

# Algorithms For Democratic Decision-Making

Jamie Tucker-Foltz • Yale University • Spring 2026

Lecture 8: **Multi-Winner Voting**

# Introduction

More common ballot format for many candidates: *approval preferences*.

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What if we need to select a committee of size 4? Tradeoff between utility and **fairness**.

# Visualization of approval sets

Candidates:

1

2

3

4

5

6

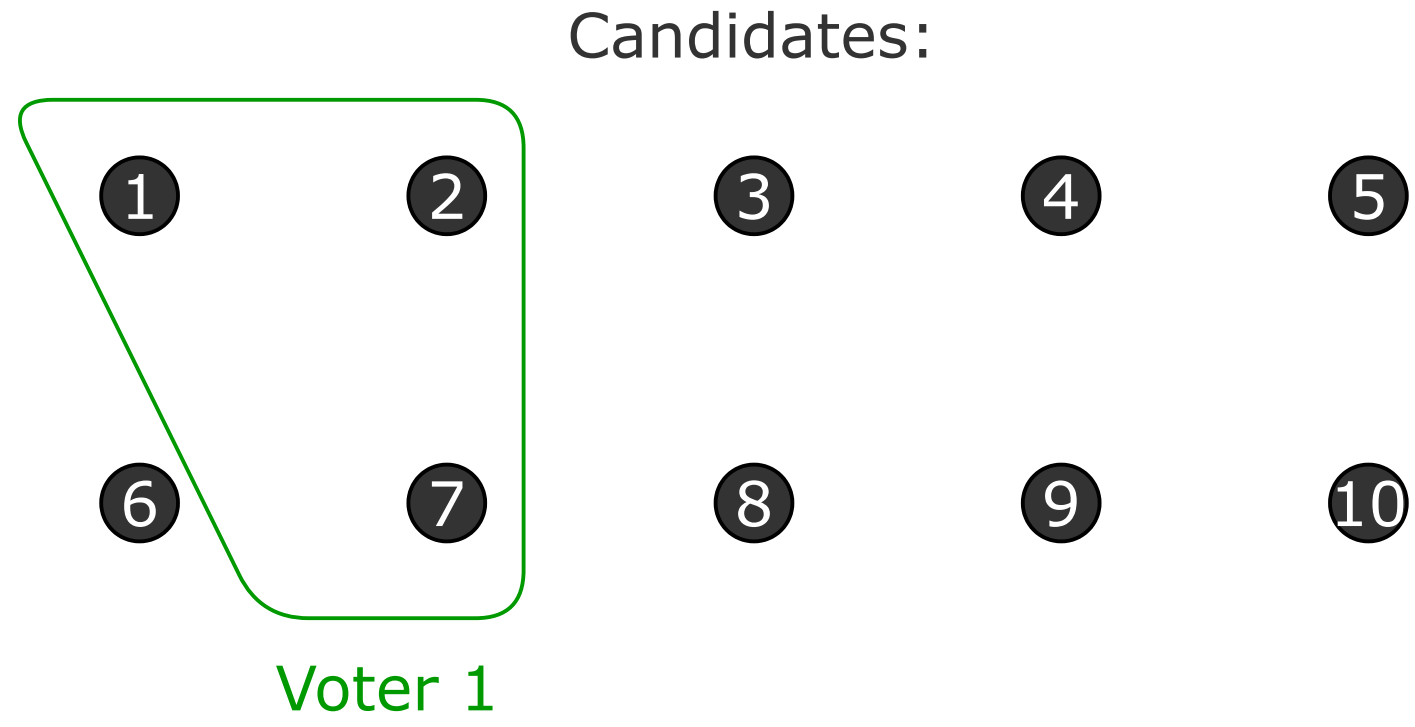
