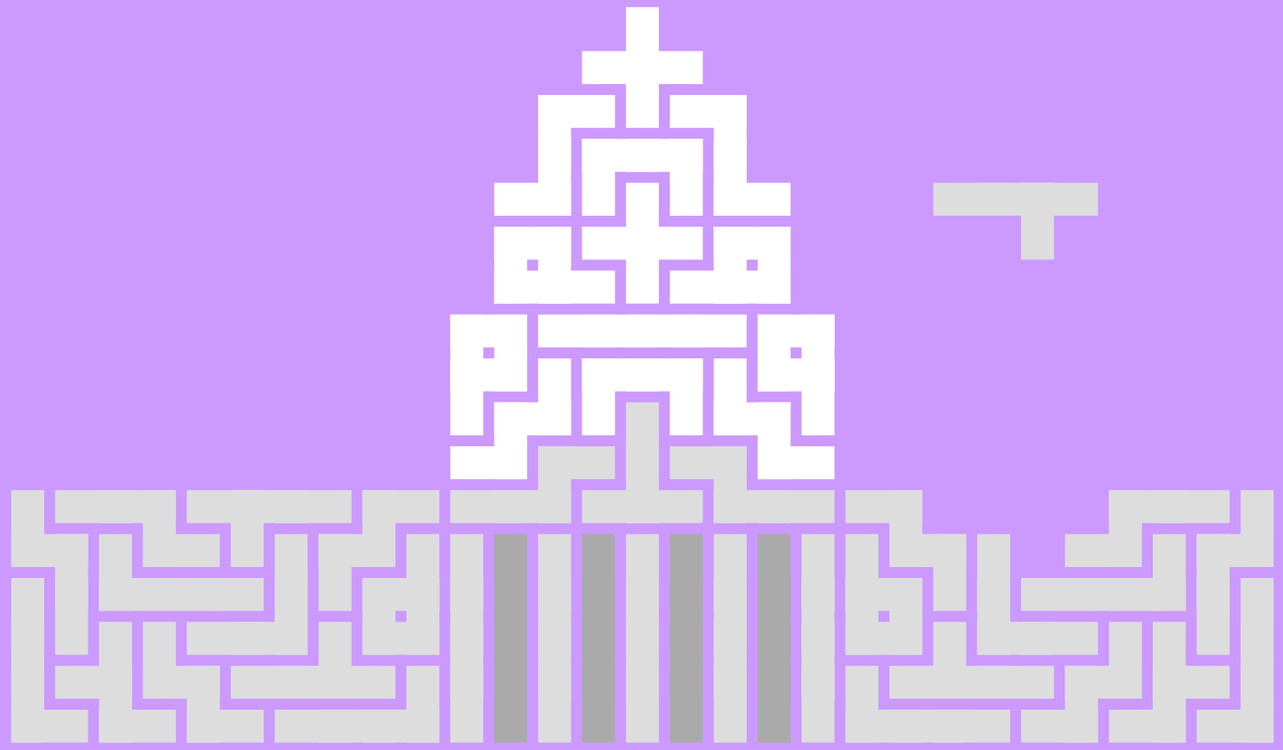


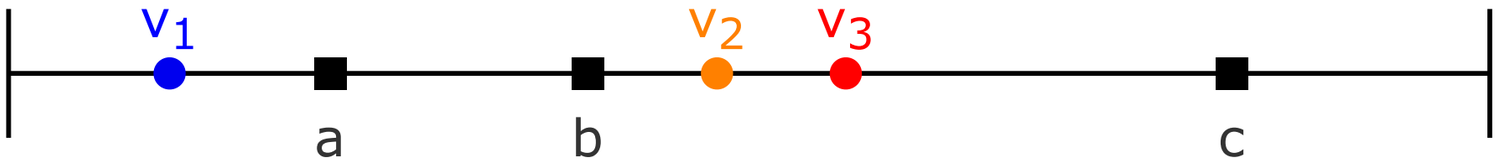
Algorithms For Democratic Decision-Making

Jamie Tucker-Foltz • Yale University • Spring 2026

Lecture 6: **Distance-Based Preferences**

Preferences on a line

Assume *n* candidates and *m* voters lie on a line.



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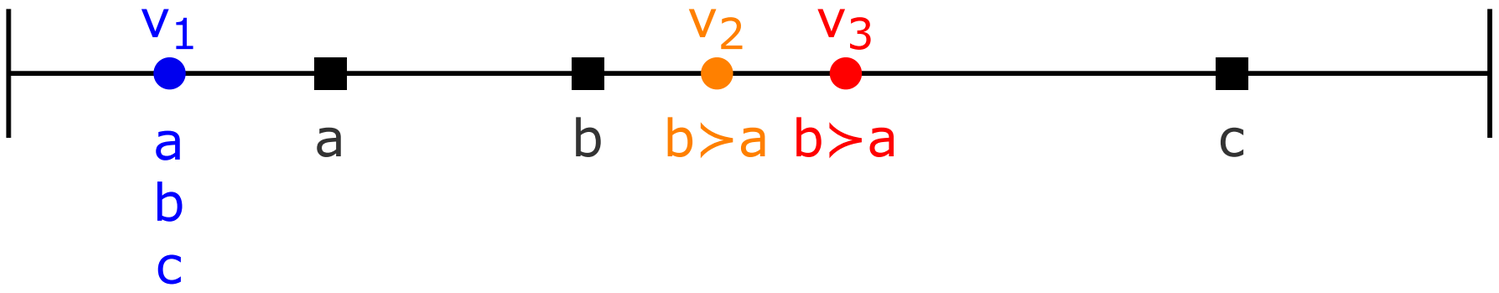

Possible assumptions on preferences:

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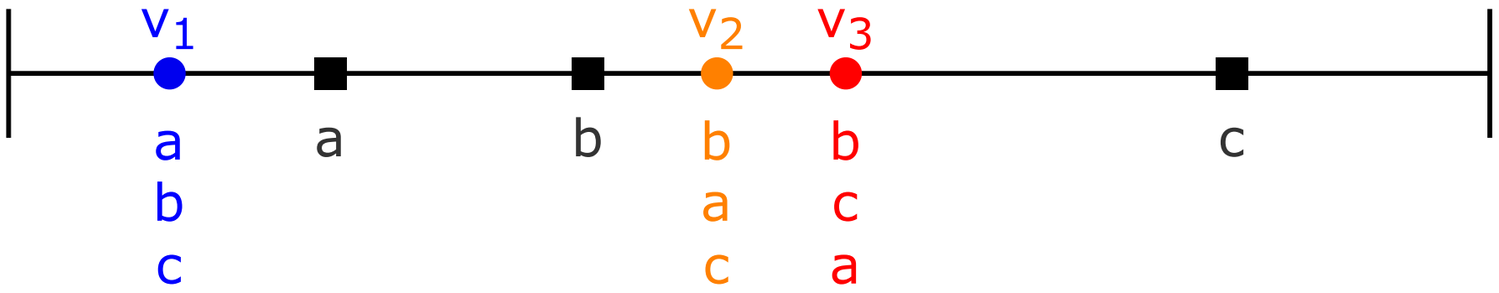

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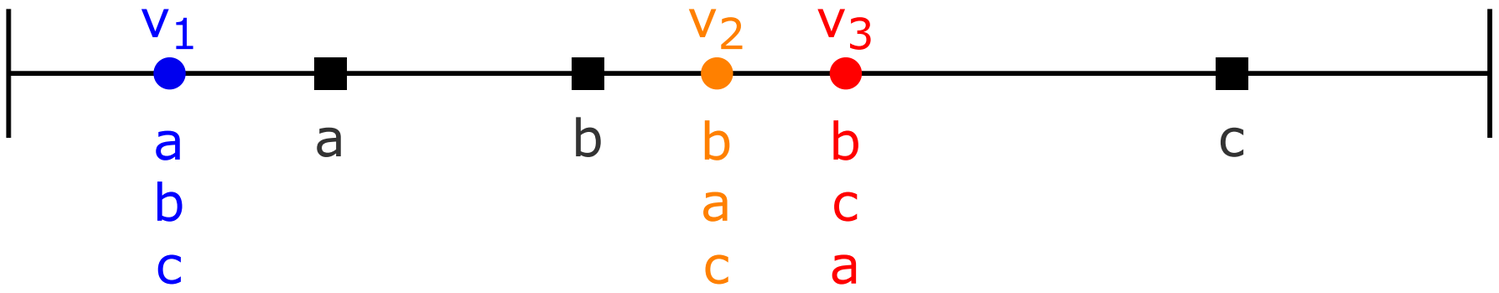

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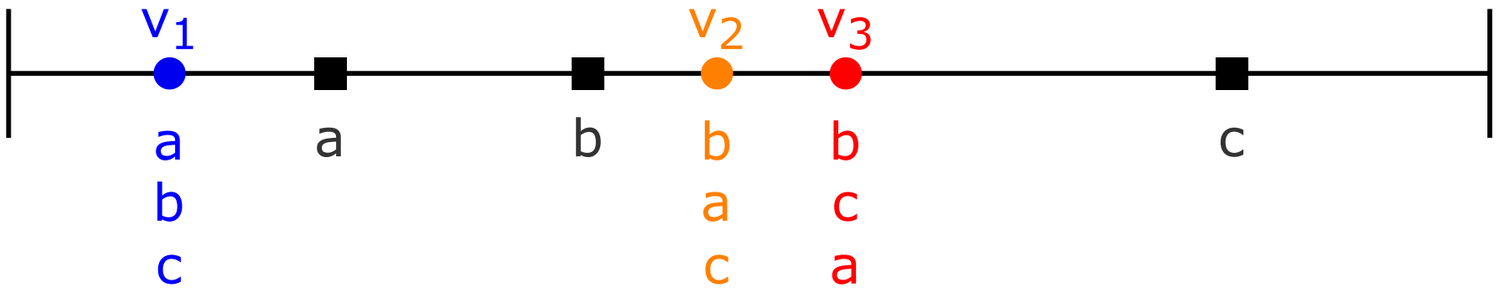

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▸ Which assumptions are necessary to guarantee that there is an anonymous, Pareto efficient, and SP voting rule?

