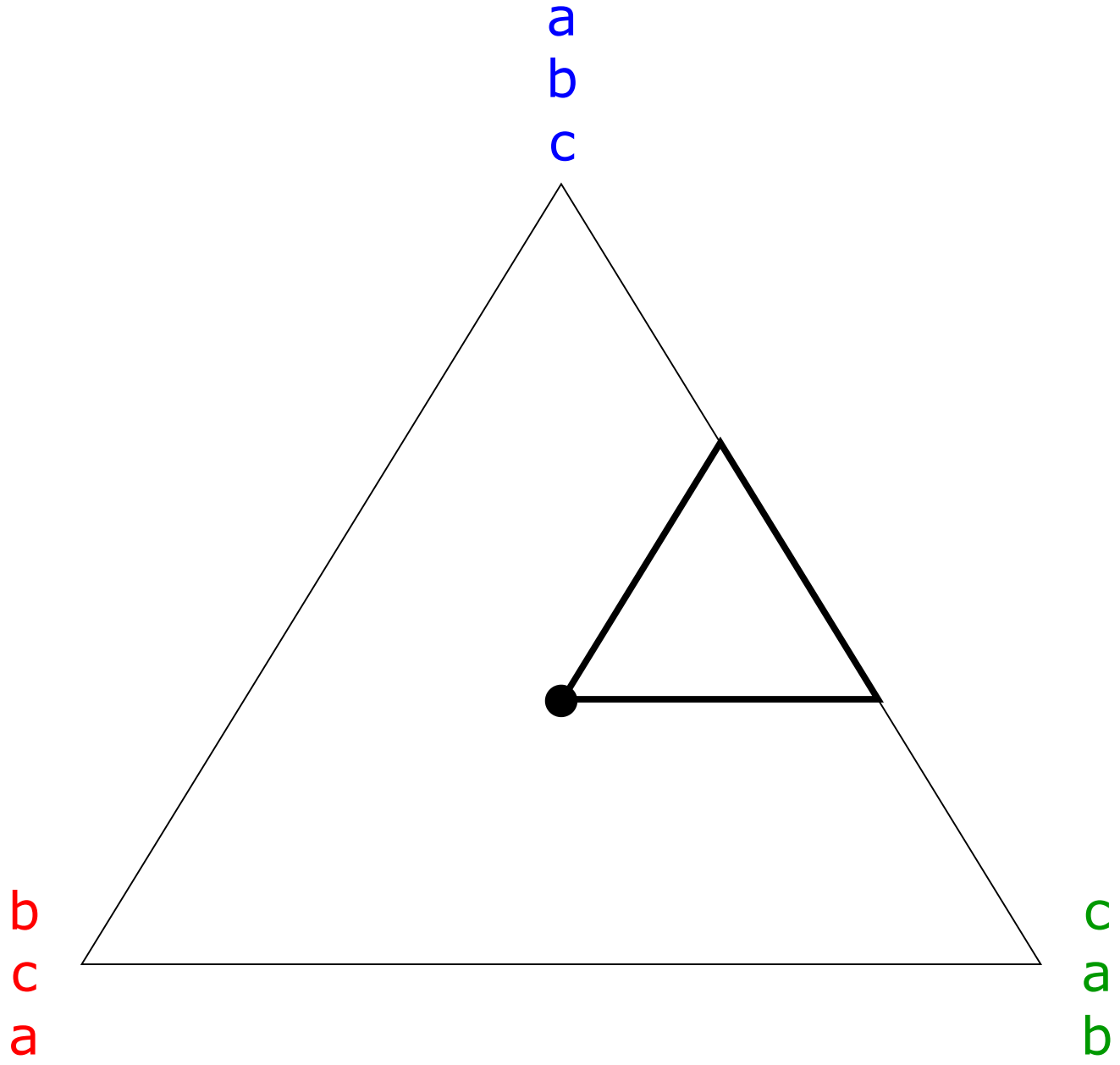
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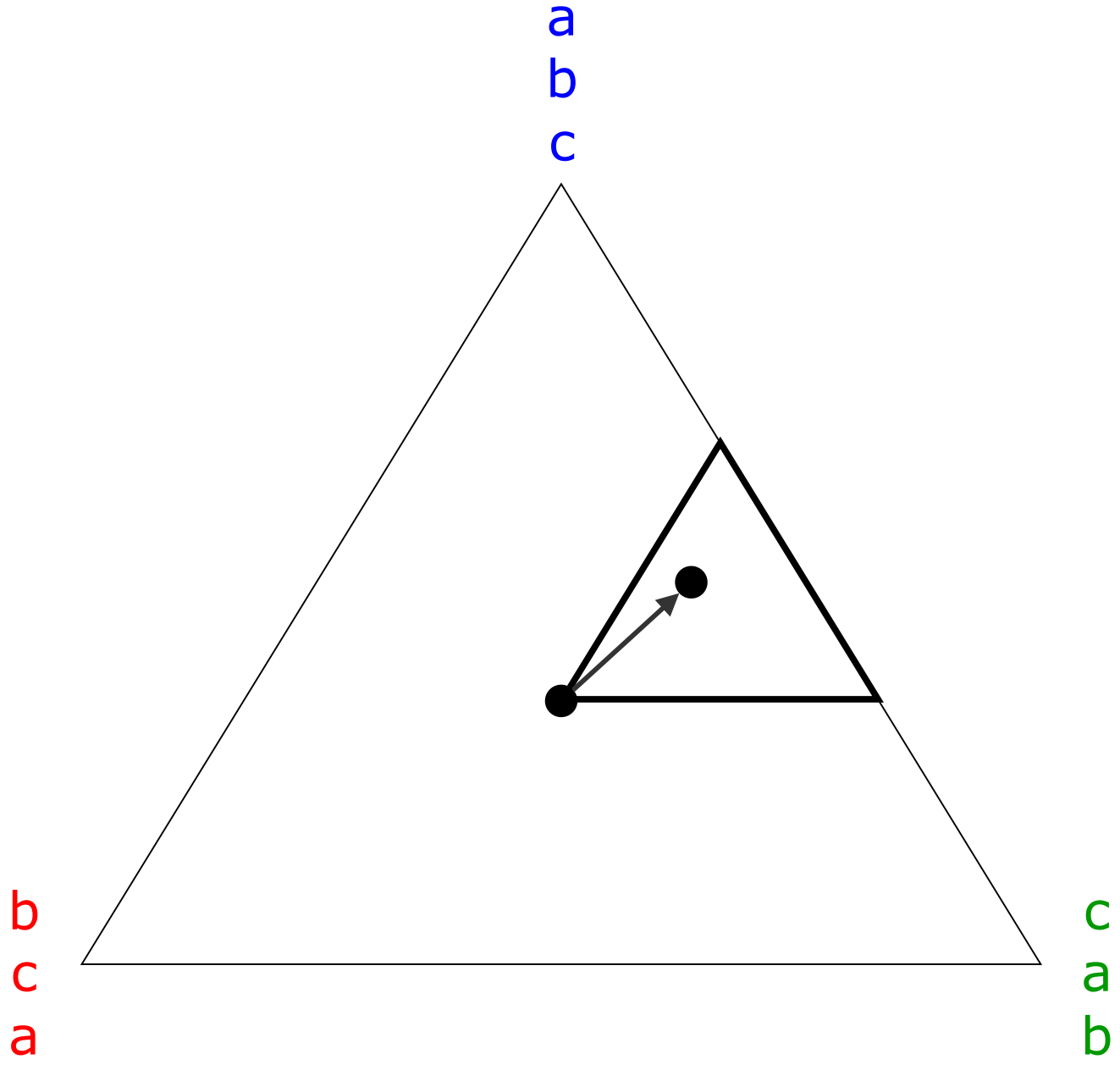
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