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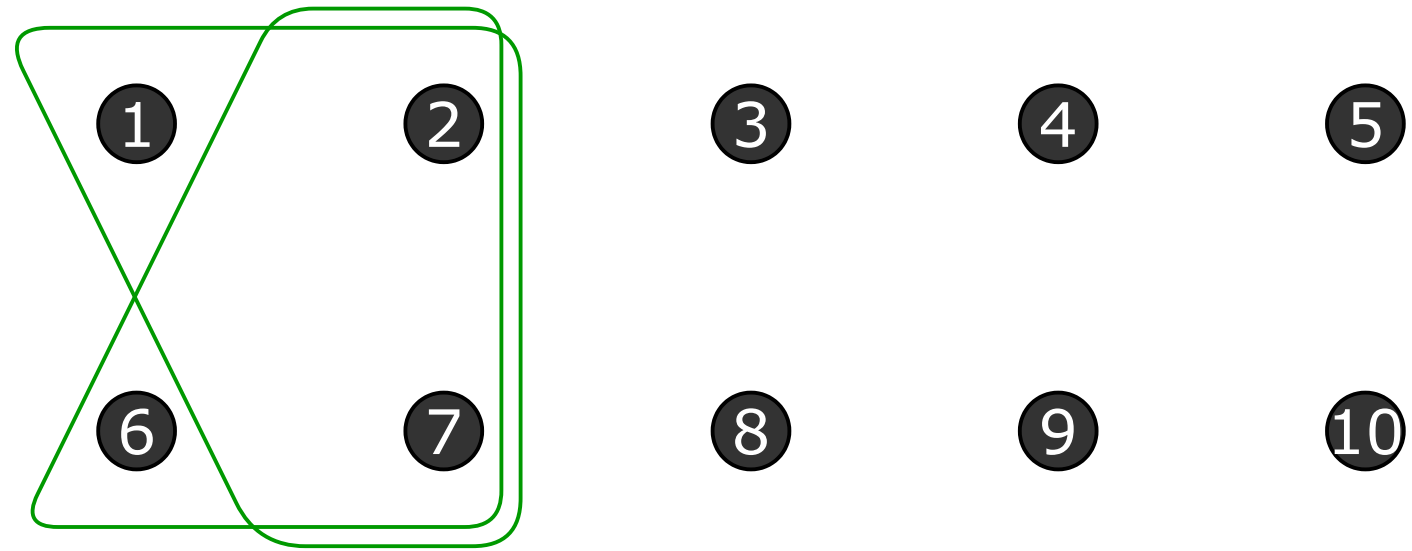
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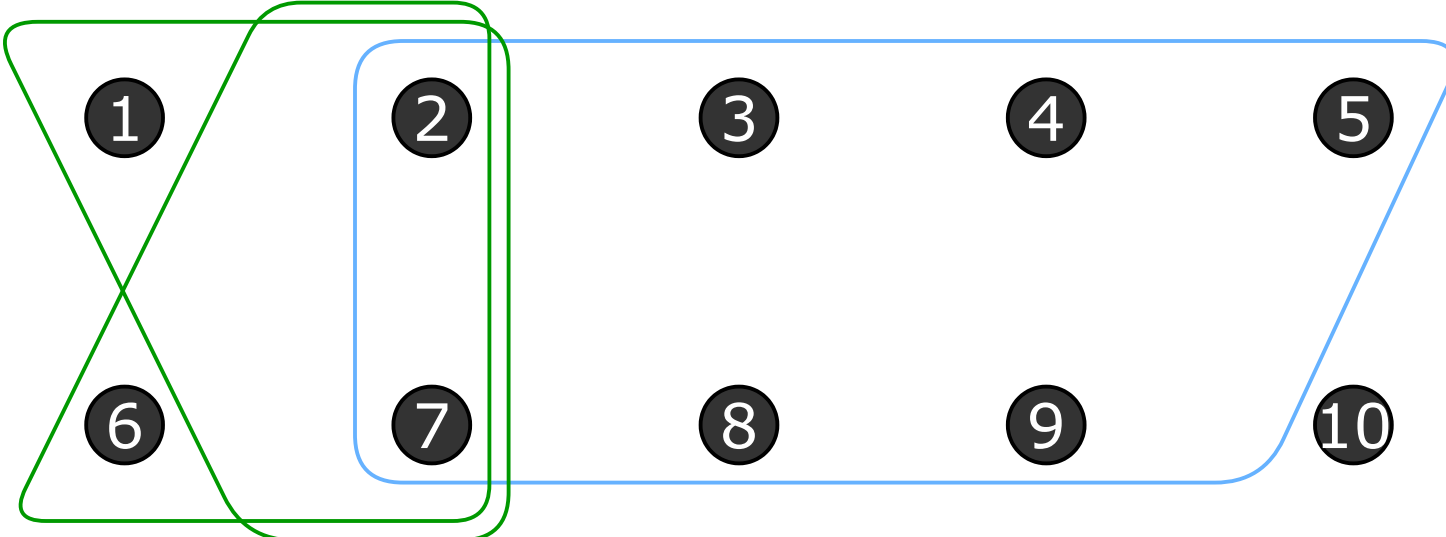
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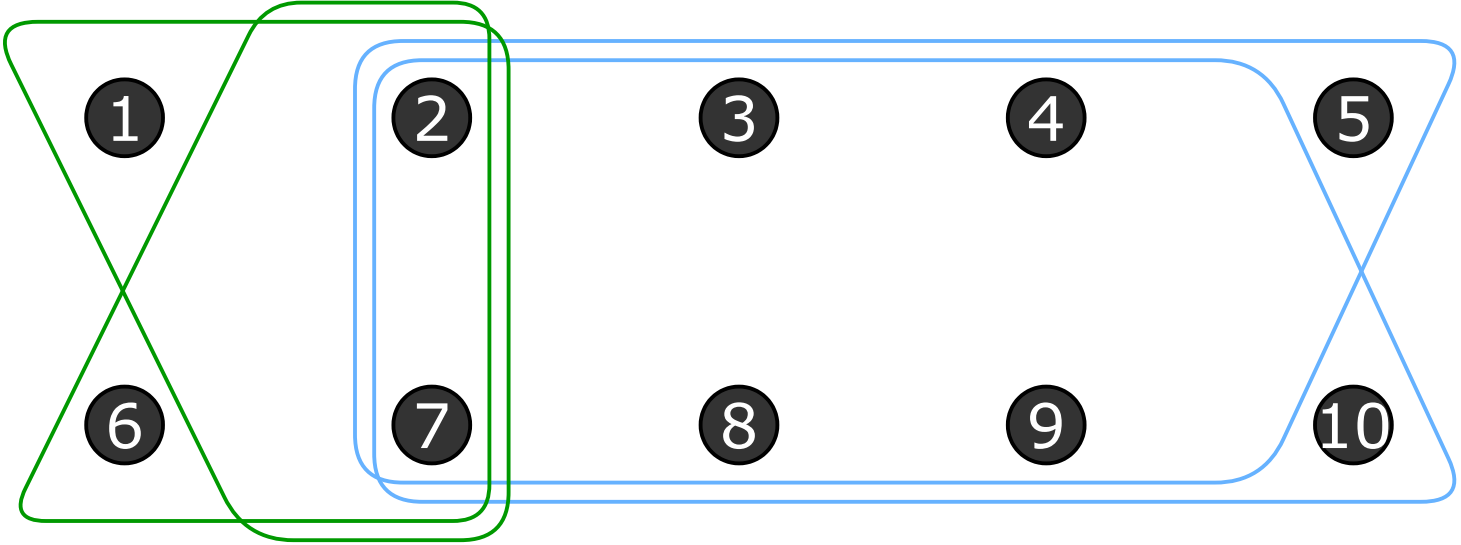
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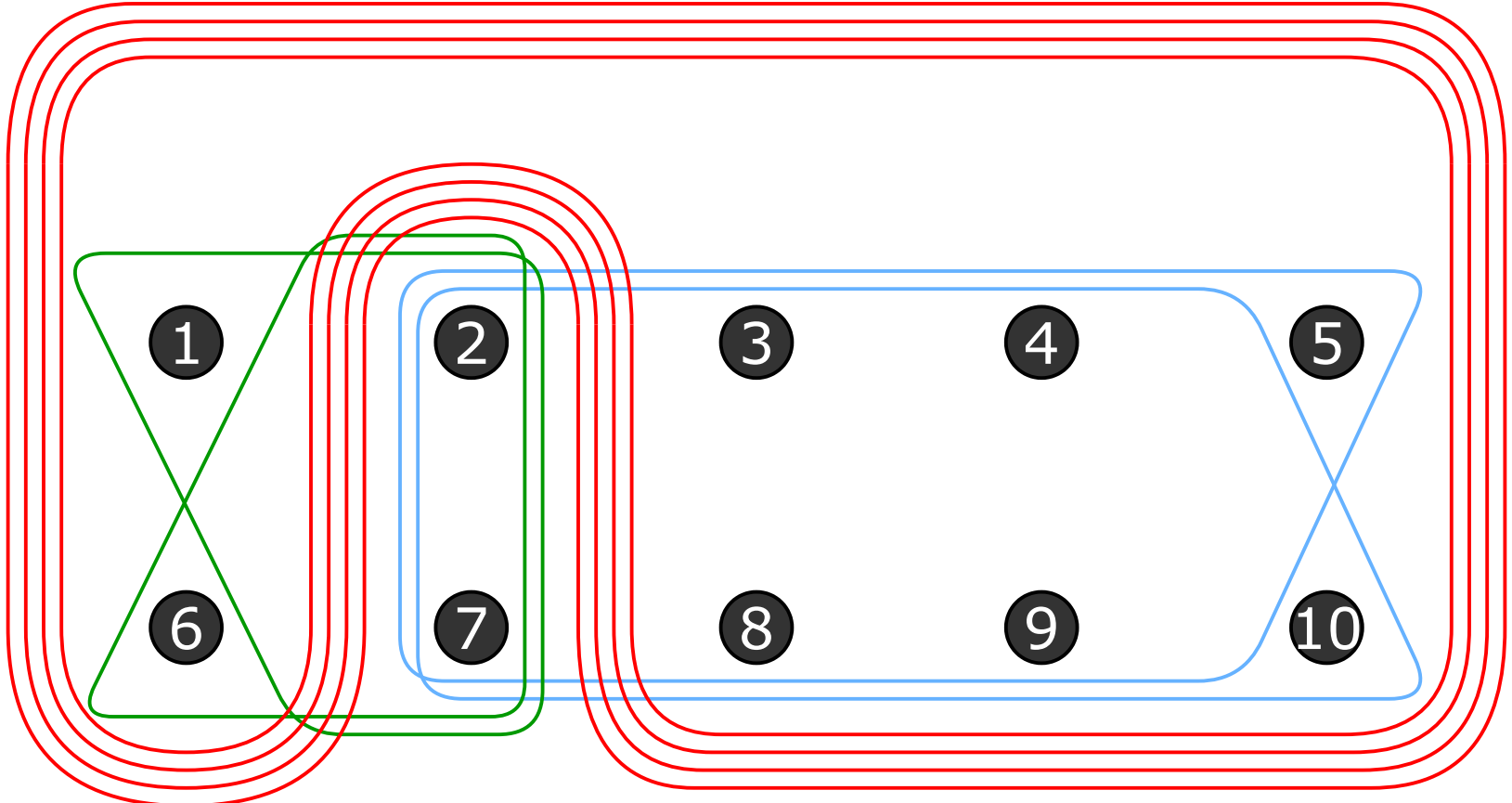
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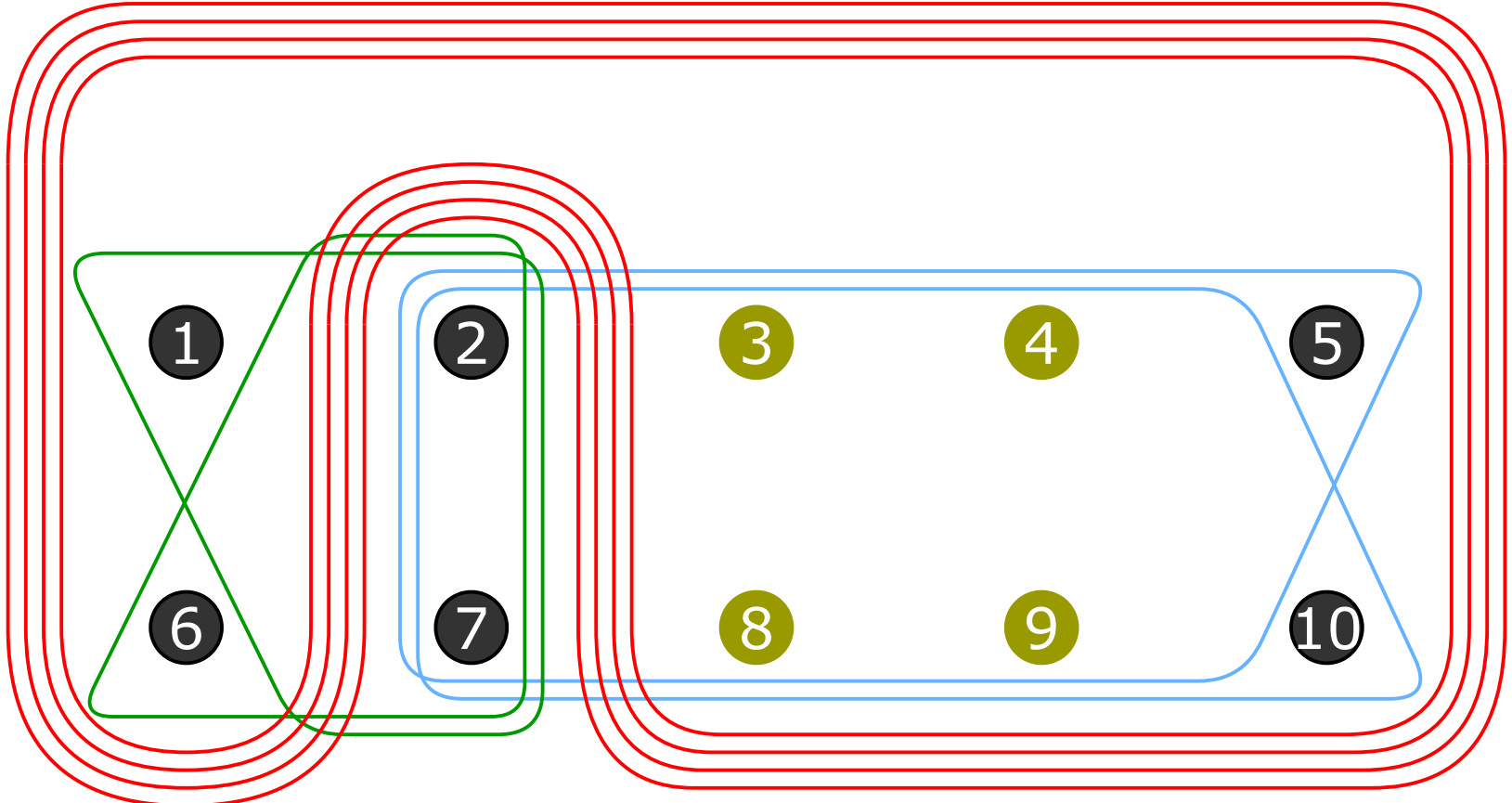
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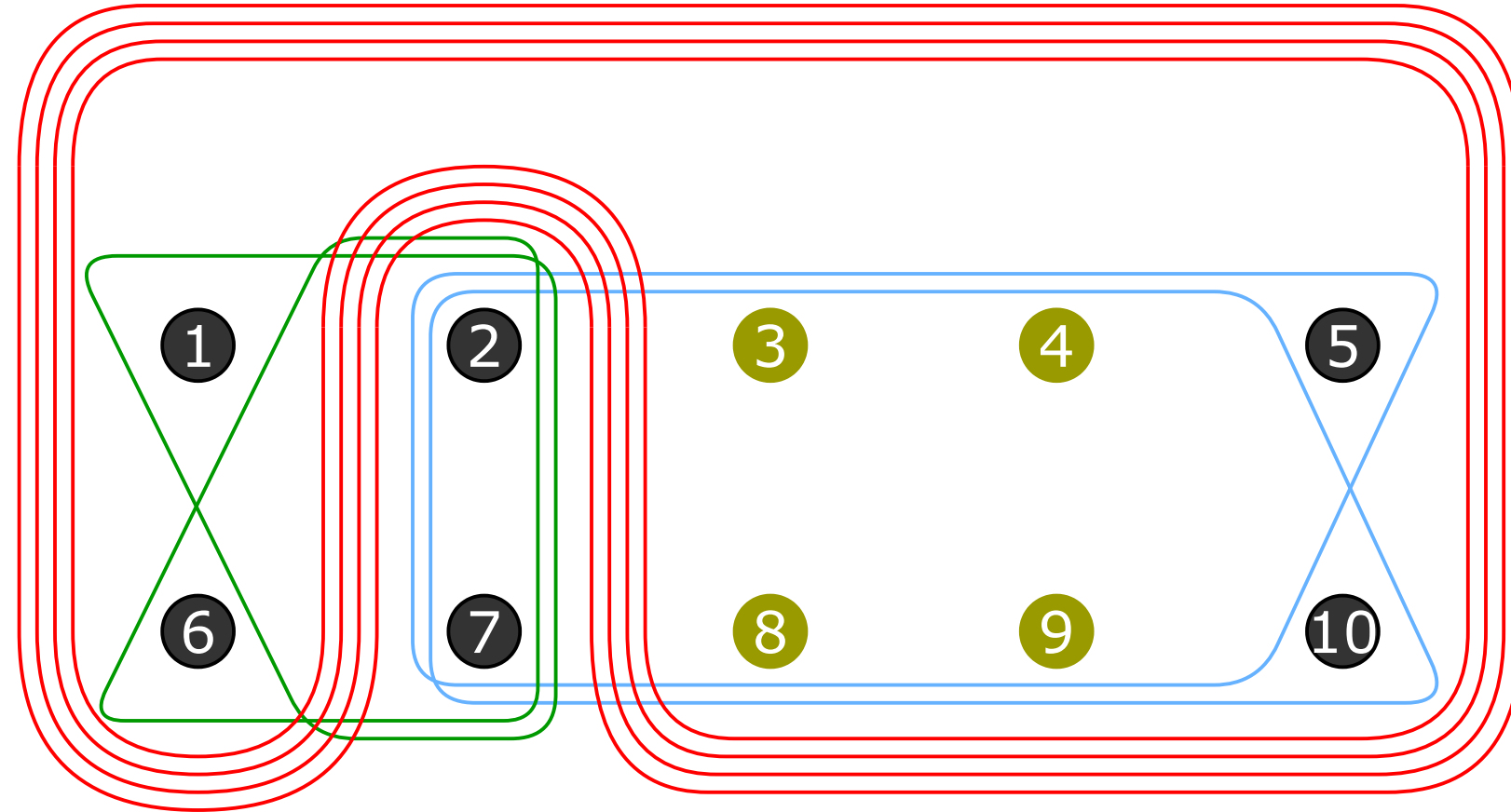