Respond at:

pollev.com/jtuckerfoltz255 or

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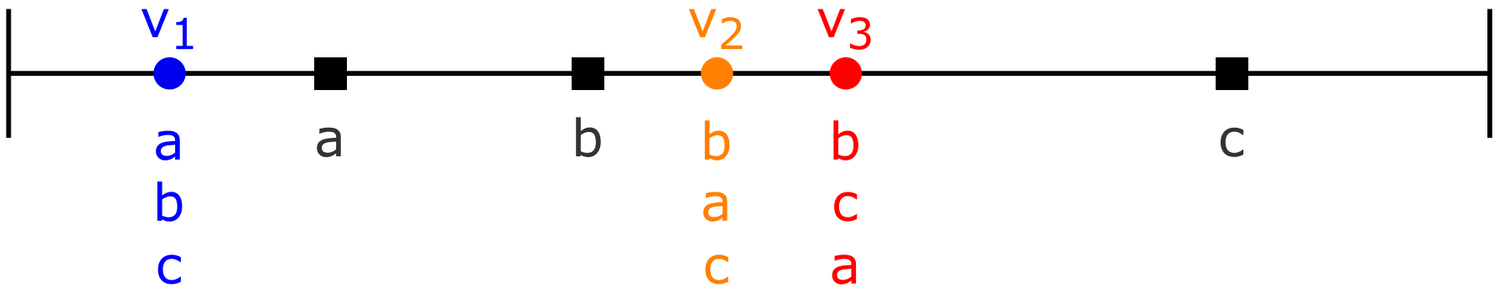

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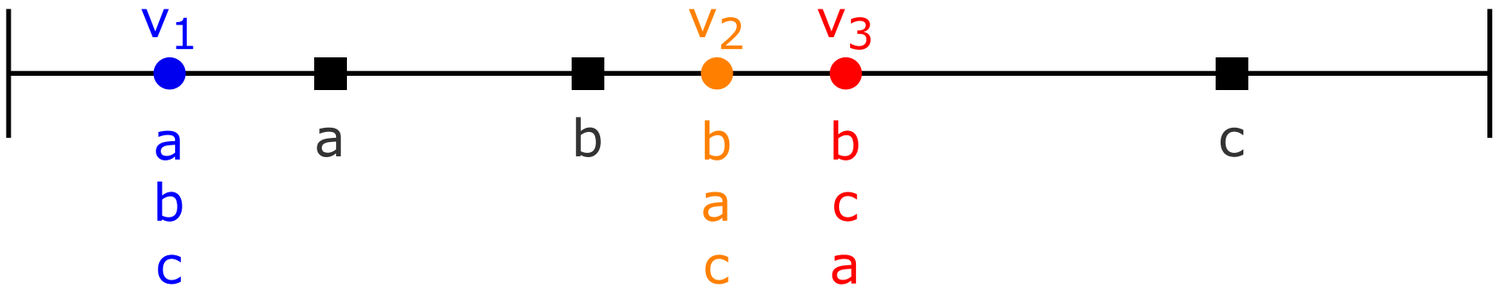

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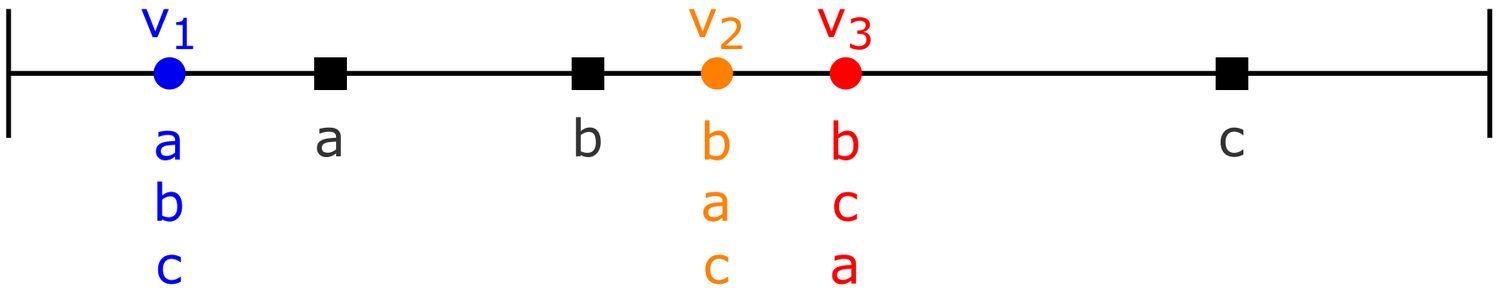

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Median Voter Theorem (Black, 1948)

Assuming 1 and 2, the candidate most preferred by the median voter is a Condorcet winner.

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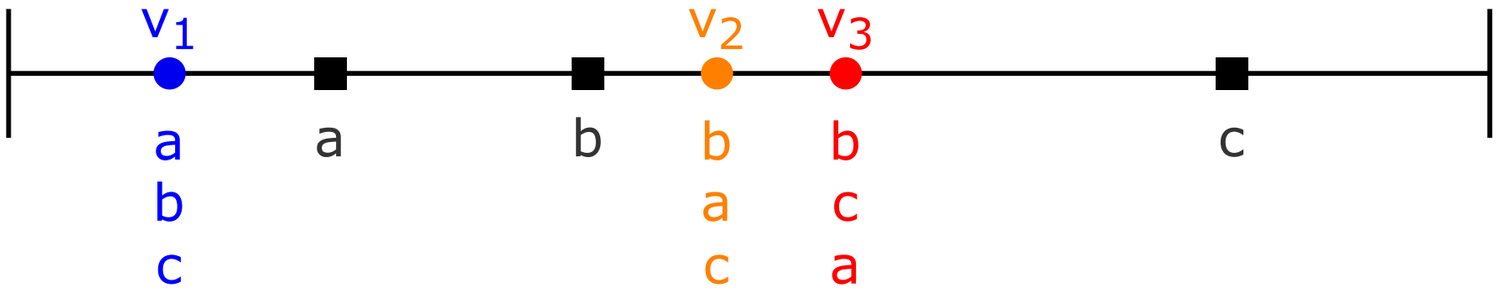

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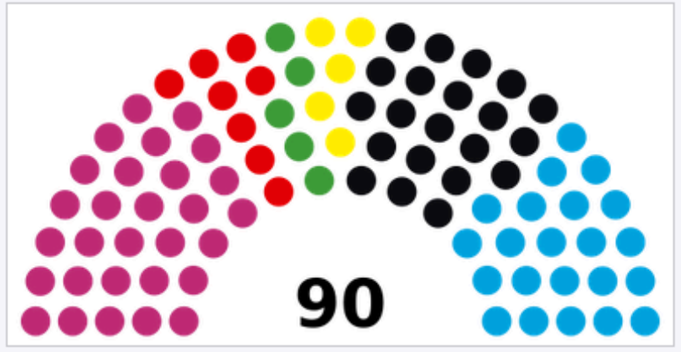
- Voters left of v_2 prefer b to c since v_2 does.
- Voters right of v_2 prefer b to a since v_2 does.

The Median Voter Theorem in practice



Composition of the Landtag of Thuringia after the 2019 state election

- The Left (Die Linke): 29 seats
- Social Democratic Party (SPD): 8 seats
- Alliance 90/The Greens (Grüne): 5 seats
- Free Democratic Party (FDP): 5 seats
- Christian Democratic Union (CDU): 21 seats
- Alternative for Germany (AfD): 22 seats

Image credit: Wikipedia

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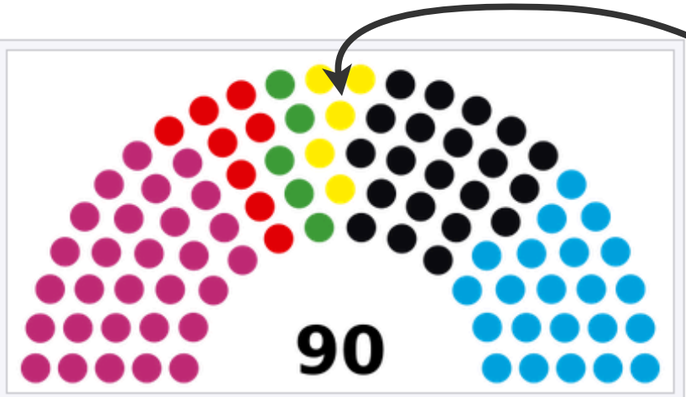

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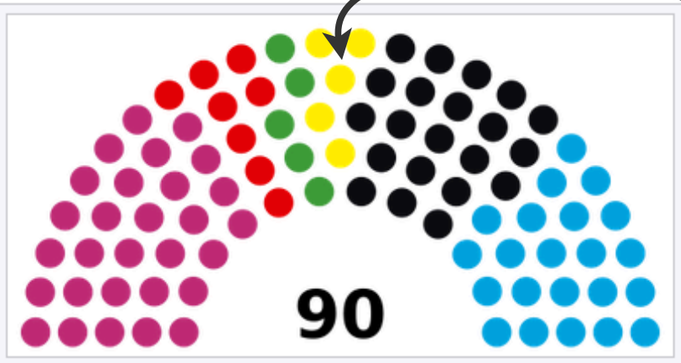

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Image credit: [Wikipedia]

BBC

Germany AfD: Thuringia PM quits amid fury over far right

6 February 2020

Share Save

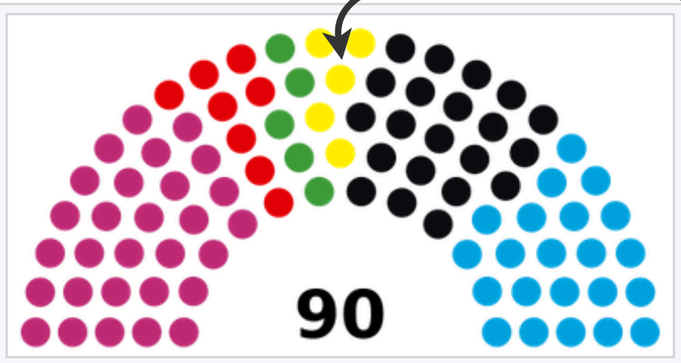
There was shock as the new state premier, Thomas Kemmerich (left), shook hands with local AfD leader Björn Höcke

A German state premier elected with the help of the far-right AfD says he is resigning to pave the way for fresh elections.

The election of liberal leader Thomas Kemmerich in the eastern state of Thuringia prompted national outrage.

"Resignation is unavoidable," he said. For years Germany's main parties have shunned Alternative for Germany (AfD).

The Median Voter Theorem in practice



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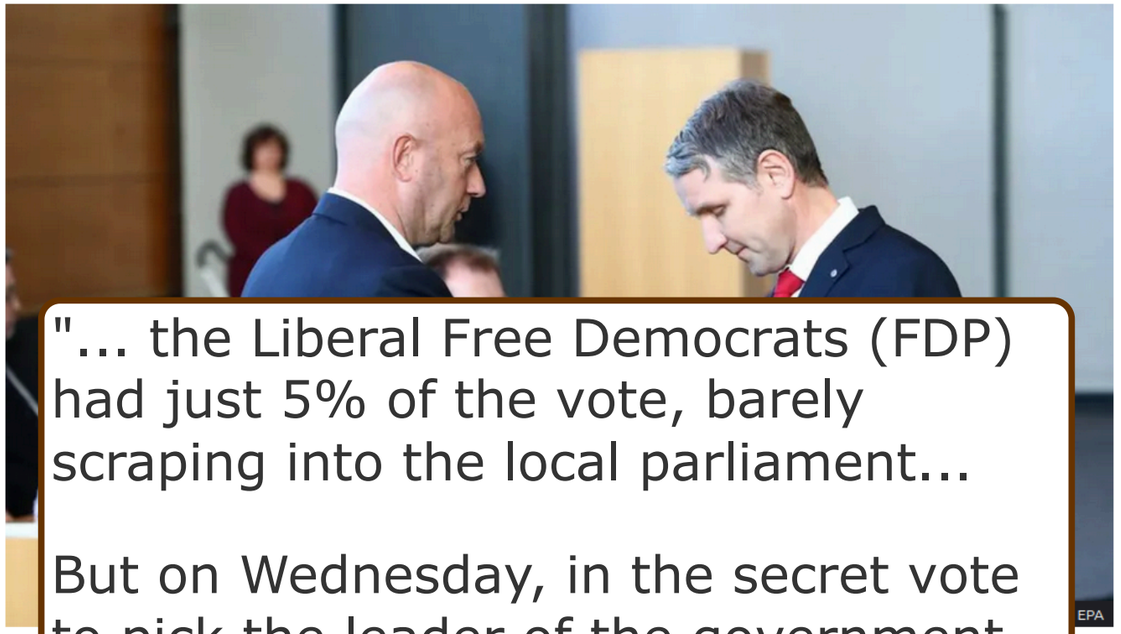
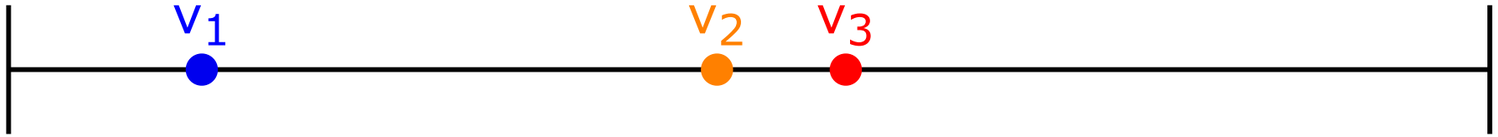


"... the Liberal Free Democrats (FDP) had just 5% of the vote, barely scraping into the local parliament...