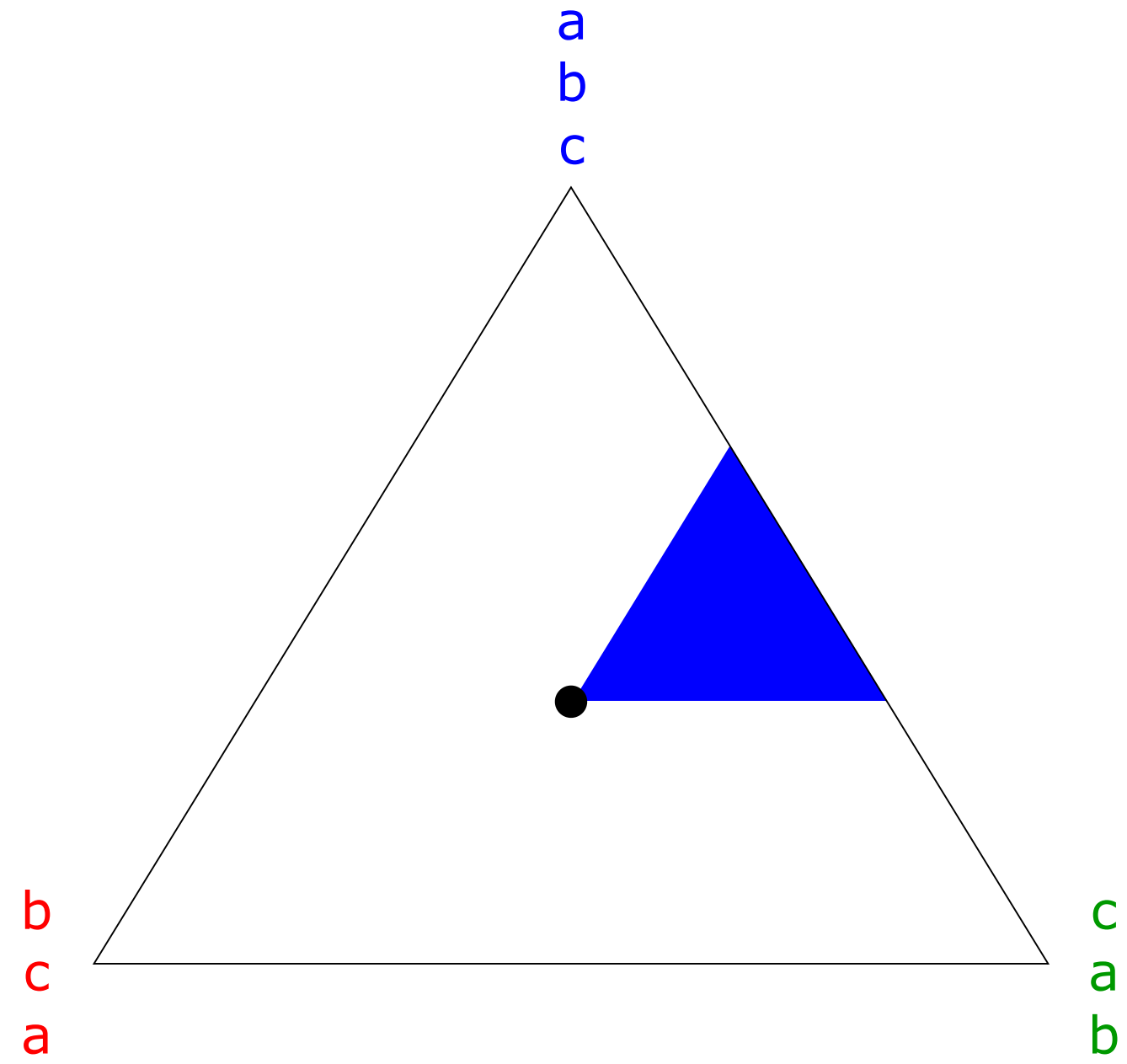
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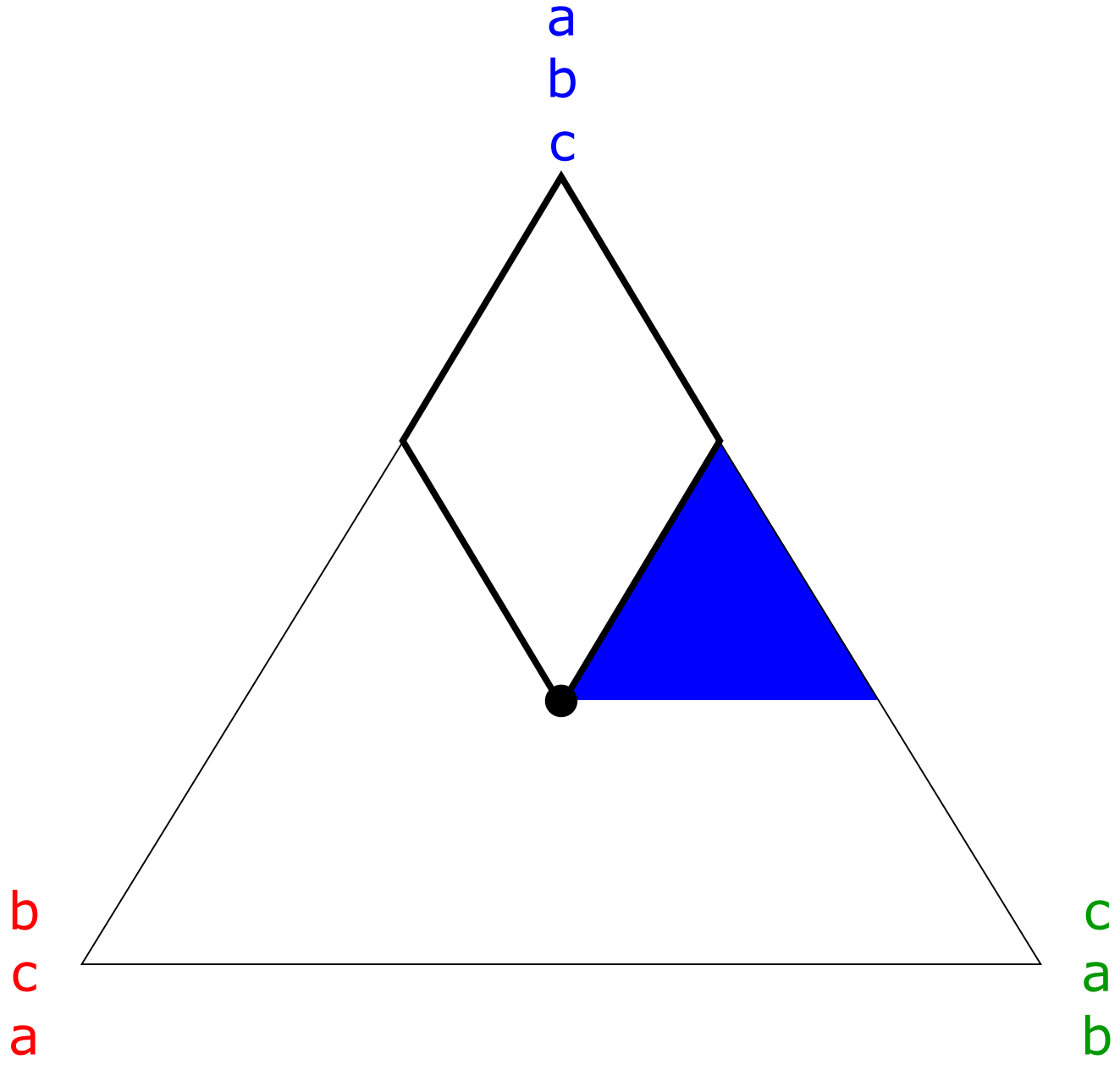