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Big idea behind committee selection fairness notions: **Voters 1 and 2** deserve 1 of the  $k$  candidates because they constitute  $1/k$  of the population.

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Respond at:

[pollev.com/jtuckerfoltz255](https://pollev.com/jtuckerfoltz255) or

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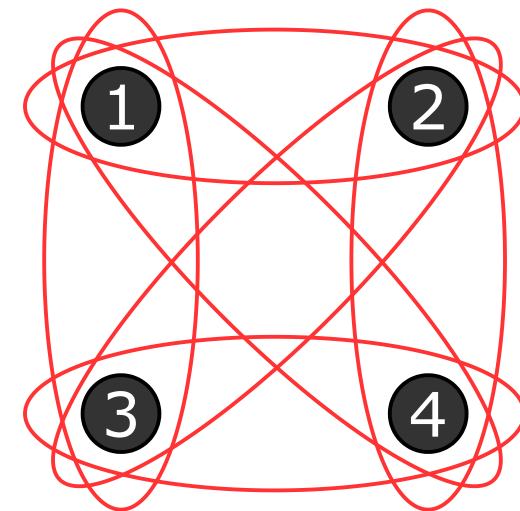
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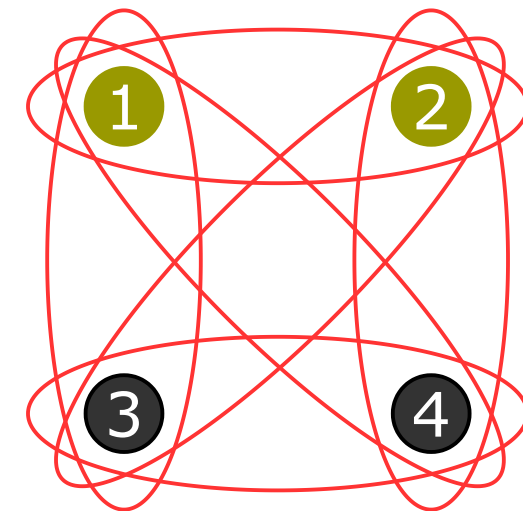