But on Wednesday, in the secret vote to pick the leader of the government, Mr Kemmerich of the FDP beat Die Linke's leader Bodo Ramelow by 45 votes to 44 - thanks to votes from the AfD." - BBC News

Facility location

Now assume the set of candidates is the entire real line, and a voter's disutility from a point is their distance.



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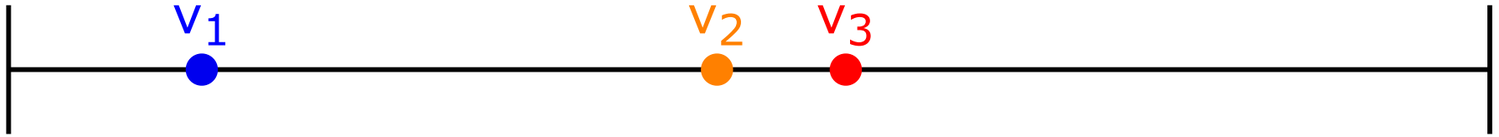

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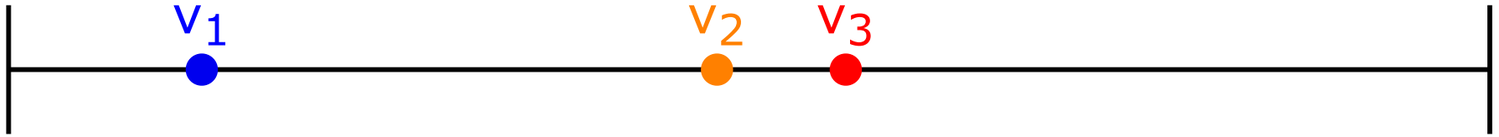

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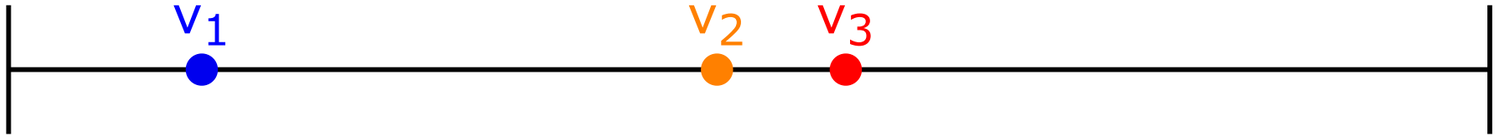

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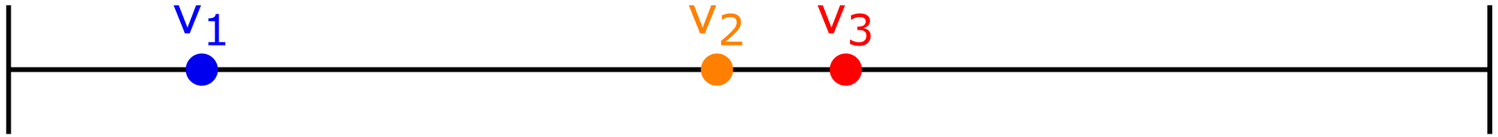

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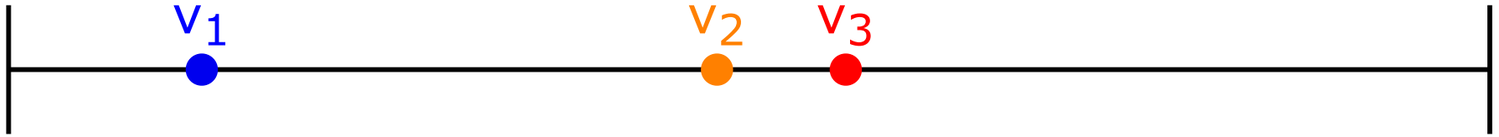

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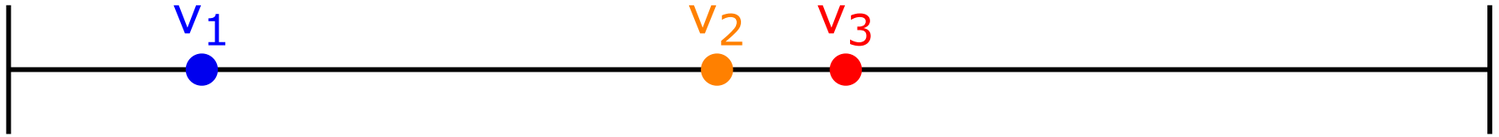

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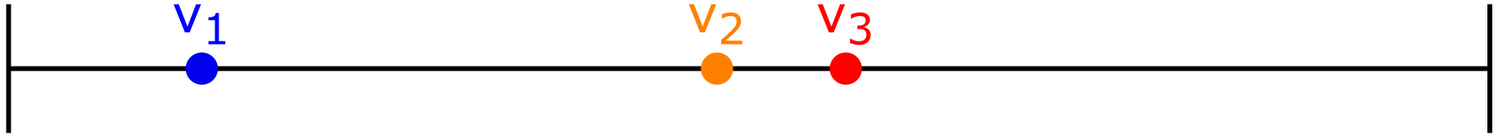

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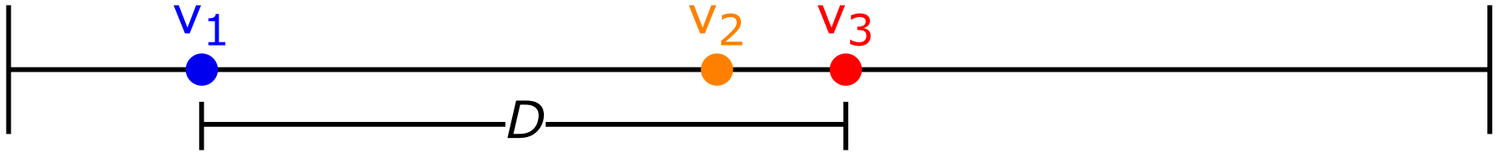

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OPT(v) = $D/2$
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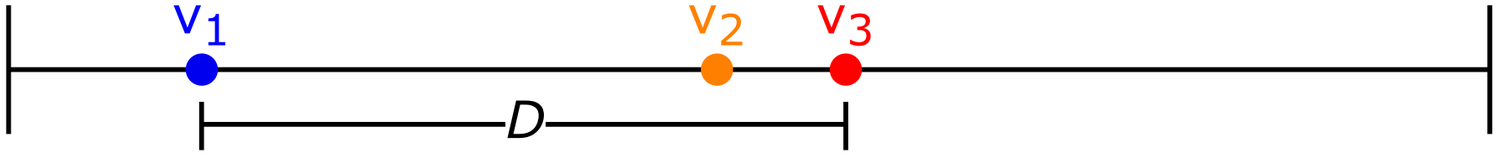

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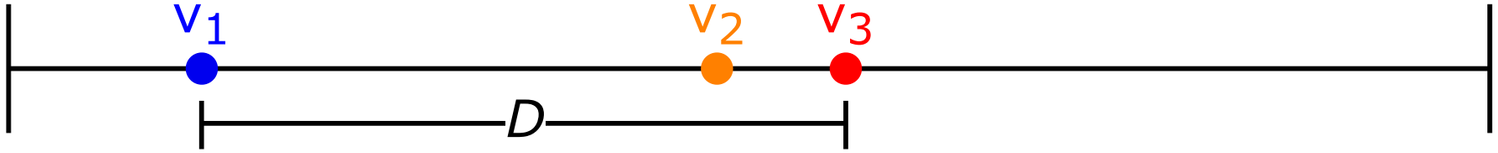

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In fact, any of Moulin's phantom mechanisms gets approximation ratio 2. Can we do better?

Optimal approximation algorithms for the min-max distance objective

Theorem (Procaccia, Tennenholtz, 2013)

If there are at least 2 agents, any deterministic SP facility location rule achieves a worst case approximation ratio of 2.

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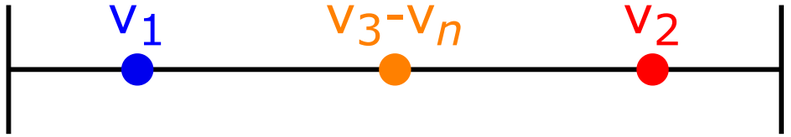

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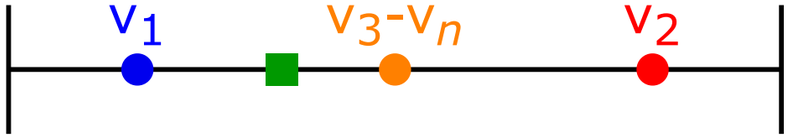

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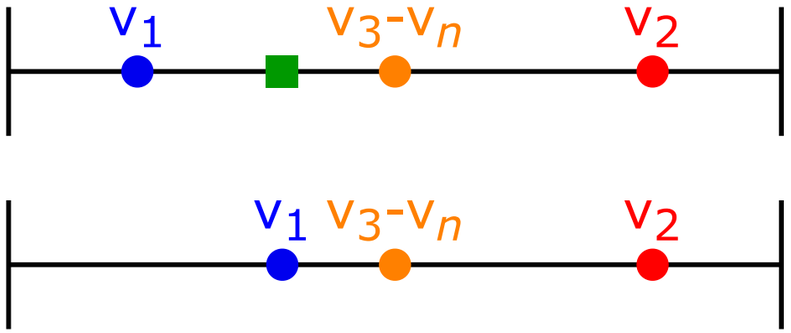

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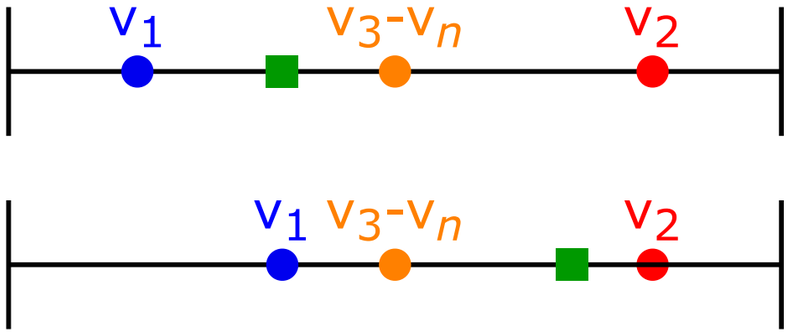

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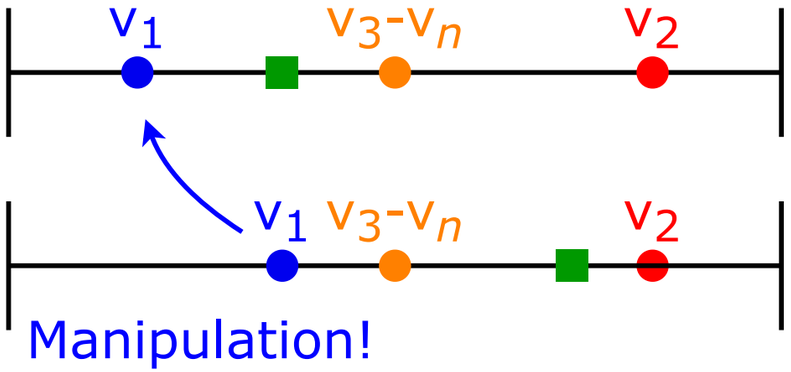