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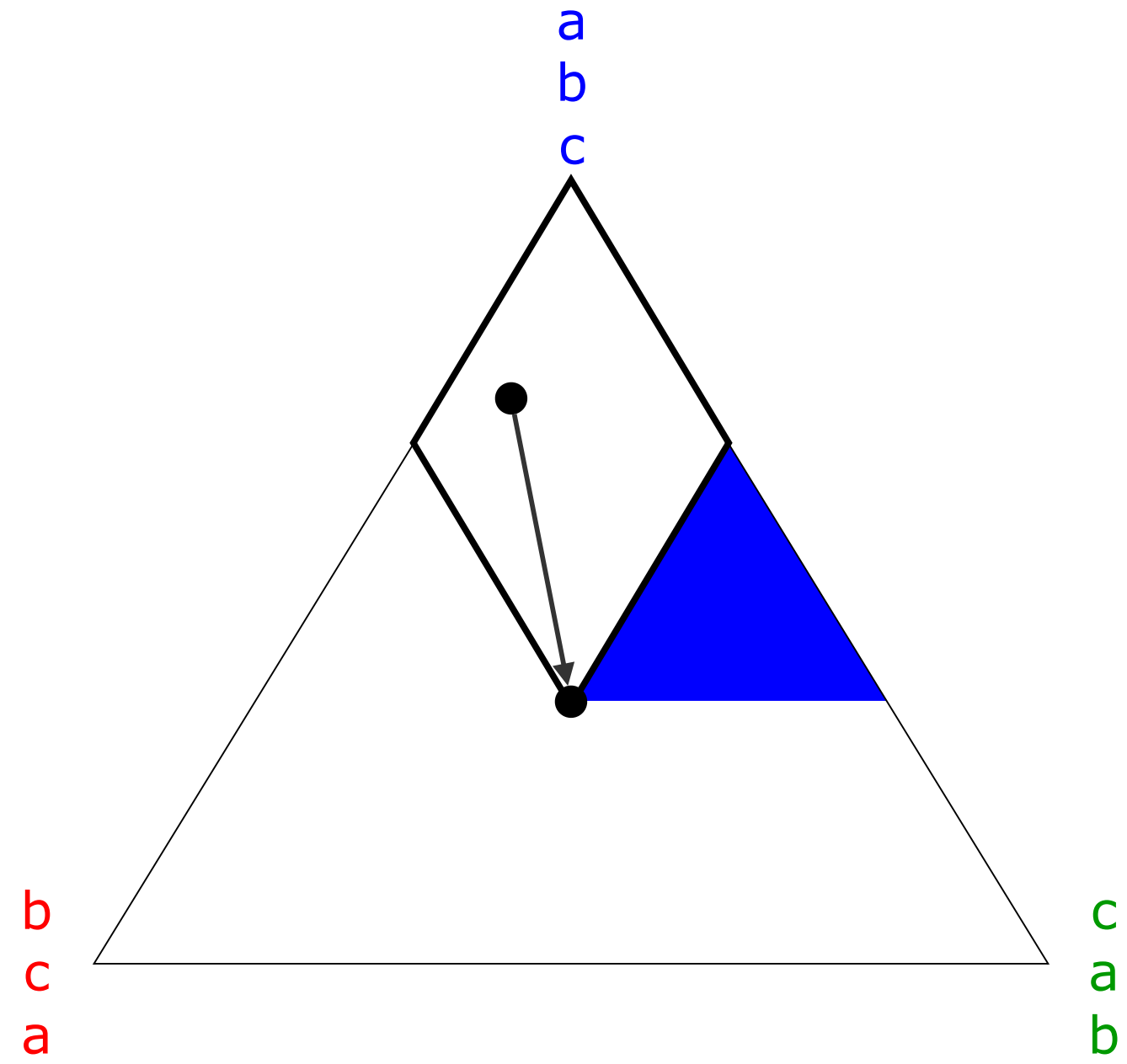
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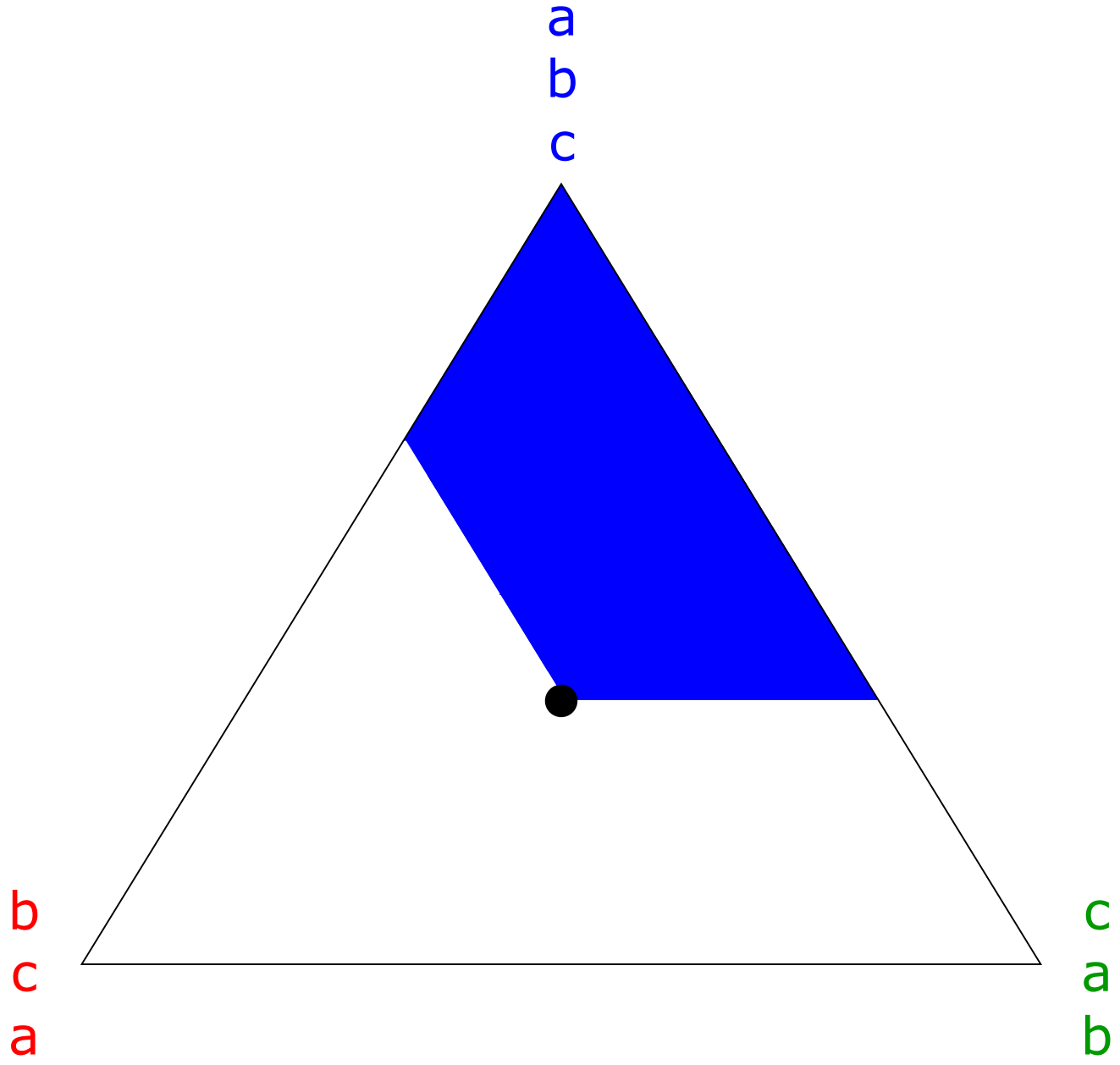