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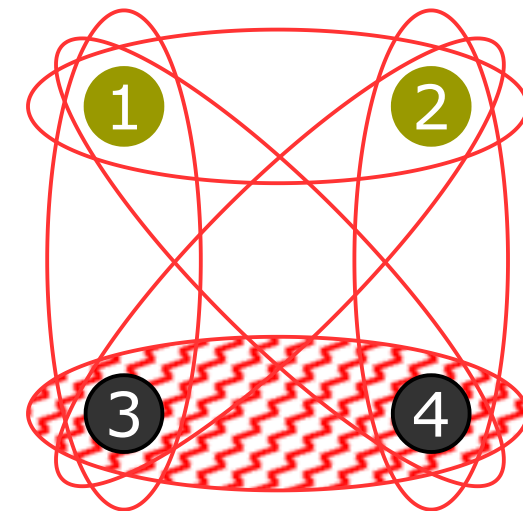
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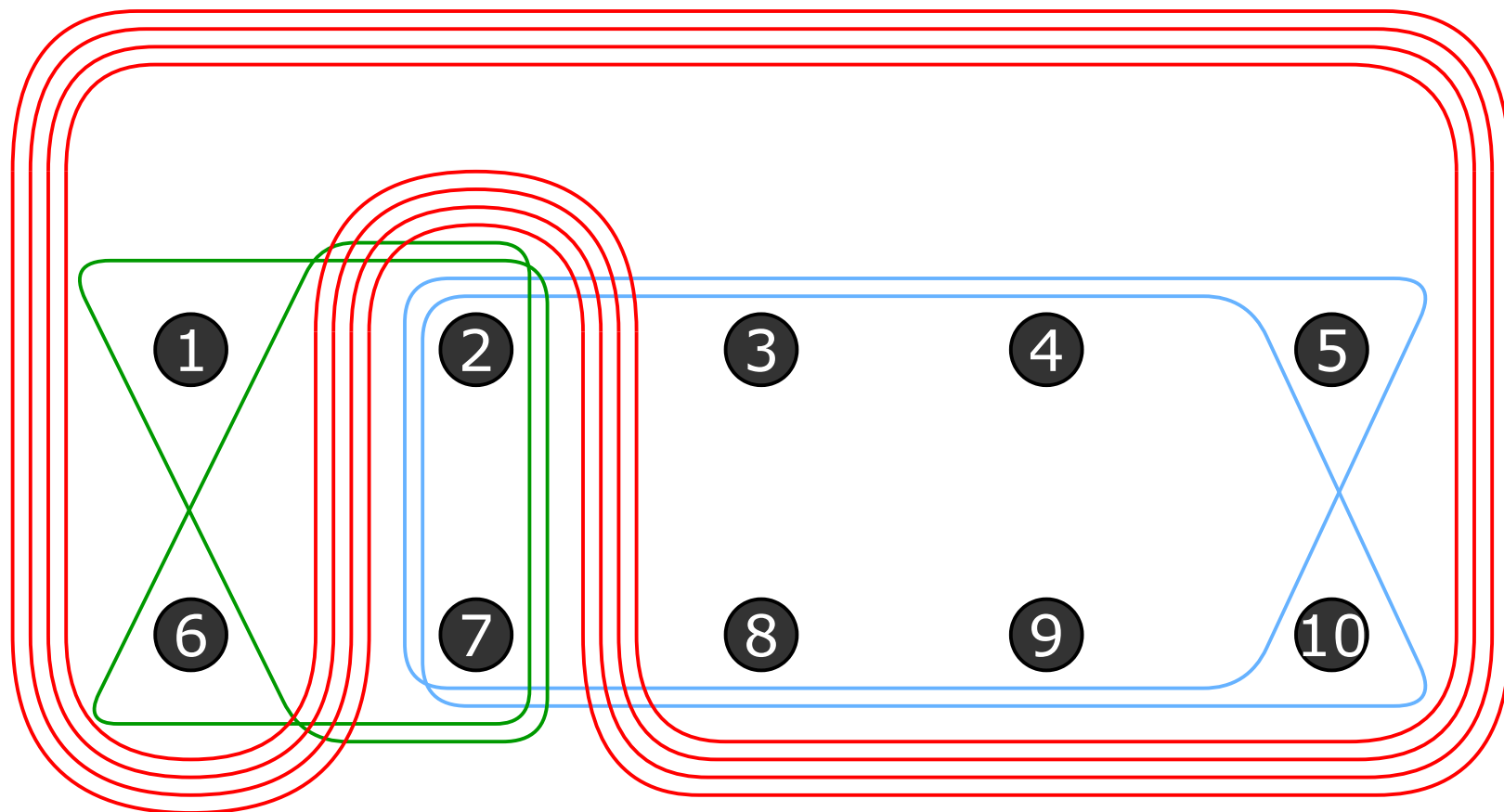
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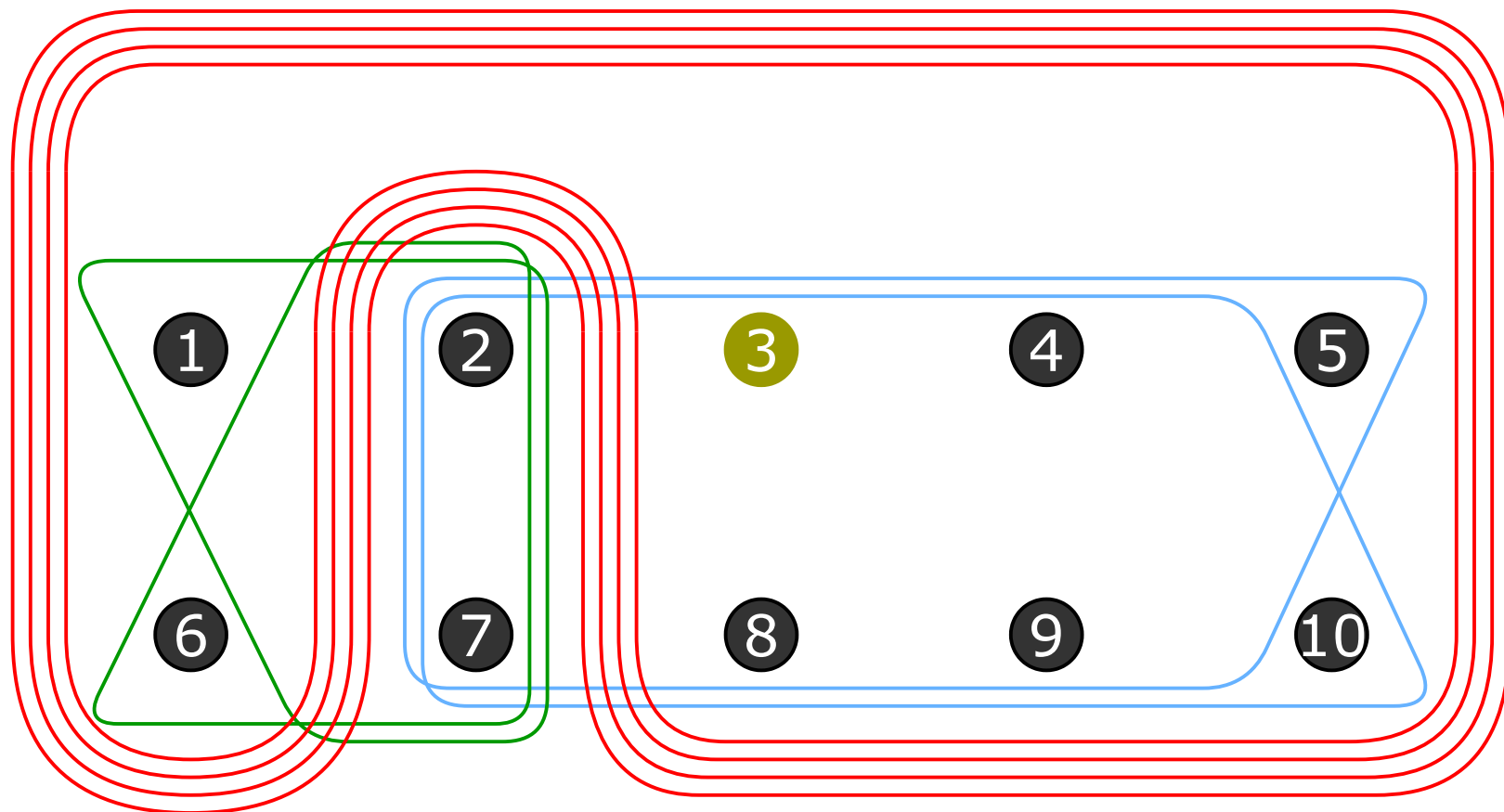
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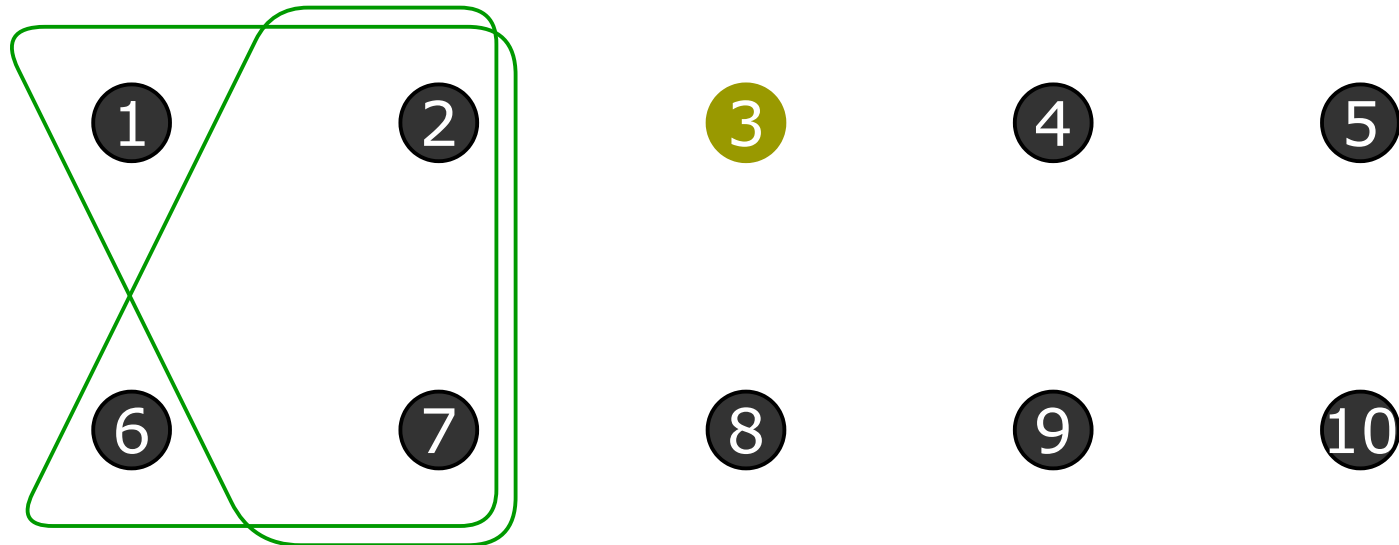
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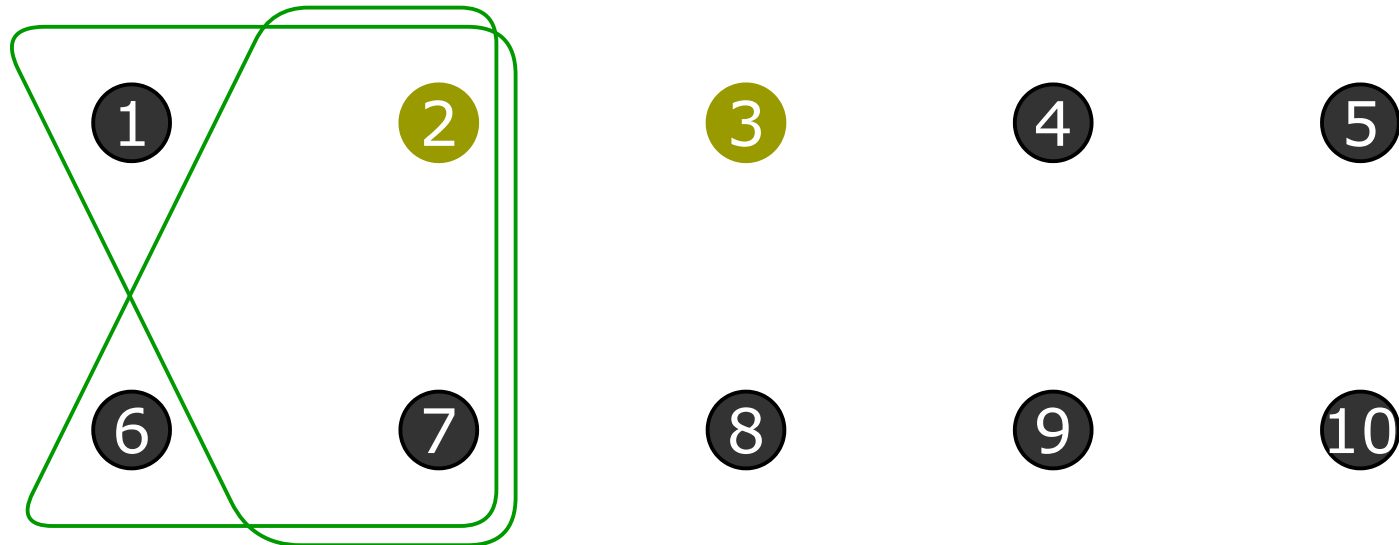
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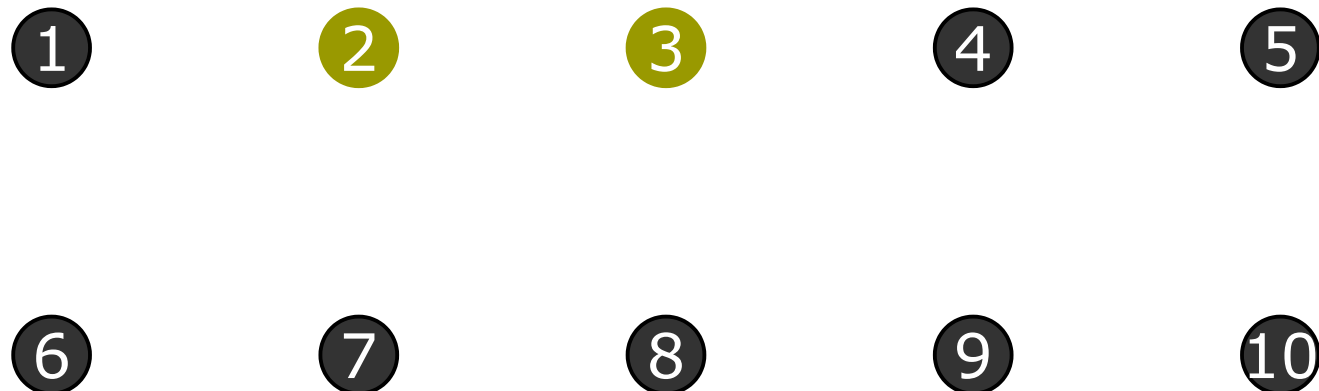