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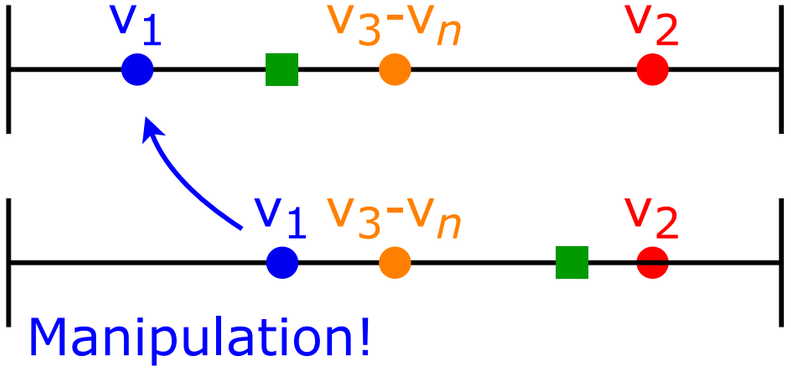

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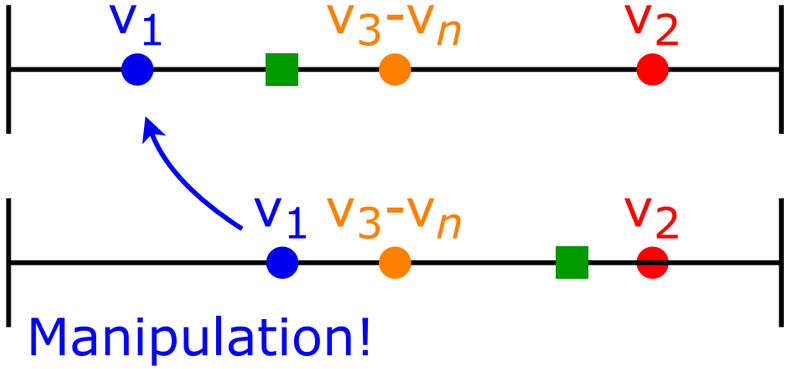

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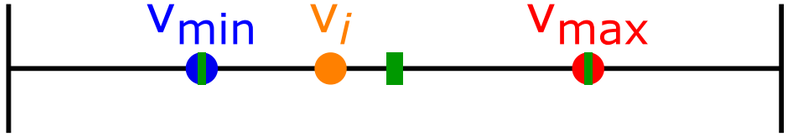

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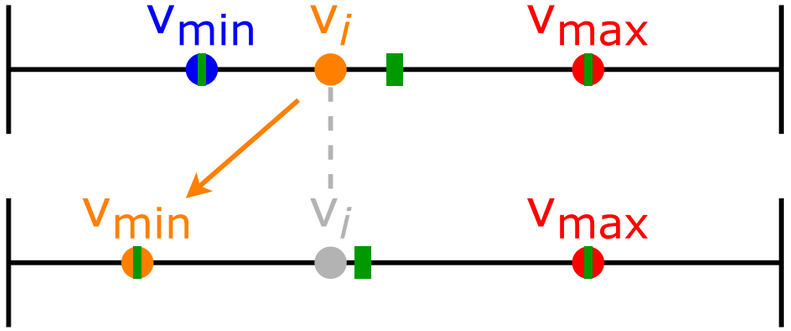

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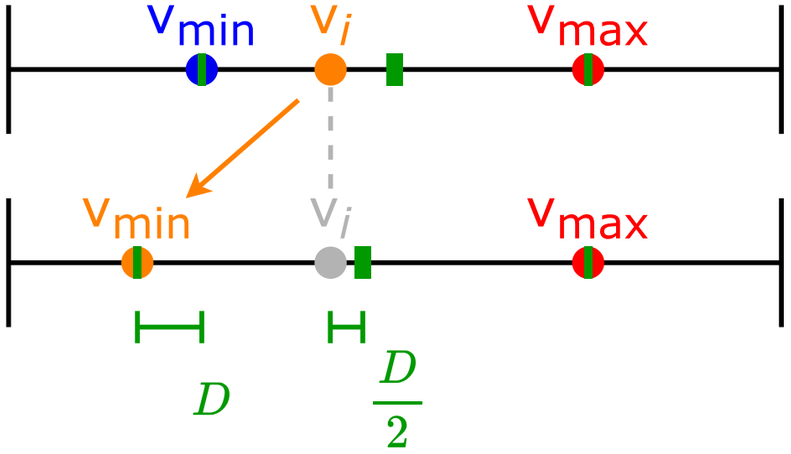

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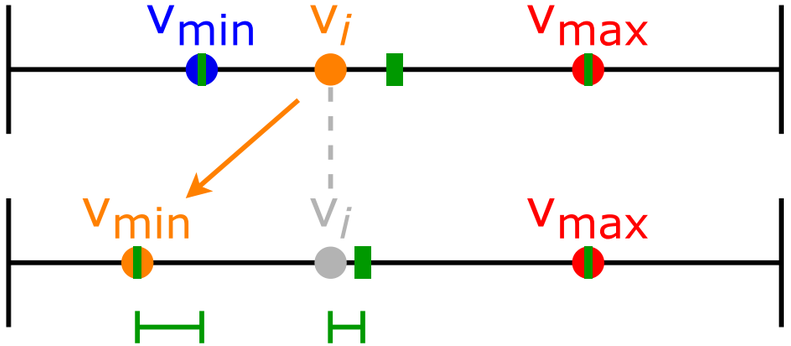

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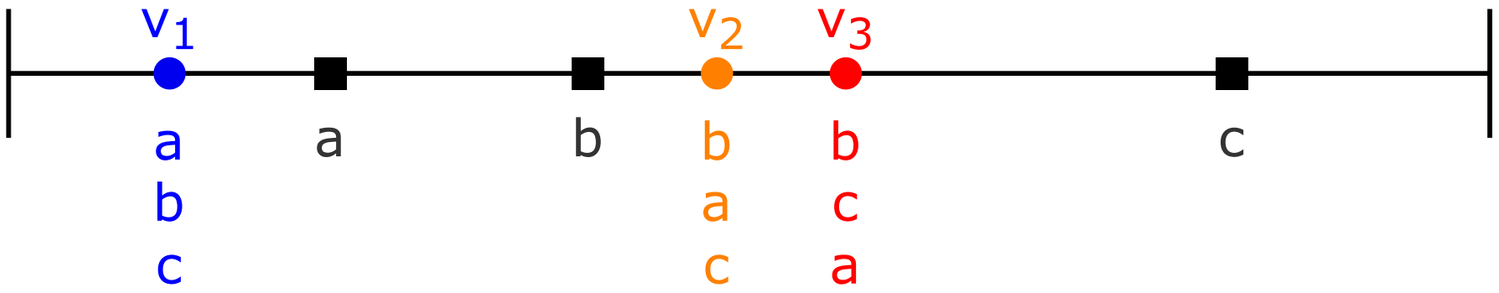

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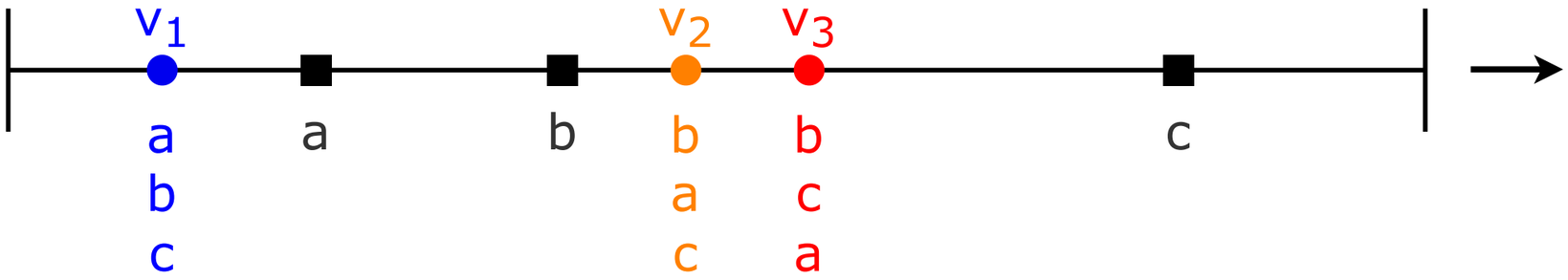

1	1	1
a	b	b
b	a	c
c	c	a

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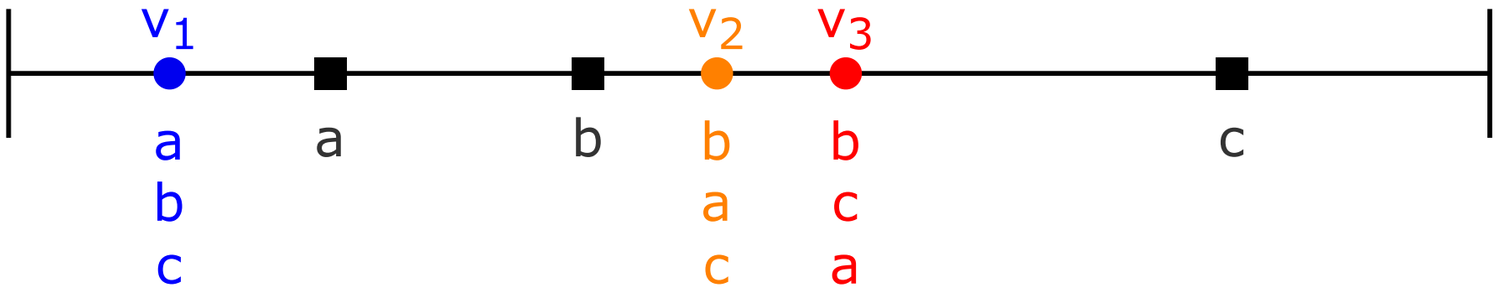

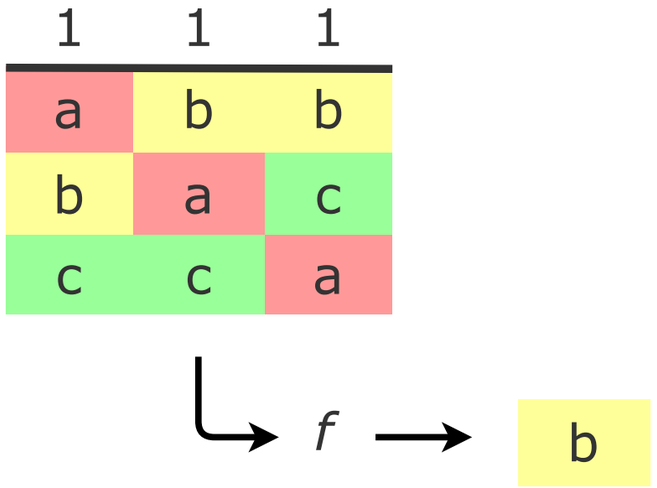

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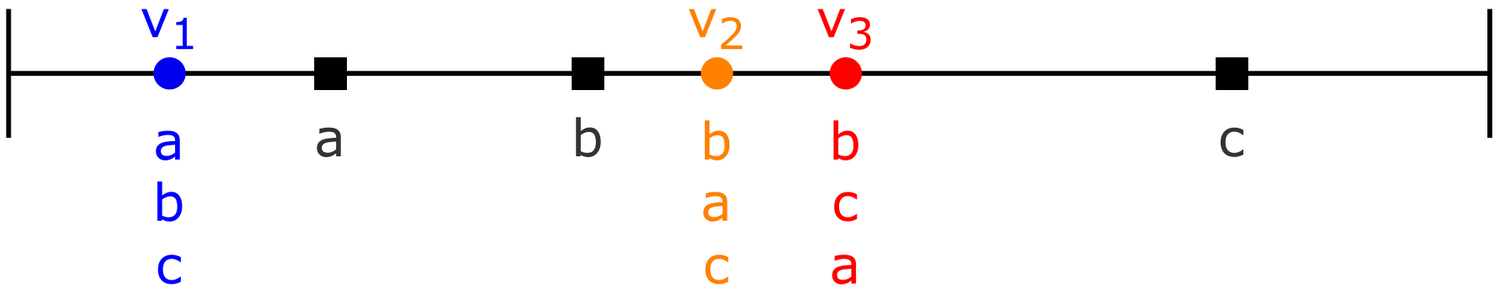