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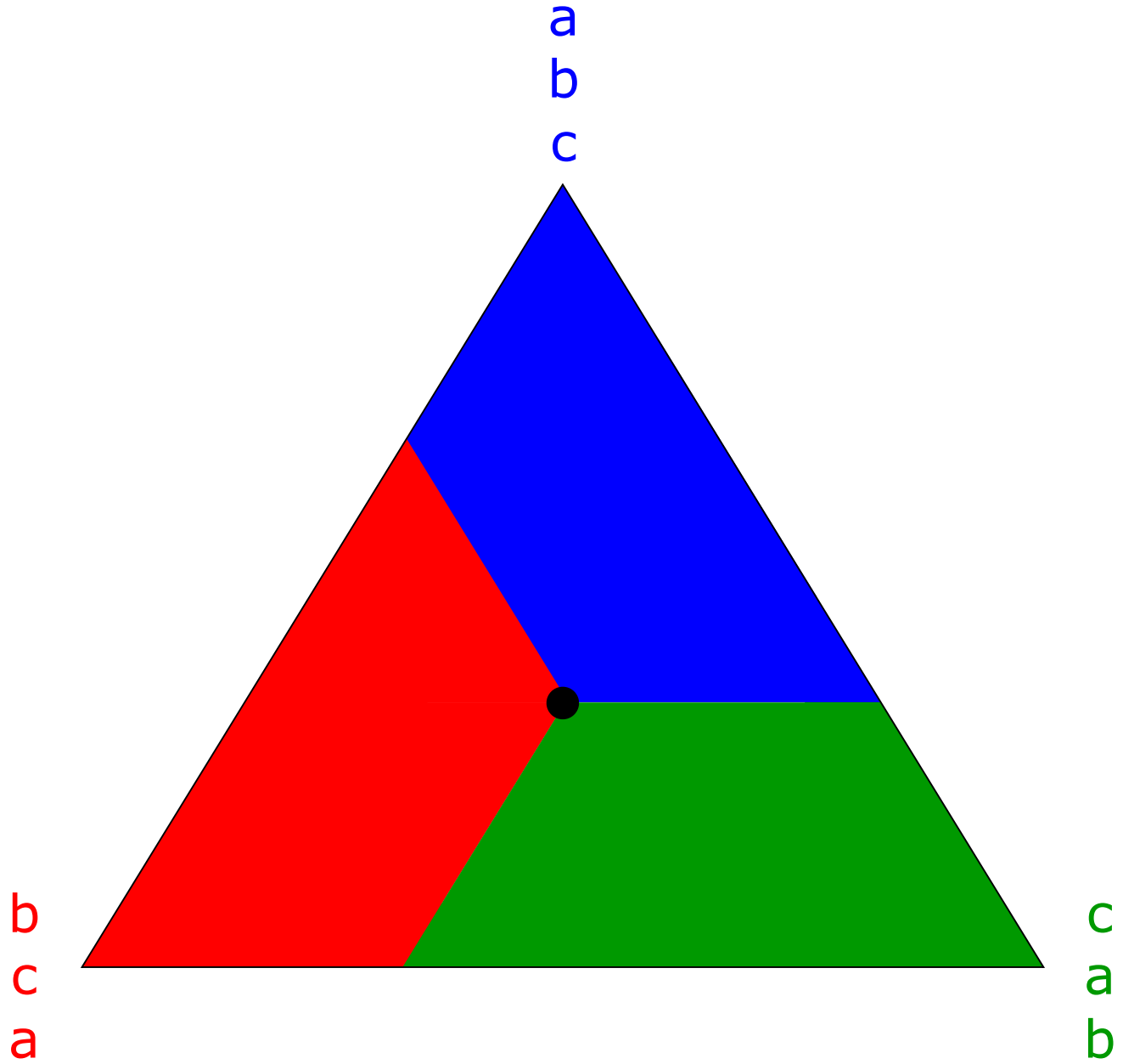
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The Borda count strikes again!