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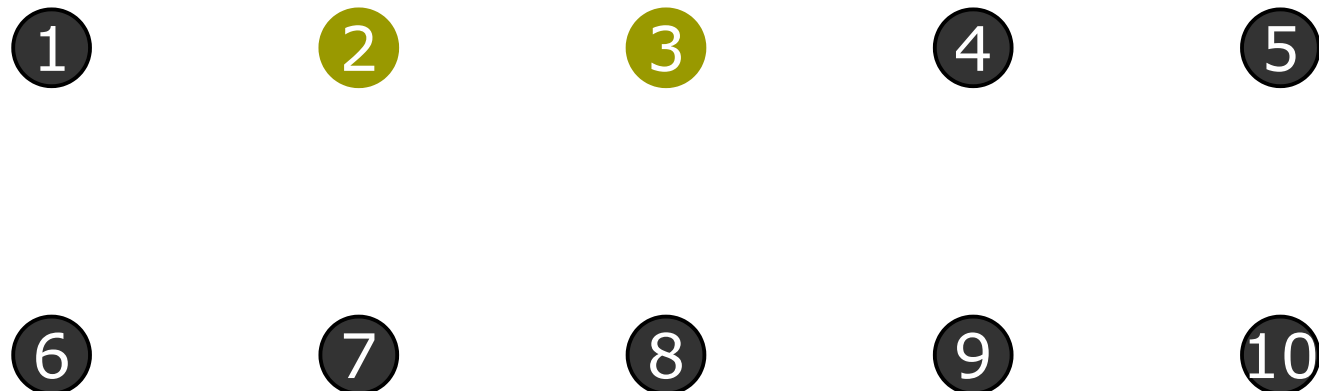
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- 2
- 3
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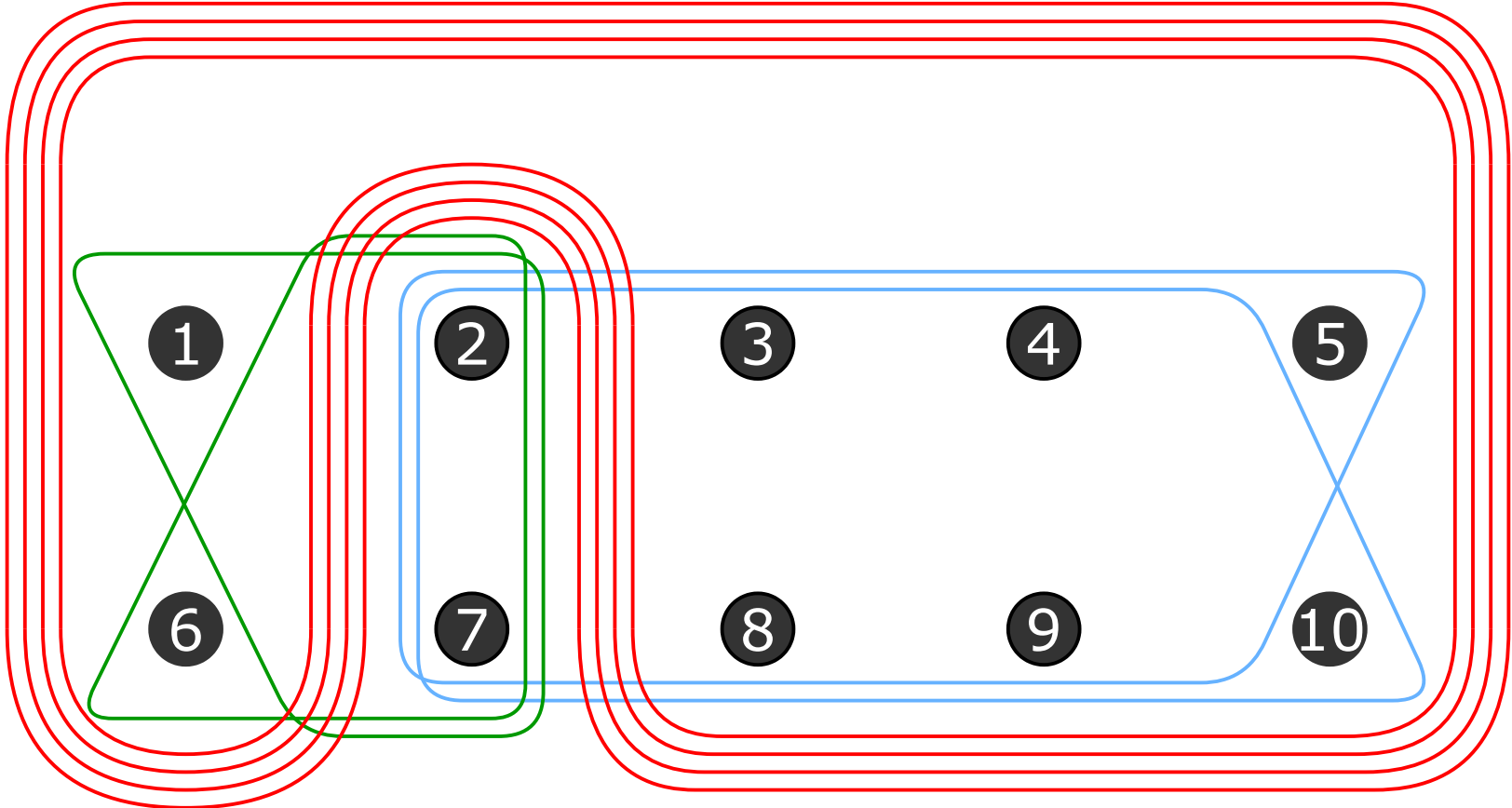
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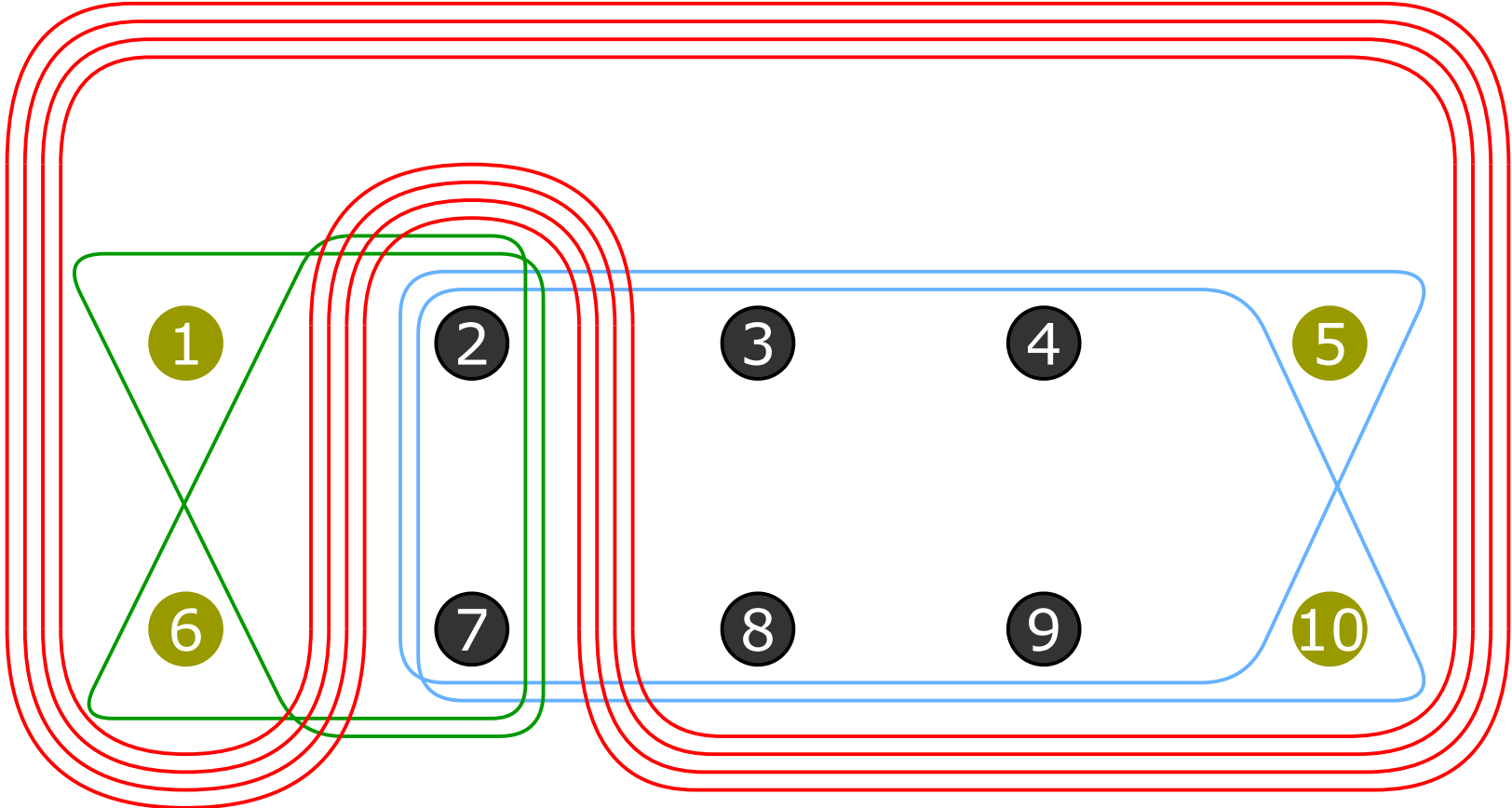
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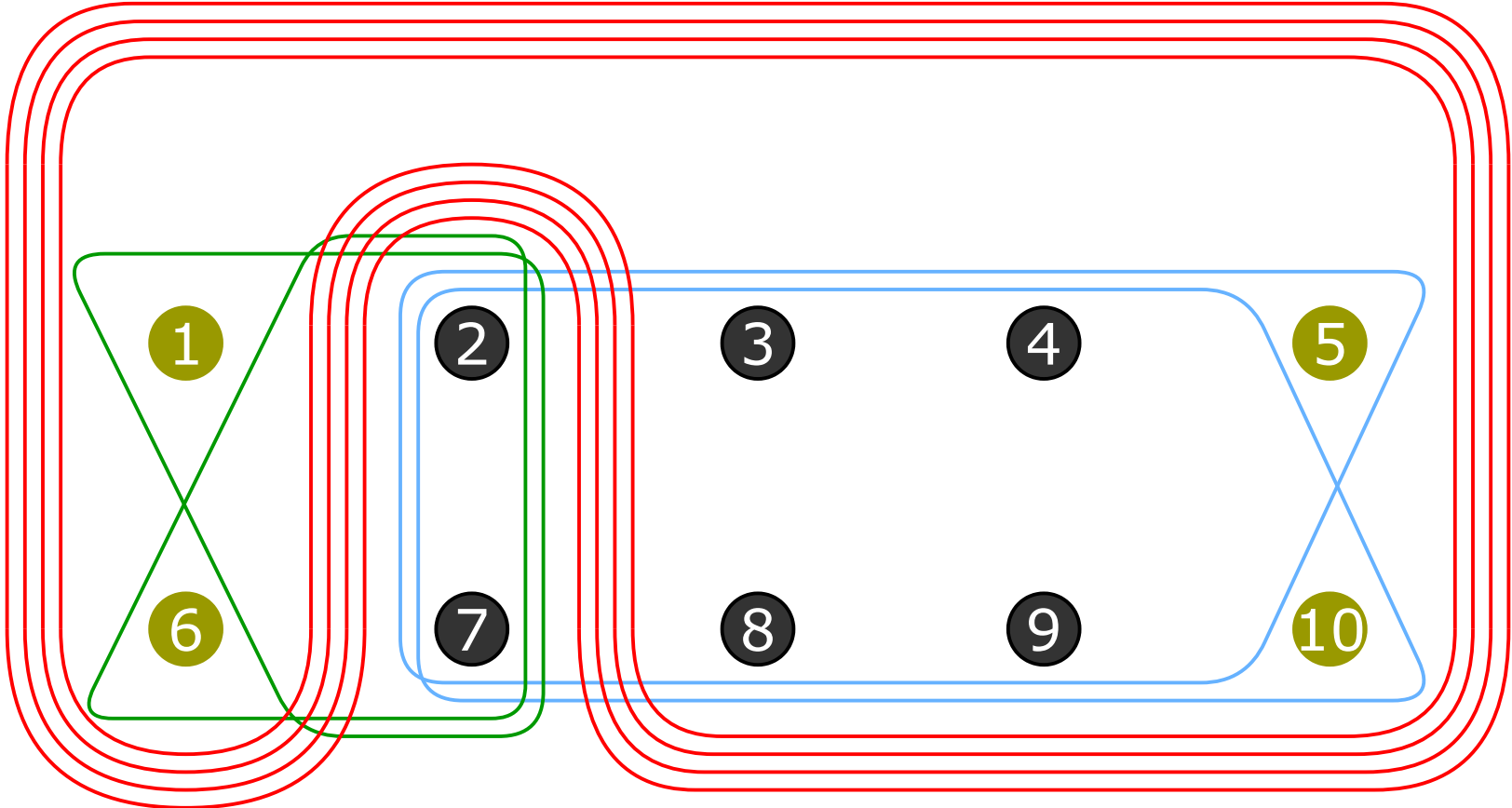
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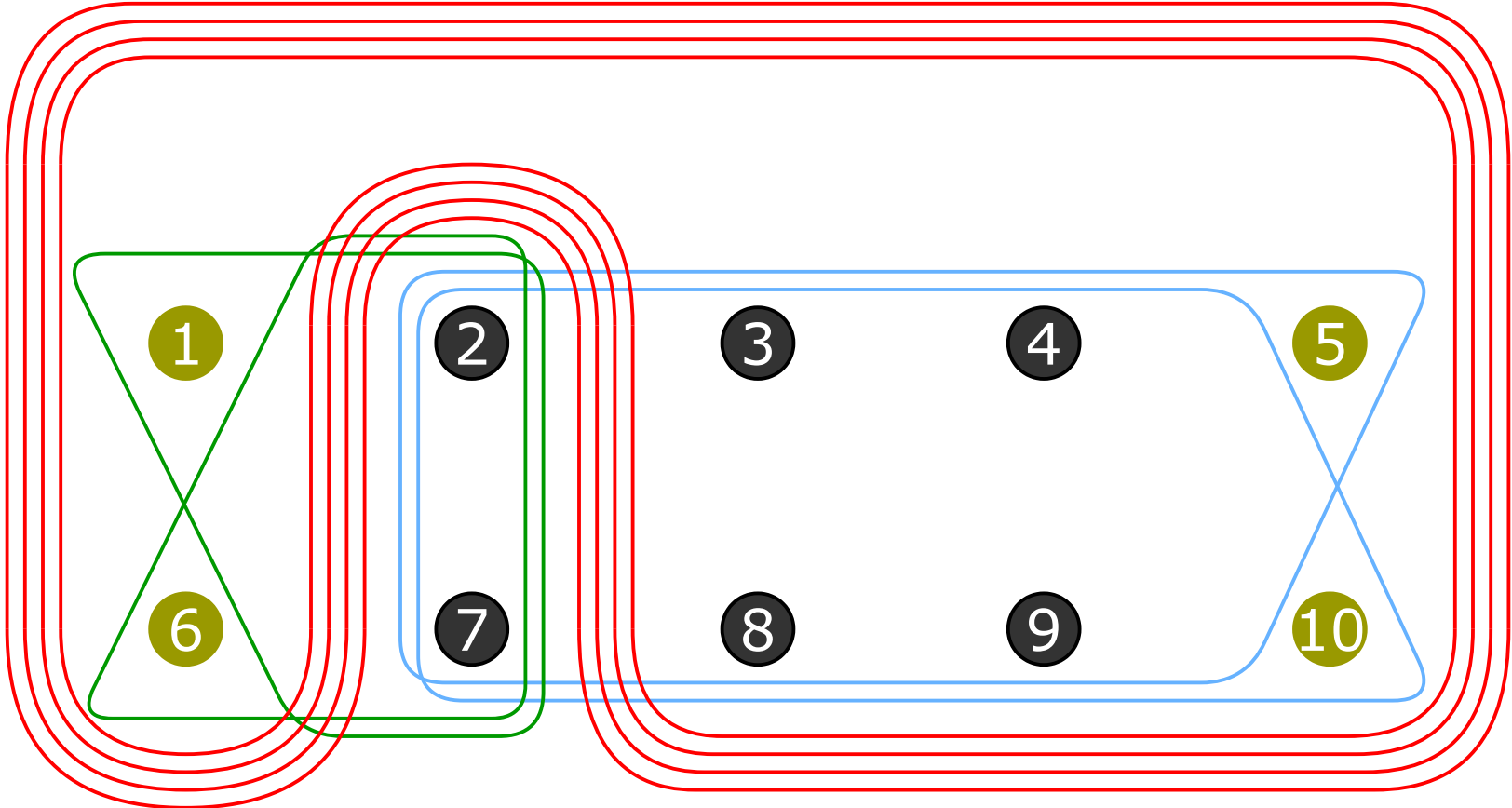