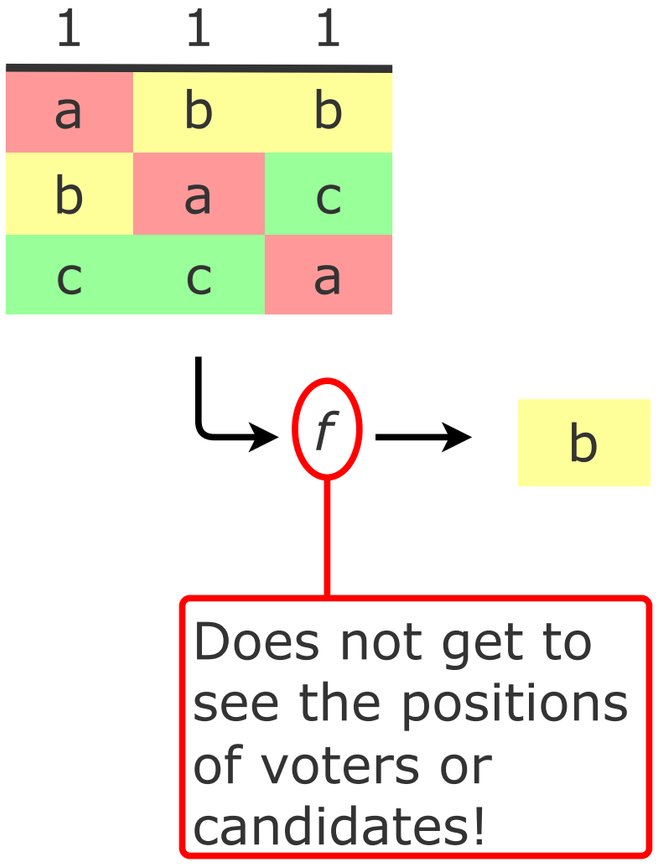

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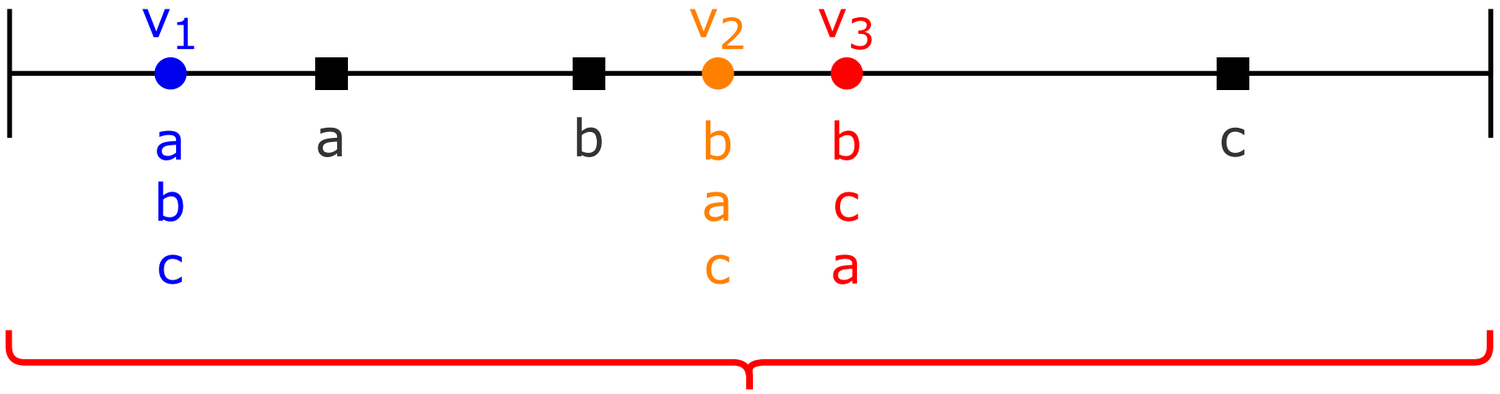

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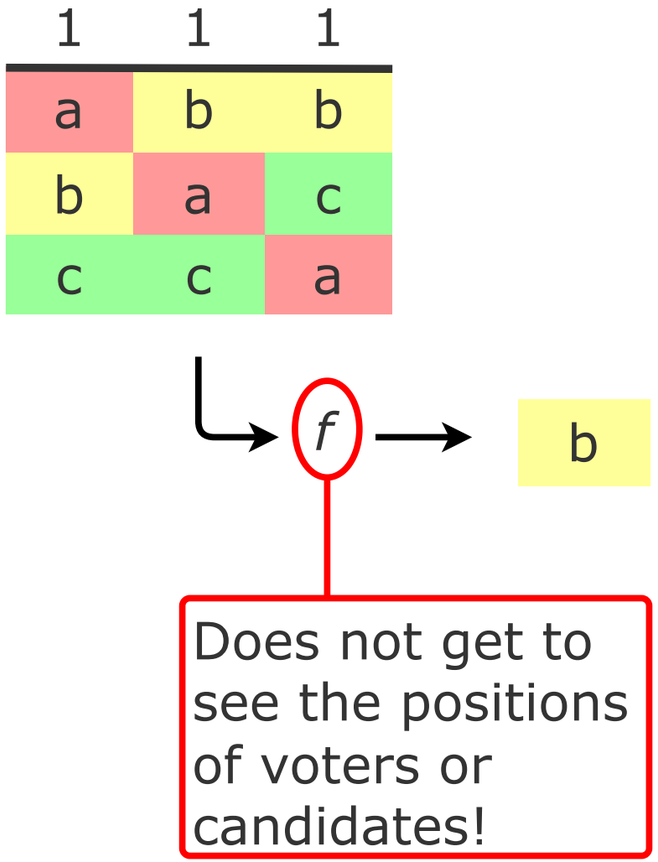

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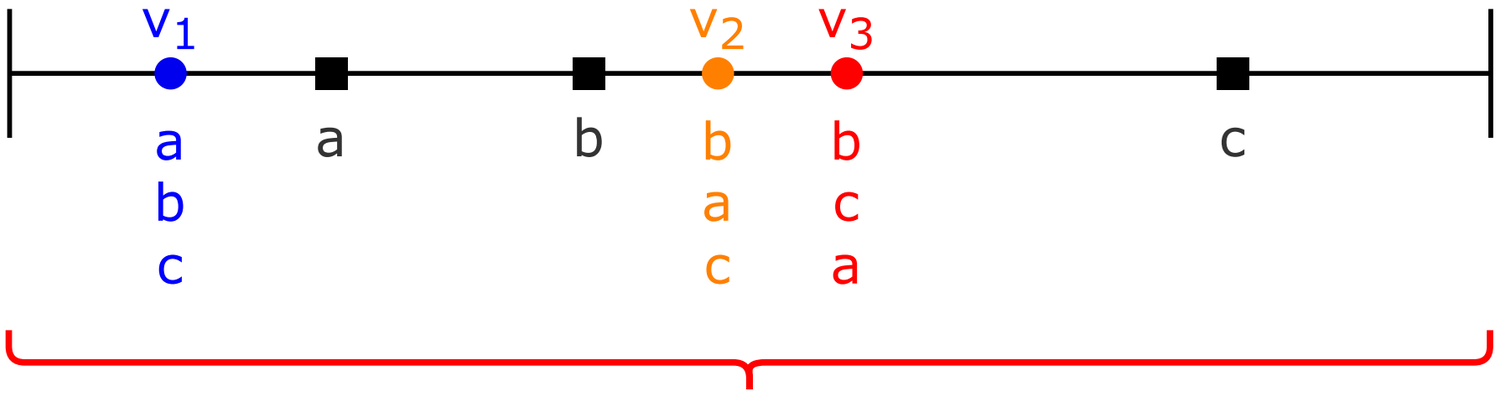

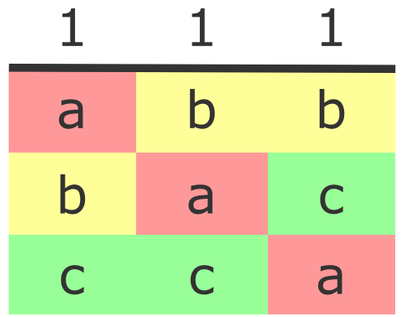

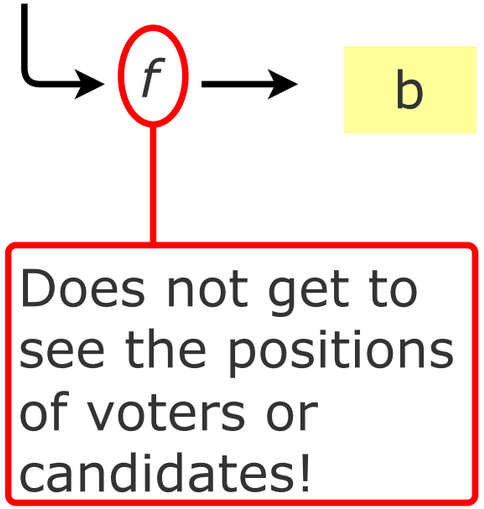

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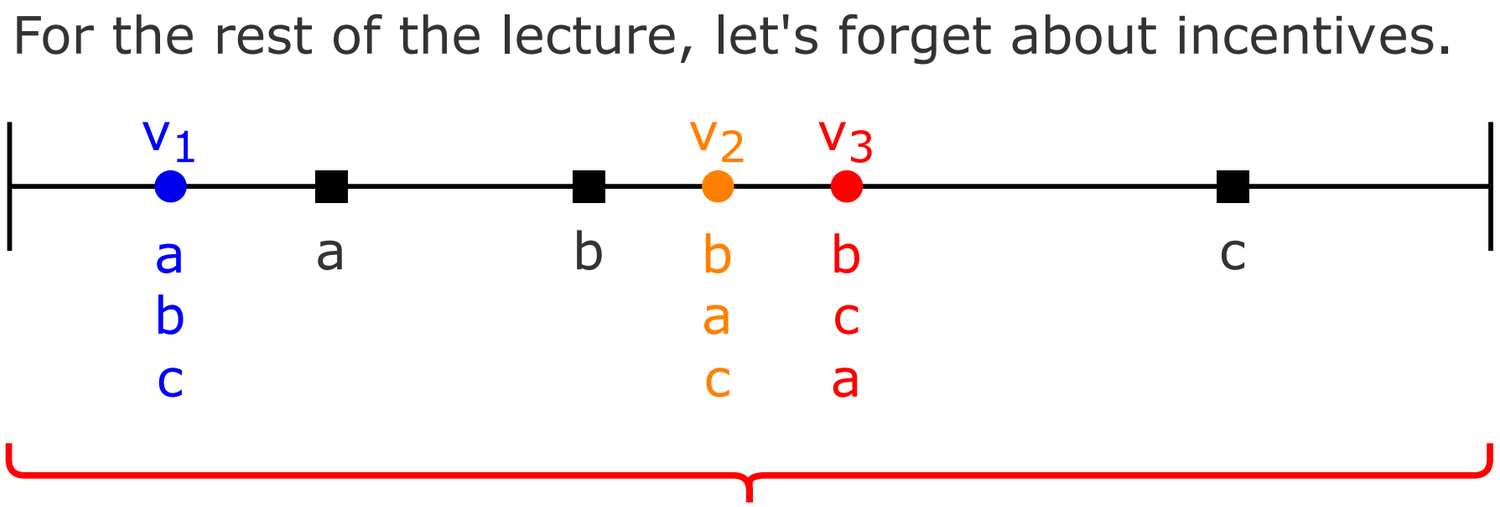