Theorem

On a domain D consisting of n orderings of n alternatives that are cyclic shifts of each other, the Borda count is an anonymous, neutral, Pareto efficient, and SP voting rule.

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i
 $i + 1$
 \dots
 $j - 1$
 j
 $j + 1$
 \dots
 n
 1
 2
 \dots
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i	...	$j + 1$	j	...	$i + 1$
$i + 1$...	$j + 1$...
...		n	...		$j - 1$
$j - 1$		1	n		j
j		2	1		$j + 1$
$j + 1$...	2		...
...		$i - 1$...		n
n		i	$i - 1$		1
1		$i + 1$	i		2
2		...	$i + 1$...
...		$j - 1$...		$i - 1$
$i - 1$...	j	$j - 1$...	i

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X Block			Y Block		
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$i + 1$...	$j + 1$...
...		n	...		$j - 1$
$j - 1$		1	n		j
j		2	1		$j + 1$
$j + 1$...	2		...
...		$i - 1$...		n
n		i	$i - 1$		1
1		$i + 1$	i		2
2		...	$i + 1$...
...		$j - 1$...		$i - 1$
$i - 1$...	j	$j - 1$...	i

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If a group of voters shares preferences over i versus j , they must be all from the X Block or all from the Y Block.

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$i + 1$...	$j + 1$...
...		n	...		$j - 1$
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j		2	1		$j + 1$
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If a group of voters shares preferences over i versus j , they must be all