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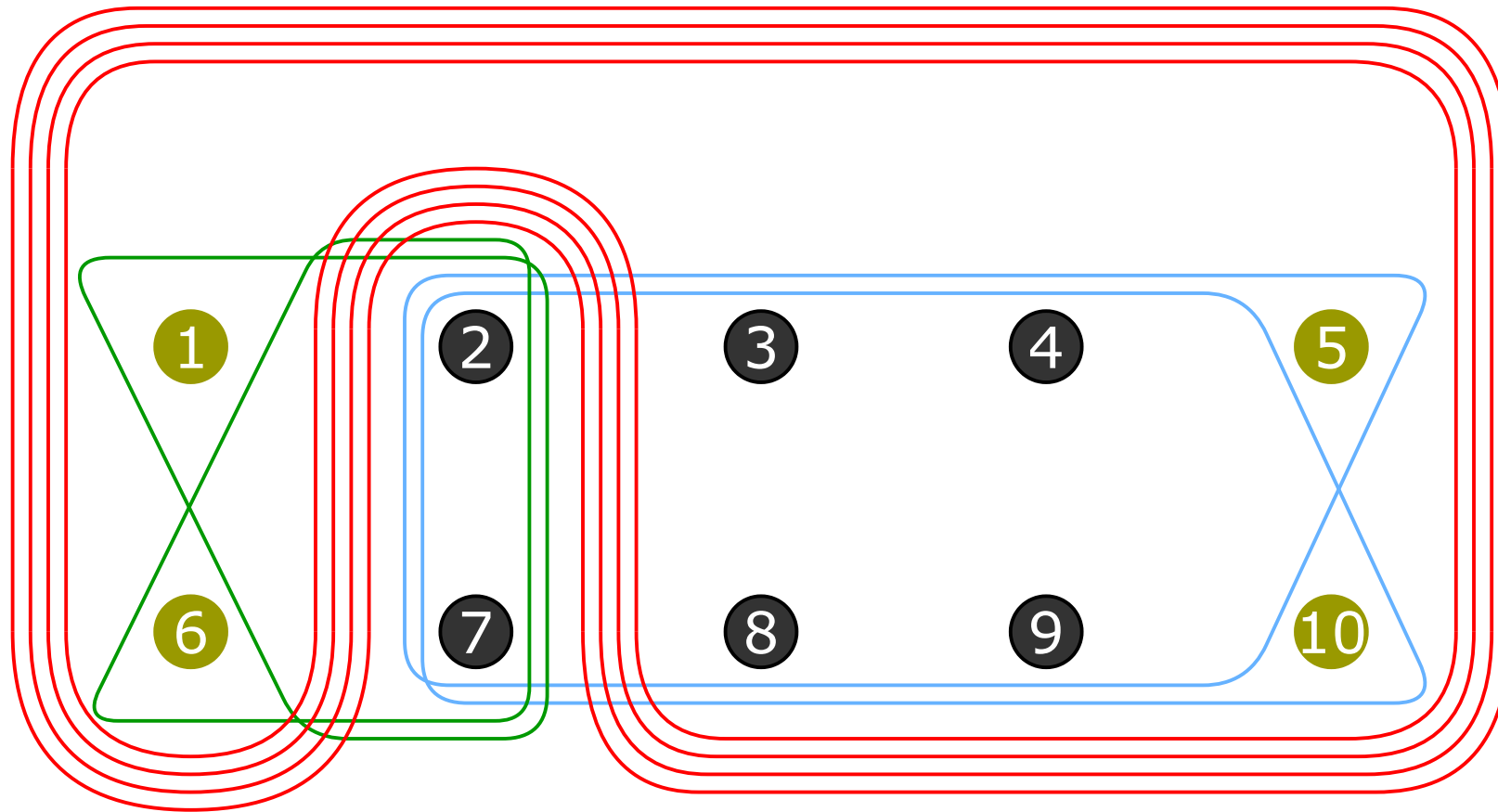
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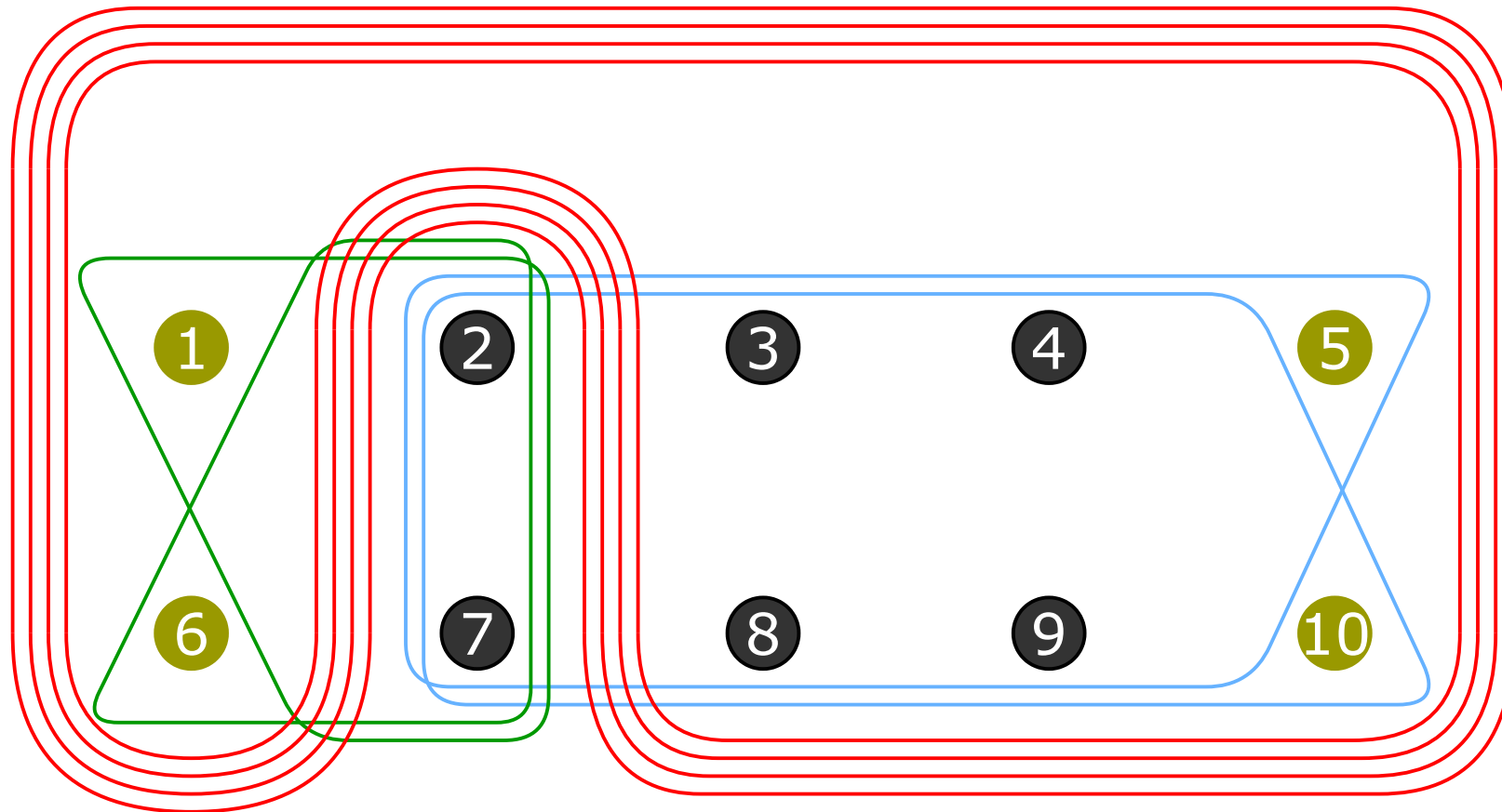
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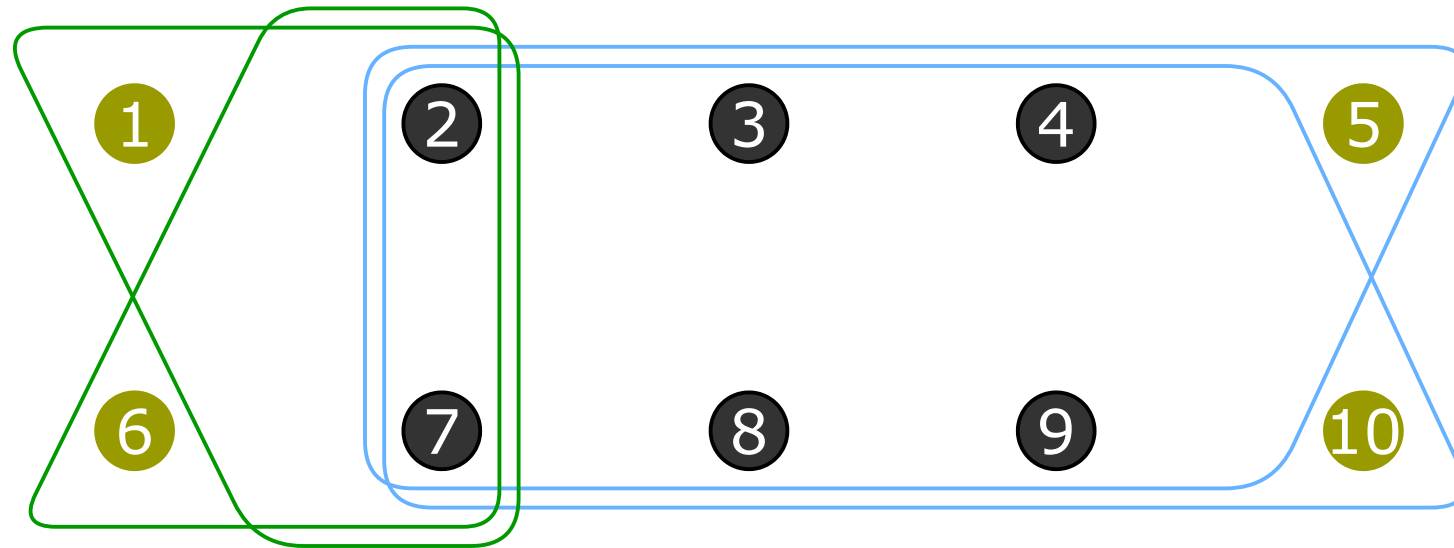