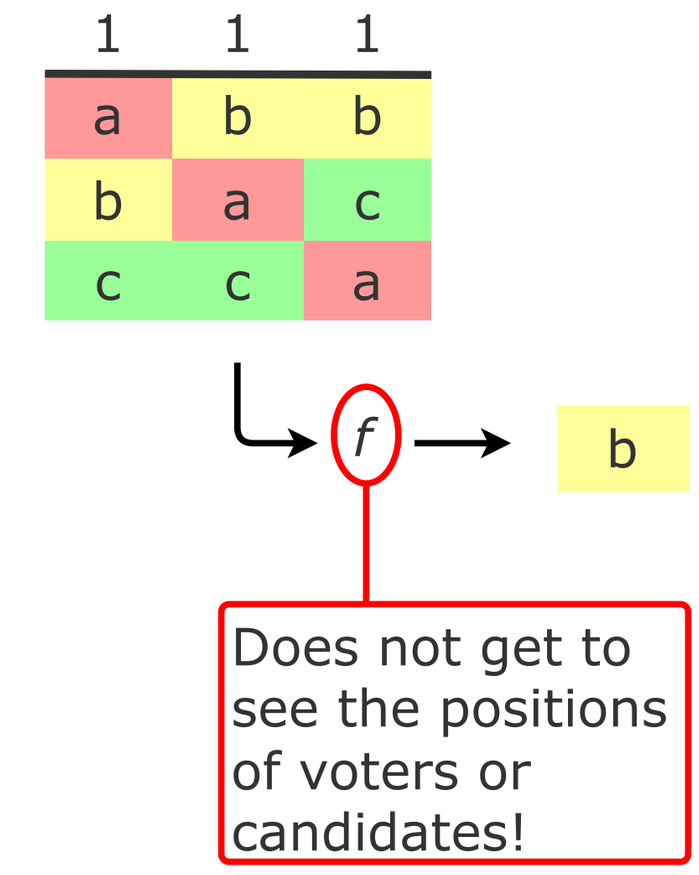

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The *distortion* of a (deterministic) voting rule f is

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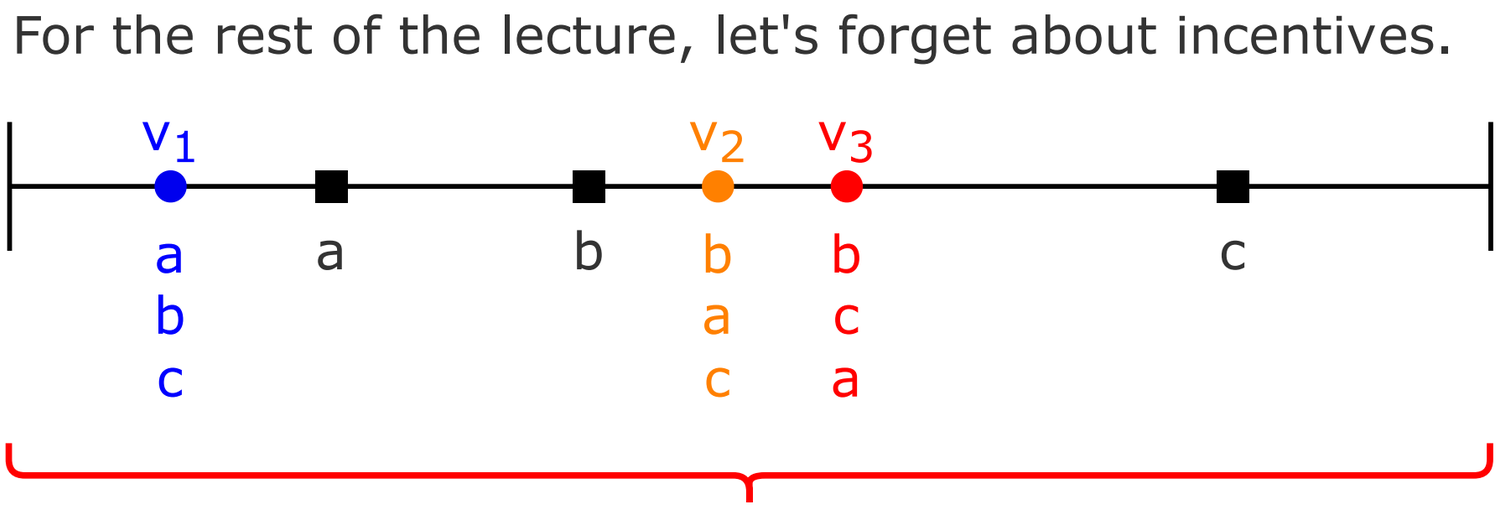

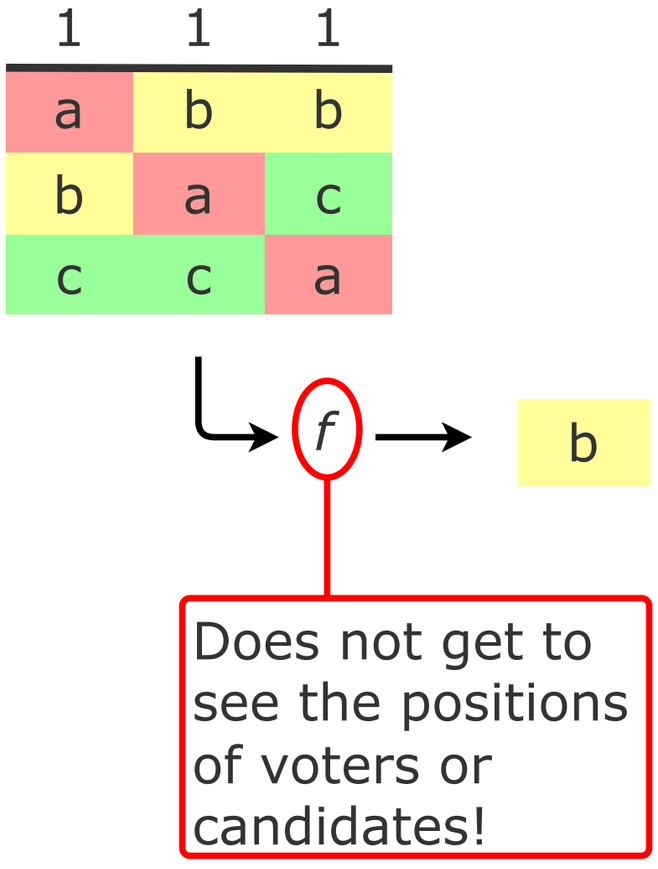

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The *distortion* of a randomized voting rule f is

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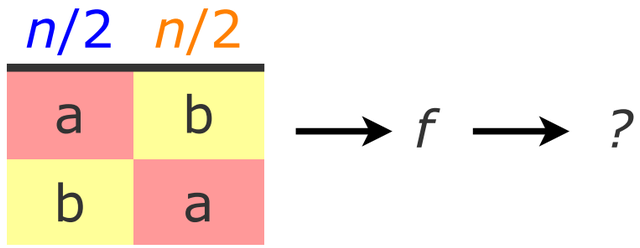
Lower bounds



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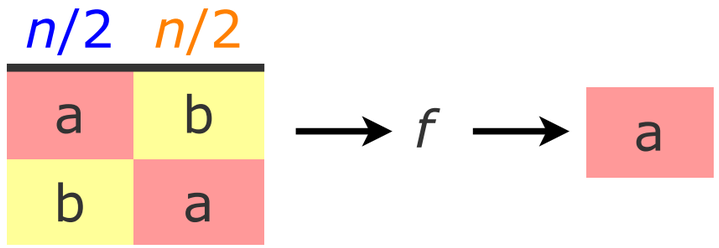

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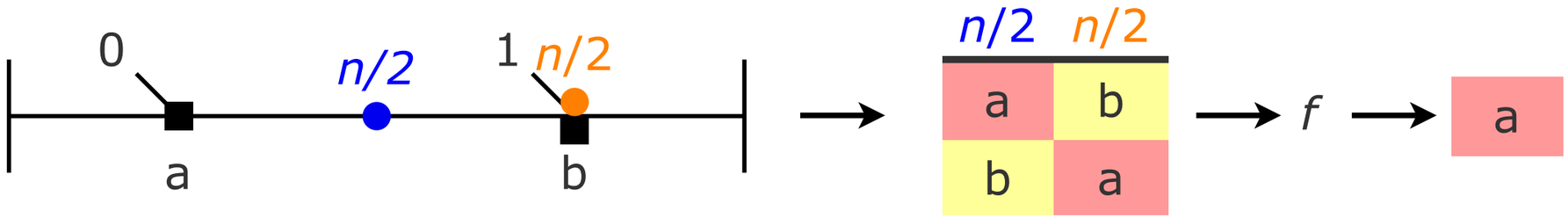

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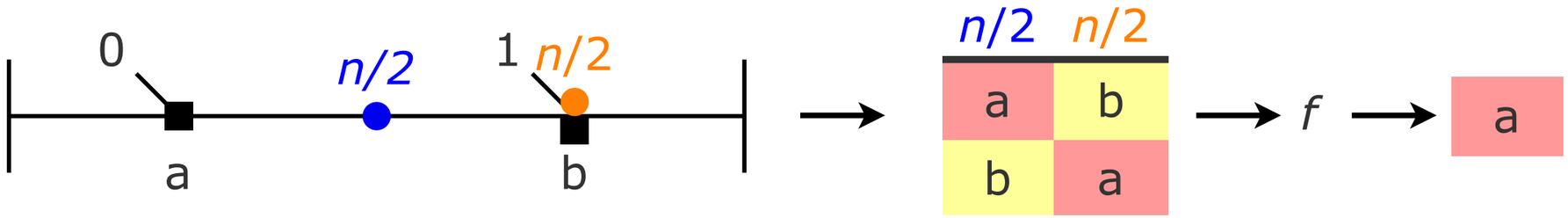

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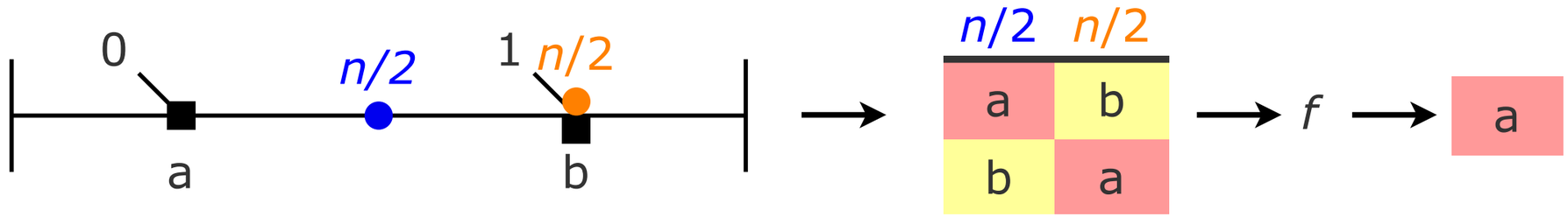

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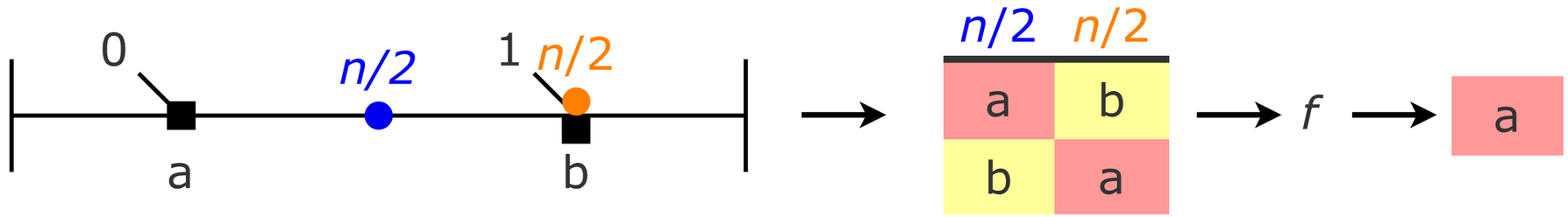