from the X Block or all from the Y Block. But sincere voting already induces the maximum difference between the Borda score of i and the Borda score of j . Thus there are no beneficial deviations.

X Block			Y Block		
i	...	$j + 1$	j	...	$i + 1$
$i + 1$...	$j + 1$...
...		n	...		$j - 1$
$j - 1$		1	n		j
j		2	1		$j + 1$
$j + 1$...	2		...
...		$i - 1$...		n
n		i	$i - 1$		1
1		$i + 1$	i		2
2		...	$i + 1$...
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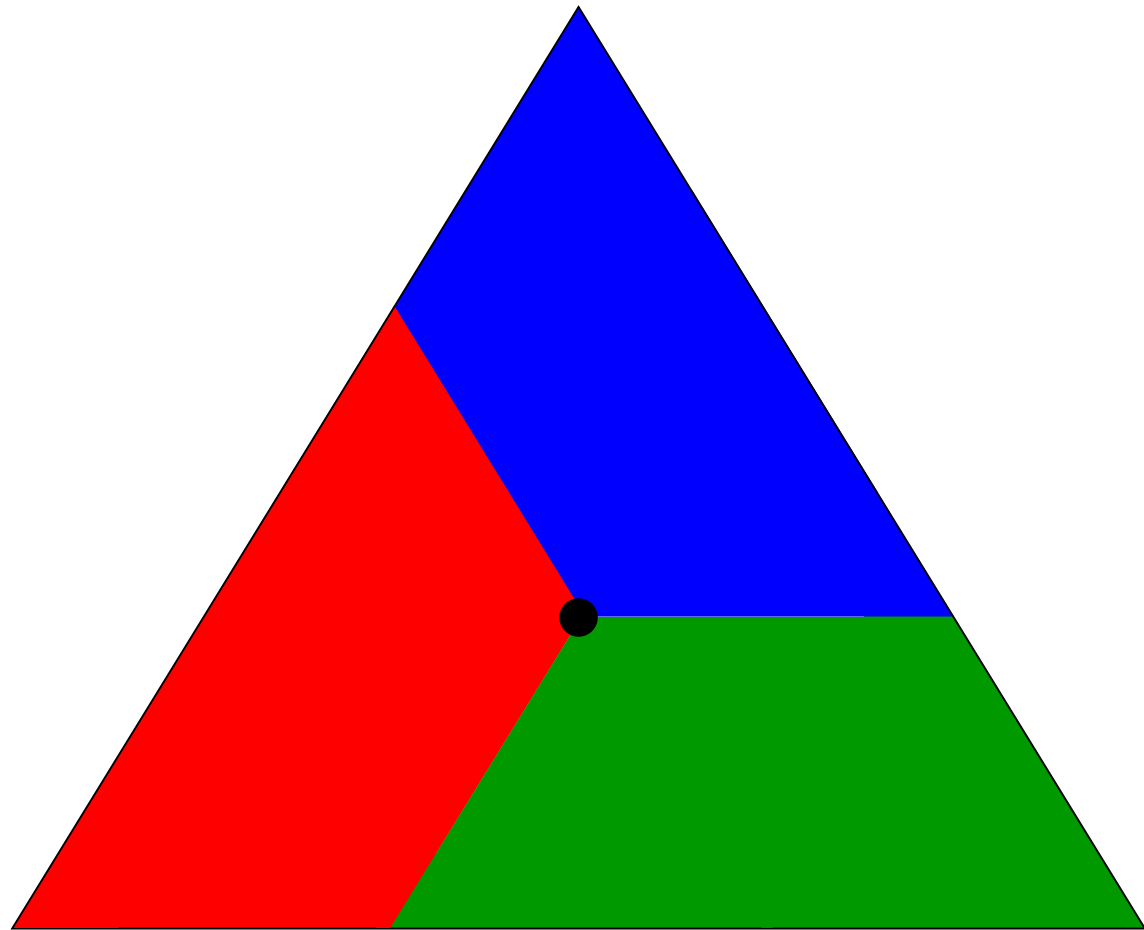
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$a \succ b \succ c$



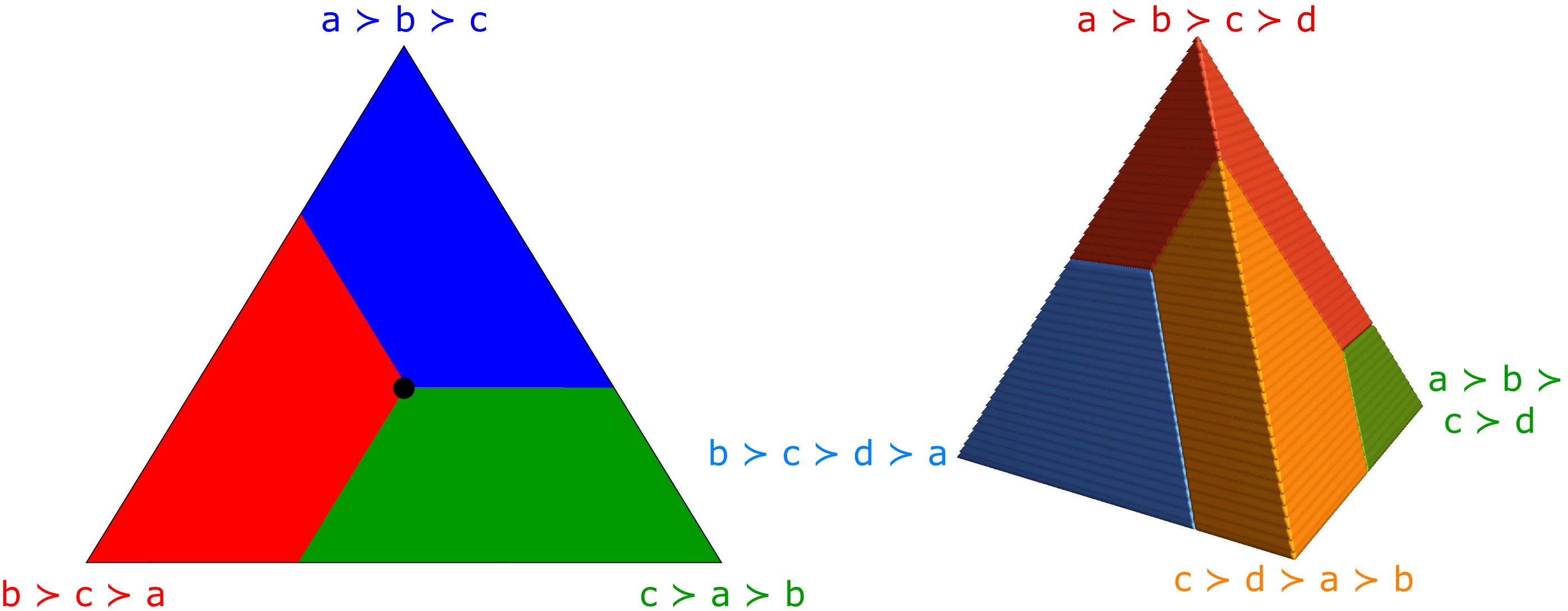
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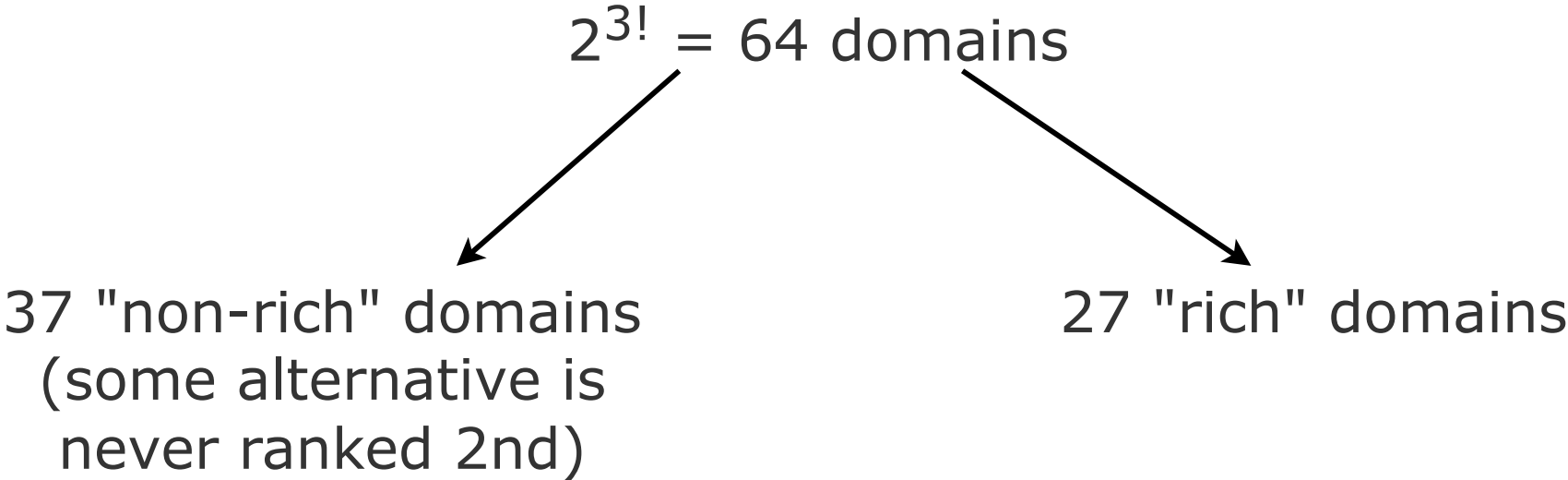