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Is it *fair*?

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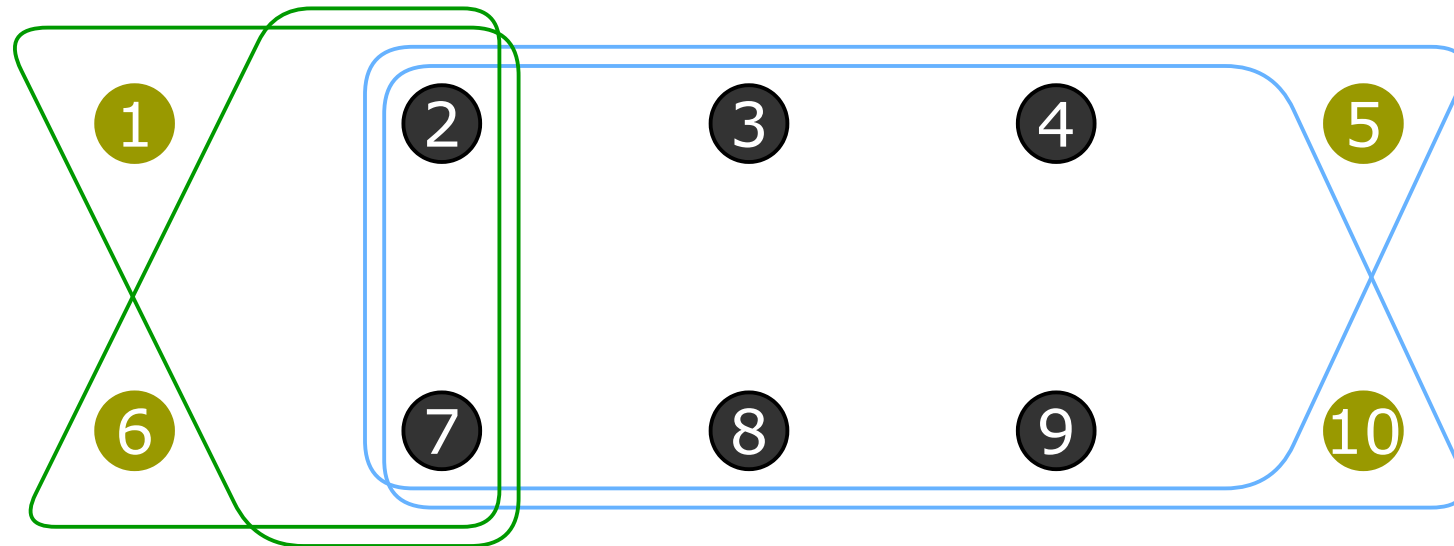


Does this committee satisfy JR?  
> Yes.

Is it Pareto efficient?  
> No,  $\{1, 4, 6, 9\}$  is better.

Is it *fair*?  
> Arguably not: the green and blue voters form half of the electorate, so they deserve to approve half of the committee.