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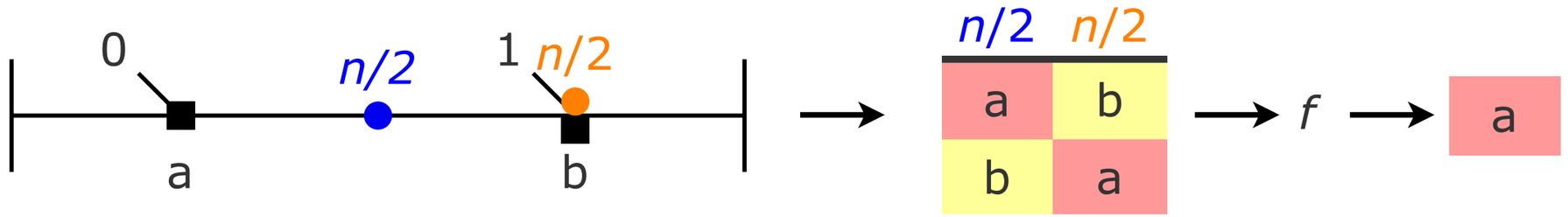

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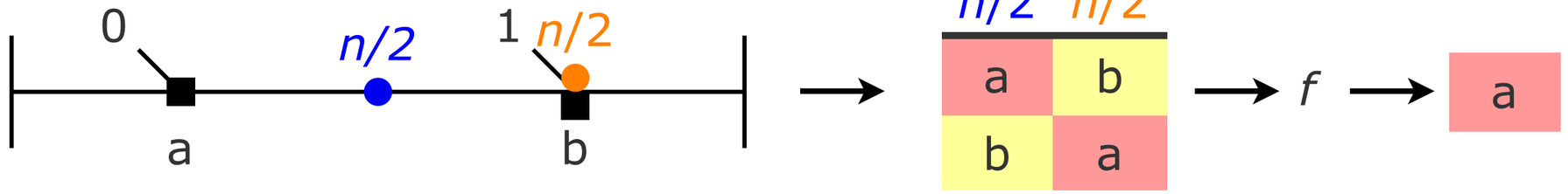

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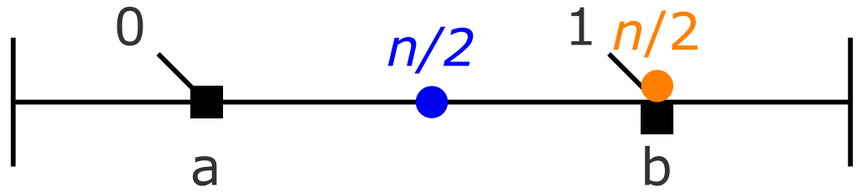

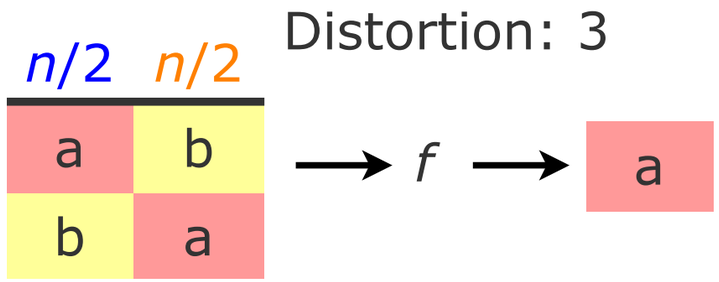

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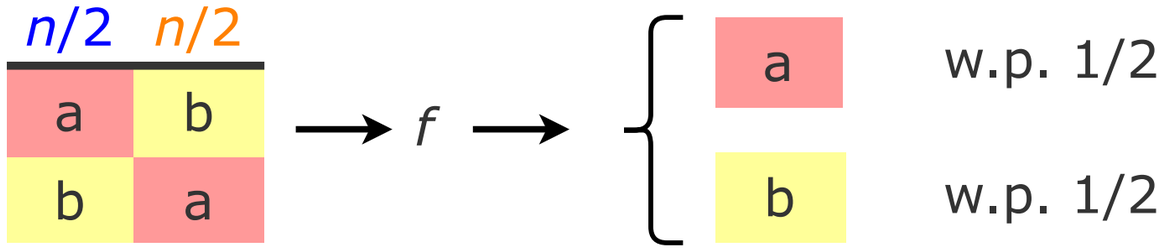

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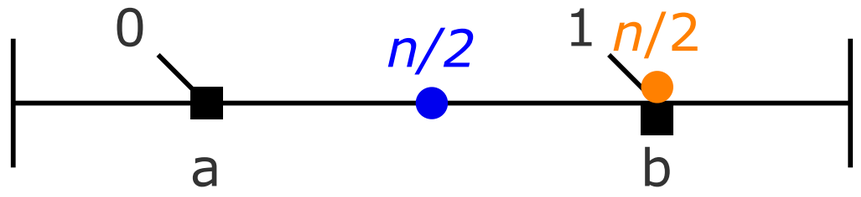

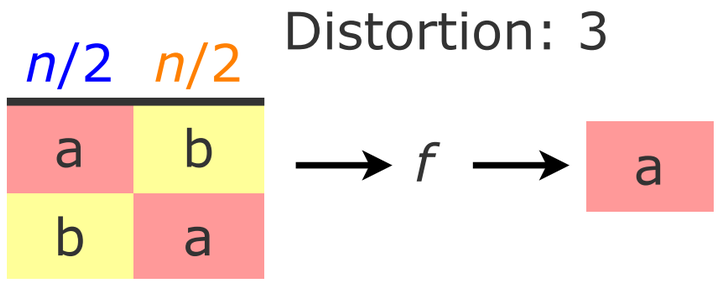

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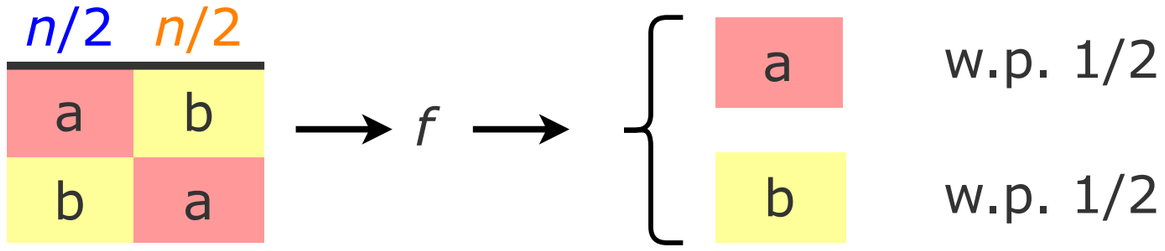

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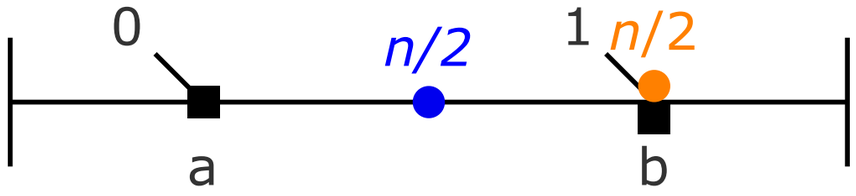

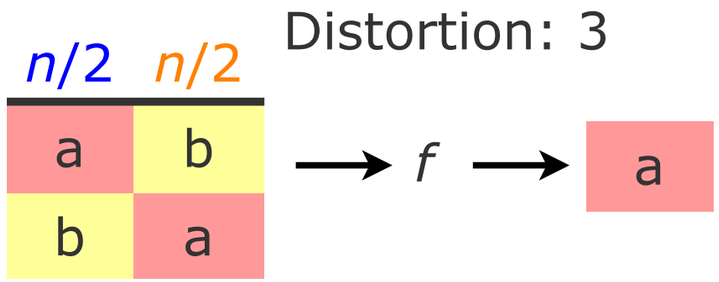

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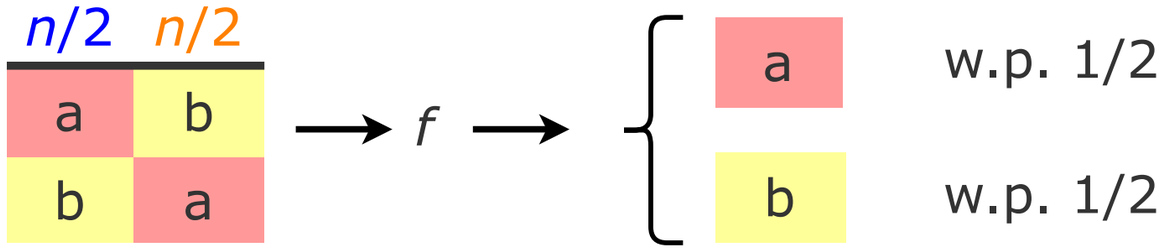

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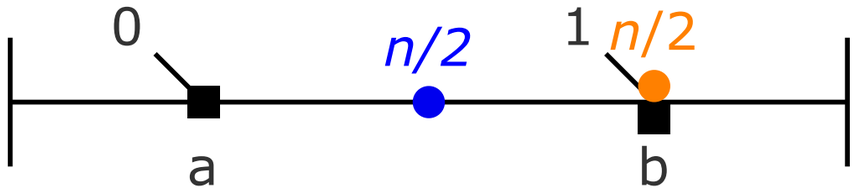

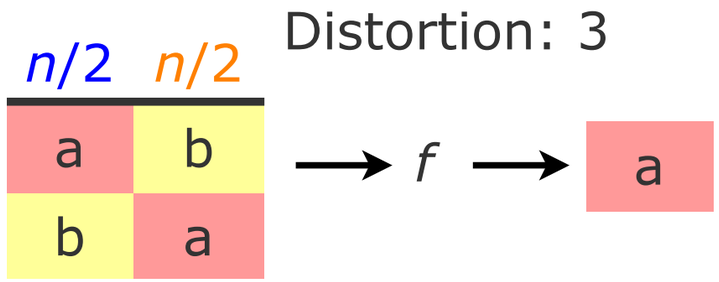

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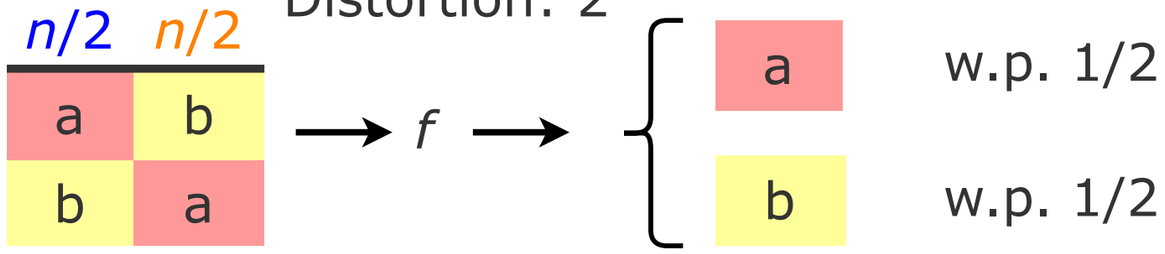
$$\sum_{i=1}^{n} d(i, f(p)) = \frac{n}{2} \cdot \frac{1}{2} + \frac{n}{2} \cdot 1 = \frac{3n}{4}$$

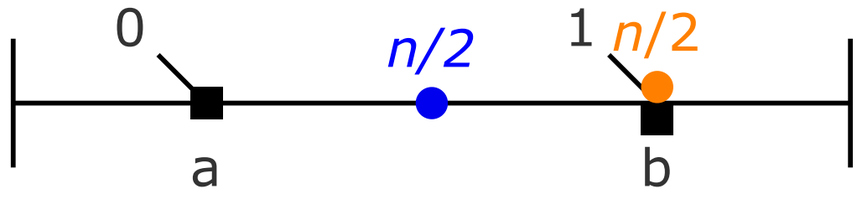
$$\min_{c \in [m]} \sum_{i=1}^{n} d(i, c) = \frac{n}{2} \cdot \frac{1}{2} = \frac{n}{4}$$



Distortion: 2

$$\mathbb{E}\left[\sum_{i=1}^{n} d(i, f(p))\right] = \frac{1}{2} \cdot \frac{3n}{4} + \frac{1}{2} \cdot \frac{n}{4} = \frac{n}{2}$$

Lower bounds



n/2	n/2
a	b
b	a

Distortion: 3

$\rightarrow f \rightarrow$ a

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Theorem

Any deterministic voting rule has metric distortion ≥3, and any randomized rule has distortion ≥2.

n/2	n/2
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Distortion: 2

$\rightarrow f \rightarrow$ { a w.p. 1/2; b w.p. 1/2

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Plurality, Borda, IRV all have unbounded distortion!