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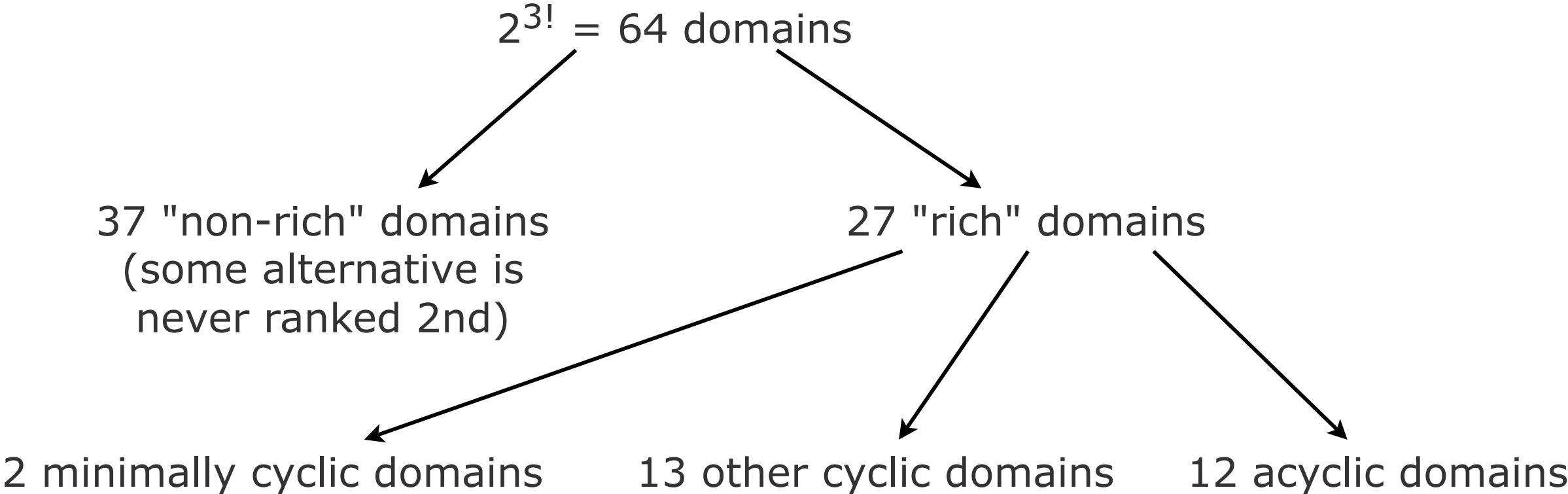
Dasgupta & Maskin's classification of domains for $m = 3$

$$2^{3!} = 64 \text{ domains}$$

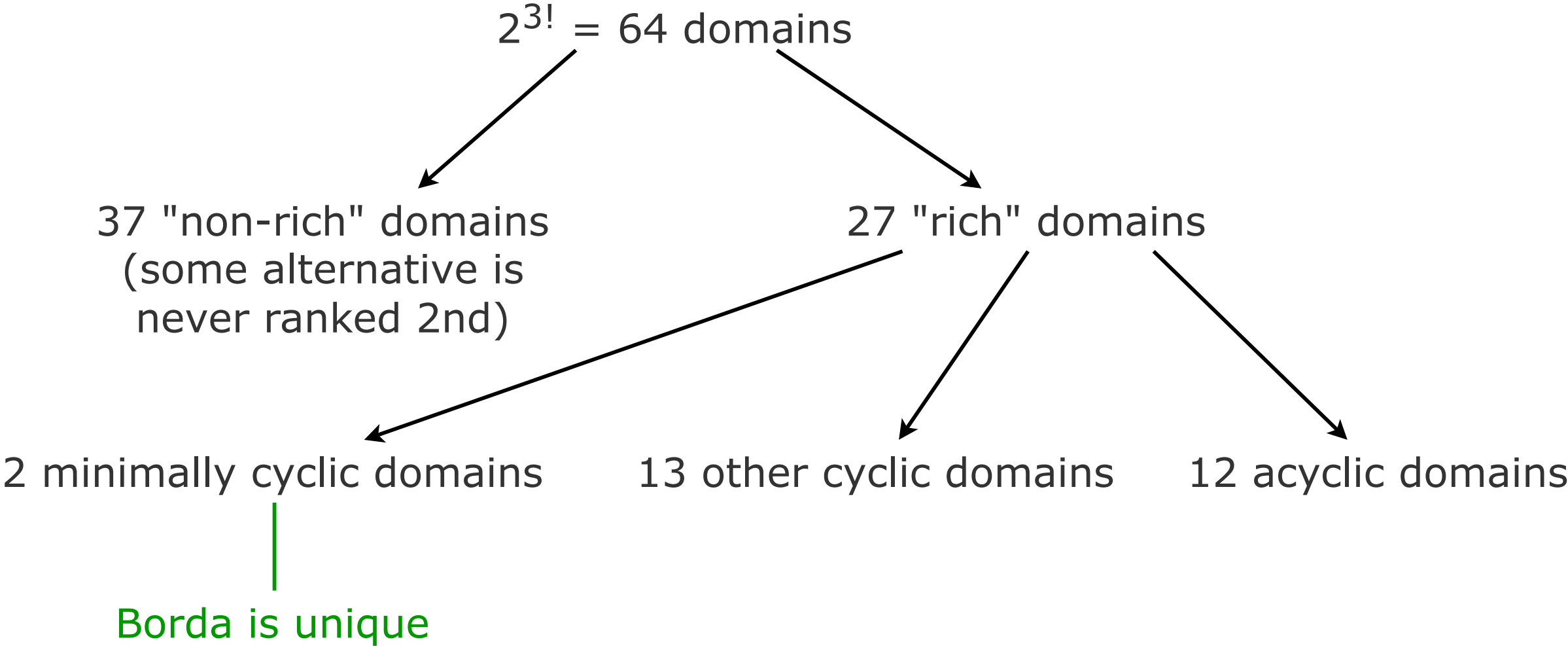
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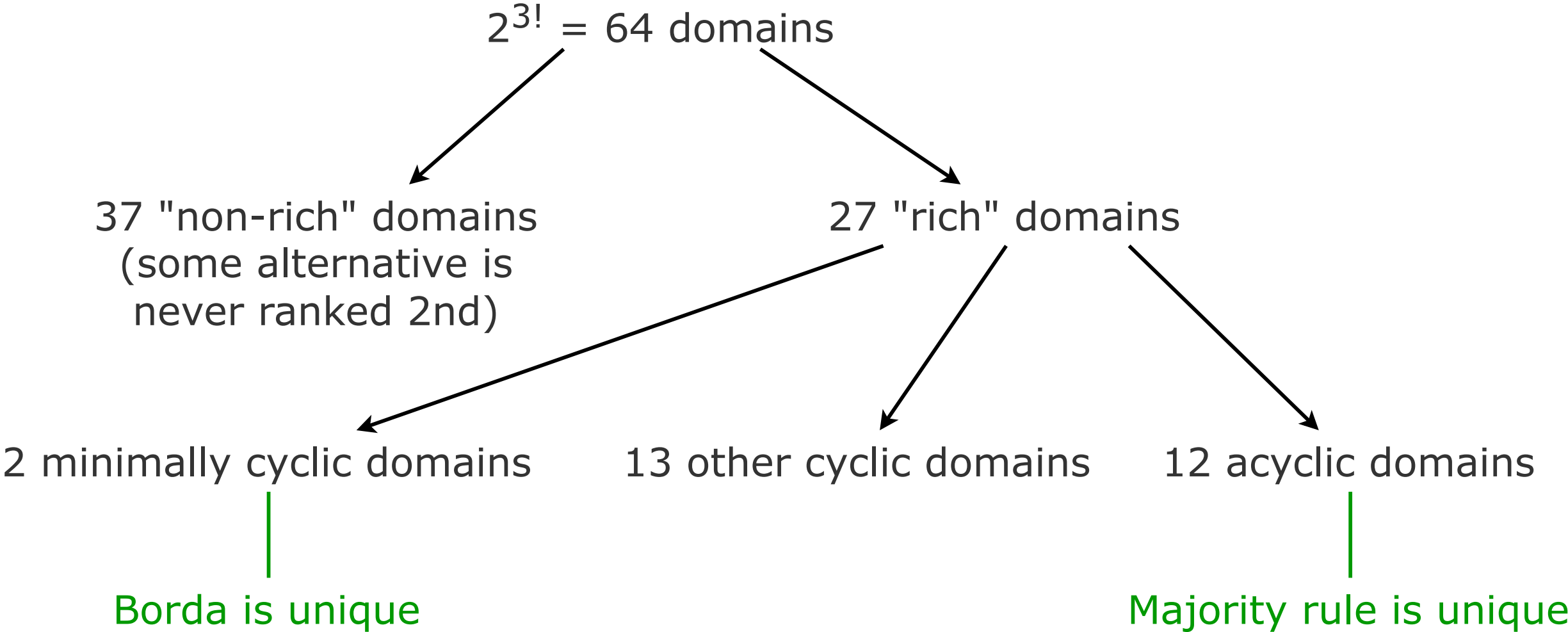
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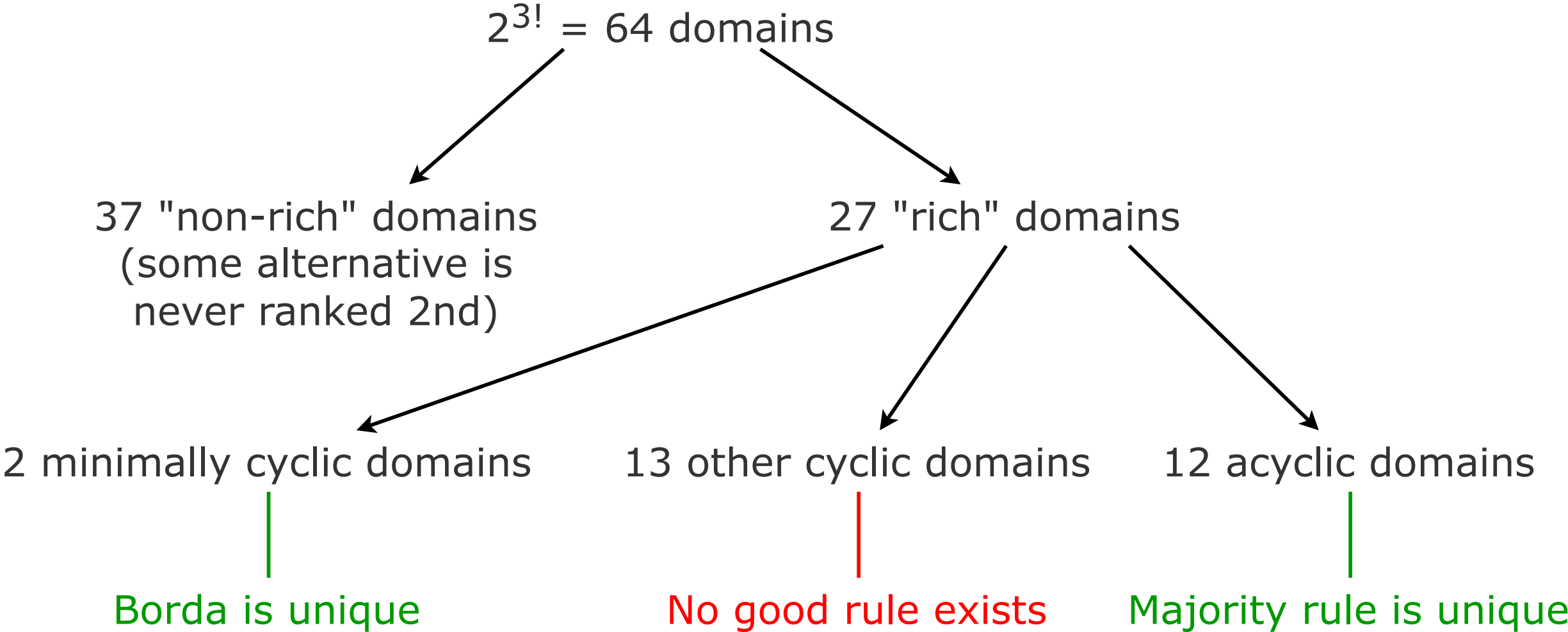
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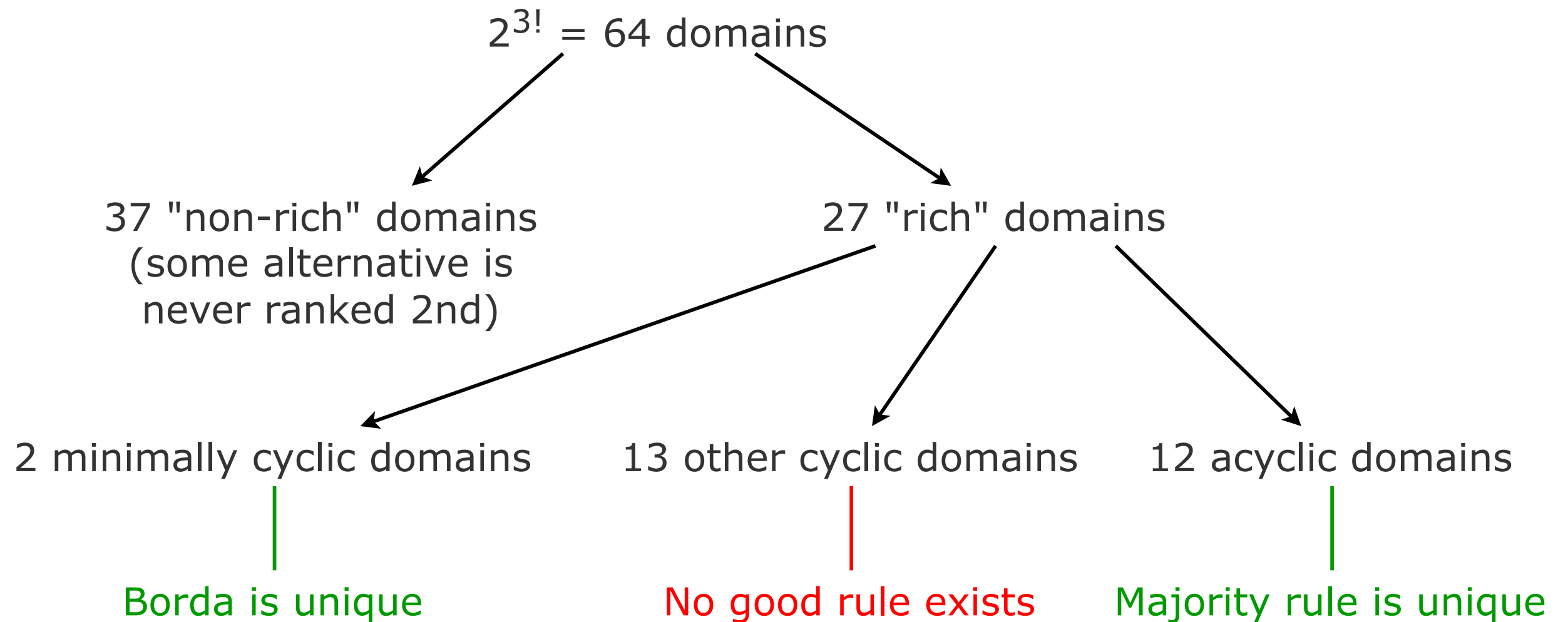
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Note: If we replace SP with IIA, then on any domain, if any rule is good, majority rule is good (Dasgupta, Maskin, 2008).