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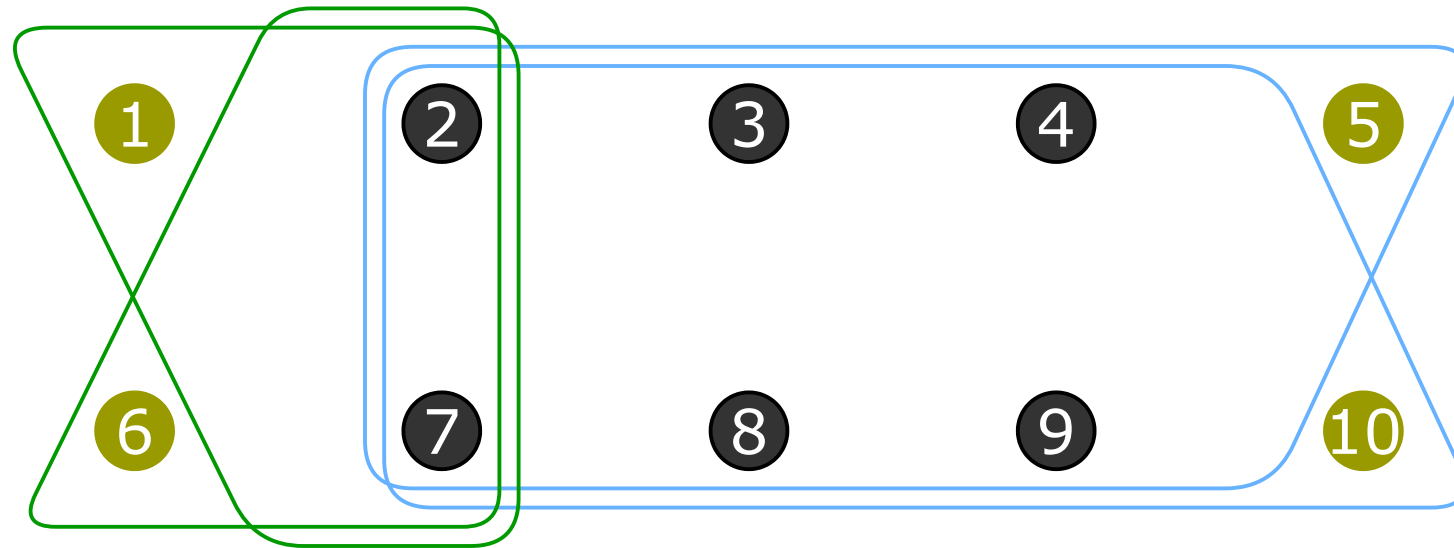
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**Core:** Violated if there is a group of at least  $\ell/k$  voters that can form an alternative committee of size  $\ell$  giving a larger number of approved candidates to each member.

# Other notions of fairness

TIME



Where is Catalonia, Spain on the map? Google Maps

### Why is Catalonia seeking independence?

Catalonia, which is Spain's richest region, has a distinct cultural heritage and language. The push for independence crystallized during the fallout of the global financial crisis in 2008. Some Catalans believe the country's rising unemployment levels and languishing economy was pulling the region down, and they began to seek greater autonomy from Madrid.



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# Feasibility of fairness axioms

Core



EJR



JR

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EJR



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**PAV:** Select the set  $C$  of  $k$  candidates maximizing:

$$\sum_{i \in \{\text{voters}\}} \frac{\text{num candidates in } C \text{ approved by } i}{j} \quad \frac{1}{j}$$

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(JR is special case where  $\ell = 1$ .)

JR **✓** Chamberlin-Courant

**Open Question:** Is there a *committee monotone* rule (increasing  $k$  doesn't kick candidates off the committee) that satisfies EJR?

**PAV:** Select the set  $C$  of  $k$  candidates maximizing:

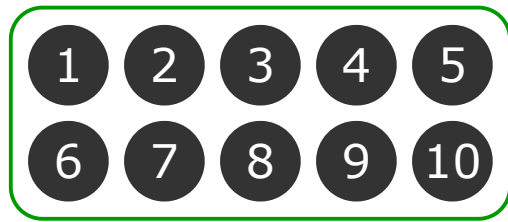
$$\sum_{i \in \{\text{voters}\}} \sum_{j=1}^{\text{num candidates in } C \text{ approved by } i} \frac{1}{j}$$

# PAV satisfies EJR - proof sketch, $k = 10$

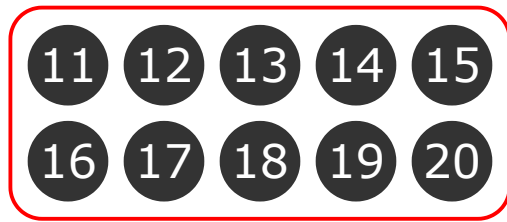
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30 voters

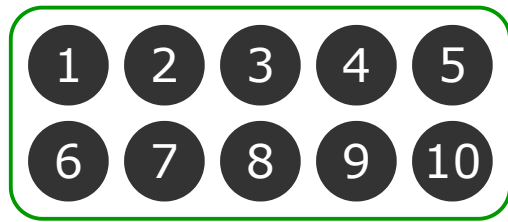


70 voters

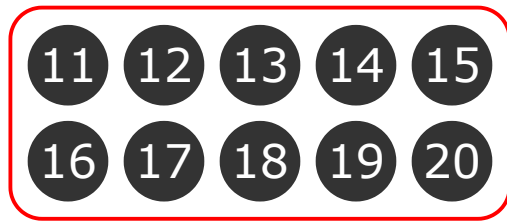
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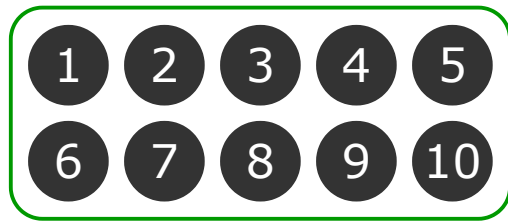
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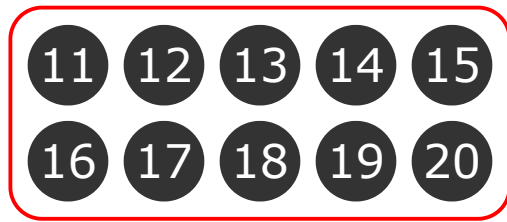
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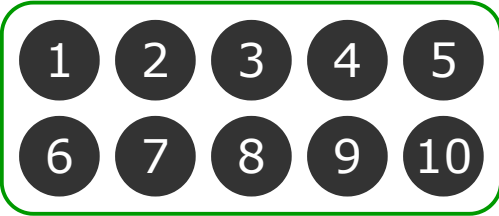


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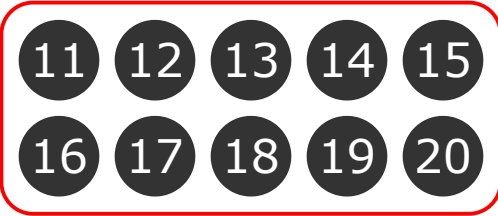
$$30 \cdot \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + 70 \cdot \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right)$$

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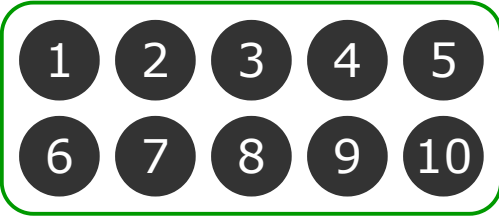
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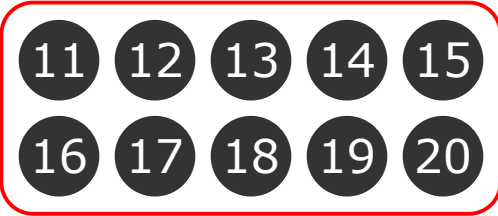
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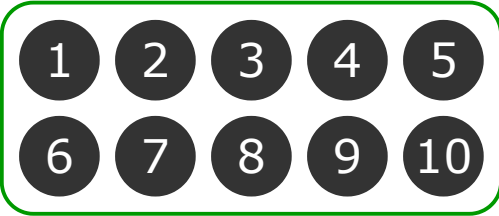


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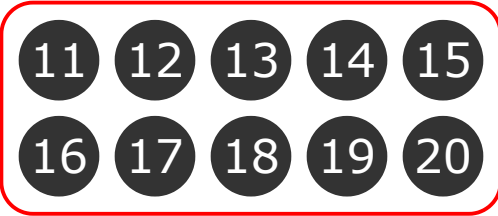
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$C$

↓ Add  $c$

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# Participatory budgeting

Like committee selection from approval ballots, but with weighted candidates!

Project	Cost
Bike path	\$700
Outdoor gym	\$400
New park	\$250
New playground	\$200
Library for kids	\$100

Example from Dominik Peters and Piotr Skowron

Suppose the total budget is \$1,100.

# Participatory budgeting

Like committee selection from approval ballots, but with weighted candidates!

Project	Cost	V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11
Bike path	\$700	✓	✓	✓	✓	✓	✓					✓
Outdoor gym	\$400	✓	✓	✓	✓	✓	✓					
New park	\$250		✓		✓	✓		✓			✓	
New playground	\$200							✓	✓	✓	✓	
Library for kids	\$100							✓		✓	✓	

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Outdoor gym	\$400	✓	✓	✓	✓	✓	✓					
New park	\$250		✓		✓	✓		✓			✓	
New playground	\$200							✓	✓	✓	✓	
Library for kids	\$100							✓		✓	✓	

Example from Dominik Peters and Piotr Skowron

Suppose the total budget is \$1,100. A greedy policy would spend the entire budget on the first two projects.

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New park	\$250		✓		✓	✓		✓			✓	
New playground	\$200							✓	✓	✓	✓	
Library for kids	\$100							✓		✓	✓	

Example from Dominik Peters and Piotr Skowron

Suppose the total budget is \$1,100. A greedy policy would spend the entire budget on the first two projects.

A better way - **Method of Equal Shares:** <https://equalshares.net/>

# Participatory budgeting

Like committee selection from approval ballots, but with weighted candidates!

Project	Cost	V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11
Bike path	\$700	✓	✓	✓	✓	✓	✓					✓
Outdoor gym	\$400	✓	✓	✓	✓	✓	✓					
New park	\$250		✓		✓	✓		✓			✓	
New playground	\$200							✓	✓	✓	✓	
Library for kids	\$100							✓		✓	✓	

Example from Dominik Peters and Piotr Skowron

Suppose the total budget is \$1,100. A greedy policy would spend the entire budget on the first two projects.

A better way - **Method of Equal Shares**: <https://equalshares.net/> This satisfies...

**EJR (for PB)**: Violated if there is a group consisting of an  $\alpha$ -fraction of voters that can spend an  $\alpha$ -fraction of the budget on projects that they all approve of, increasing the number of approved projects for every member of the group.