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Copeland's Rule has metric distortion at most 9.

Proof. This follows from the following two claims:

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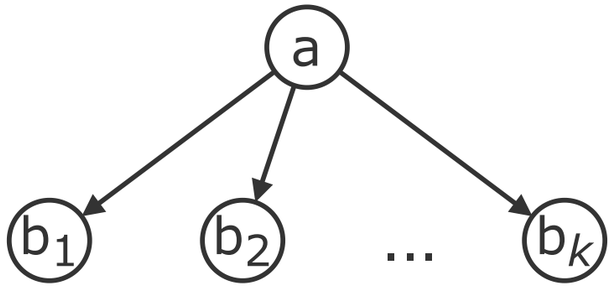

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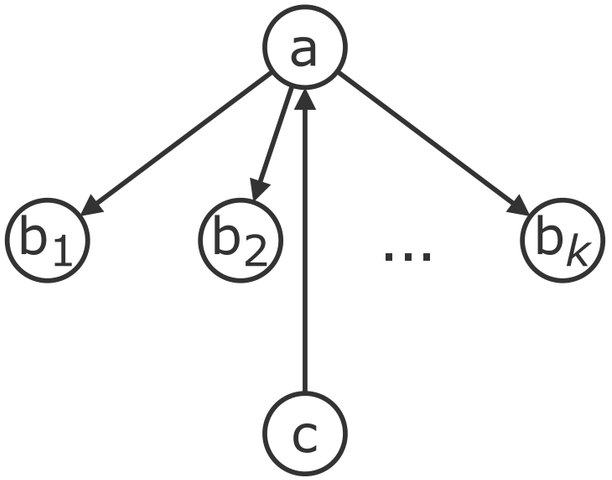

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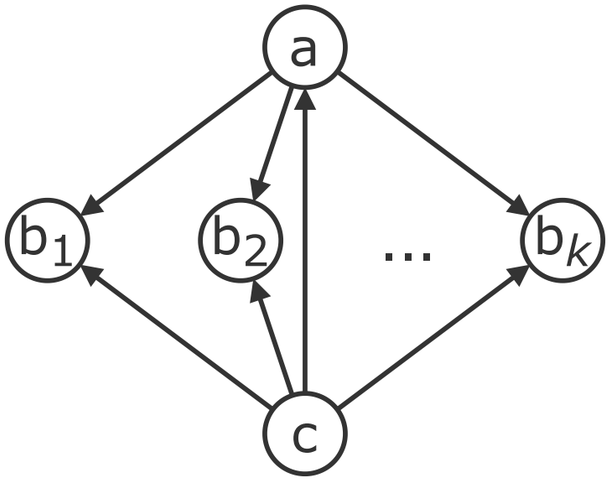

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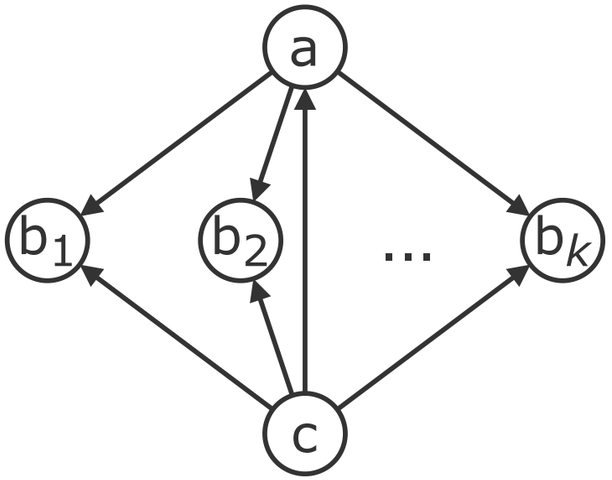

Then c should have been the Copeland winner, not a!

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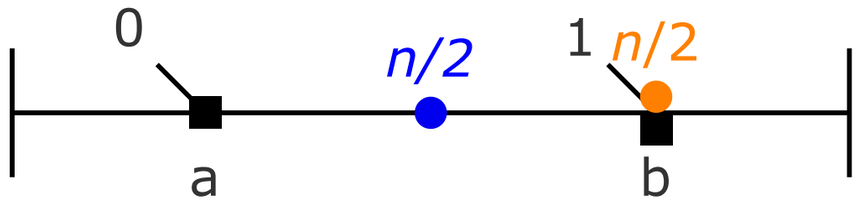

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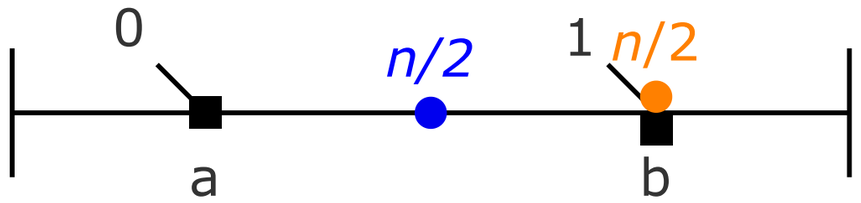

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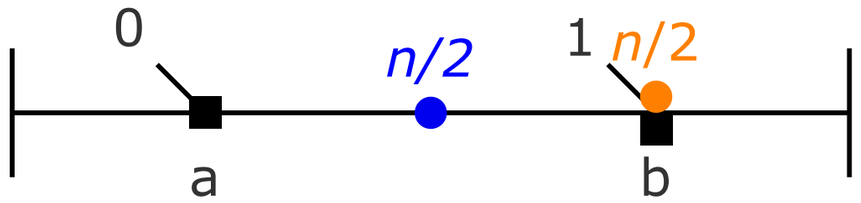

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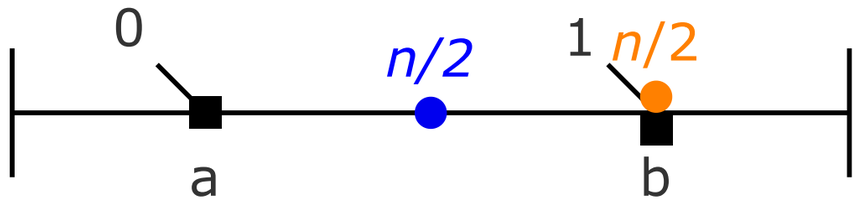

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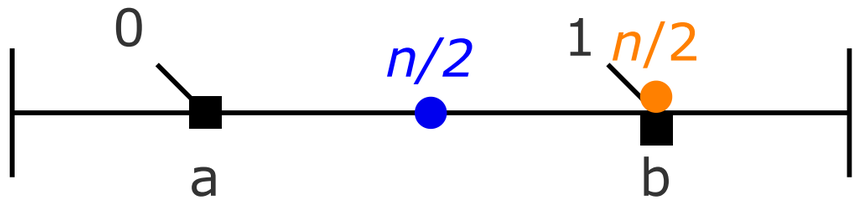

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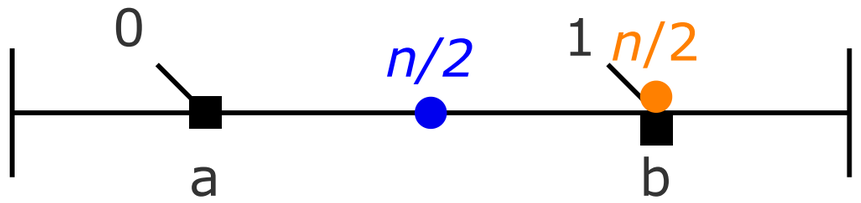

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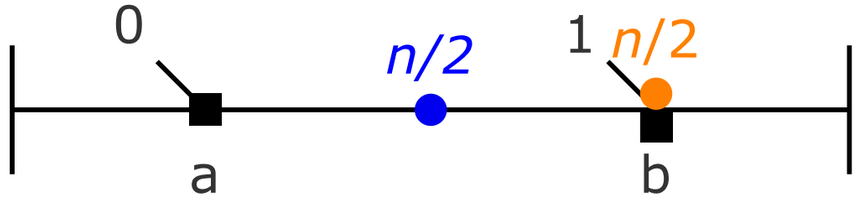

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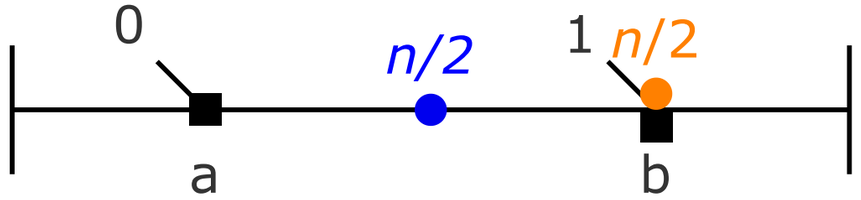

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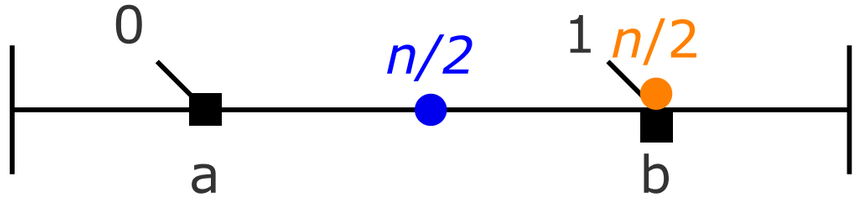

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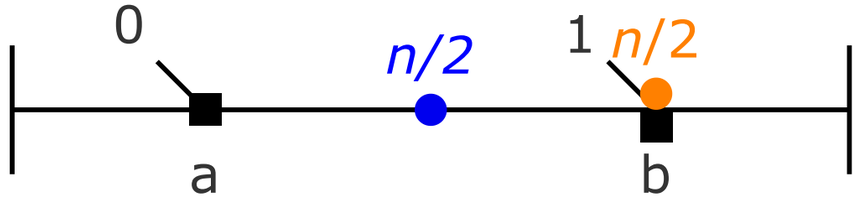

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Theorem (Anshelevich, Bhardwaj, Elkind, Postl, Skowron, 2018)

*Copeland's Rule has metric distortion **exactly 5**.*

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Deterministic	3	3
Randomized	2	?

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Open Question: What is the optimal distortion of any randomized voting rule?

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Main ingredient of proof is the following game between Alice and Bob. Both players look at all of the votes cast, and simultaneously choose a candidate. Then a random voter arrives. Whoever's candidate that voter prefers wins the game.

Maximal Lotteries: A randomized voting rule, where the probability that candidate a is selected is the probability that Alice picks a at equilibrium.

This also has distortion 3, but can be brought down to distortion 2.753 by mixing with other rules.