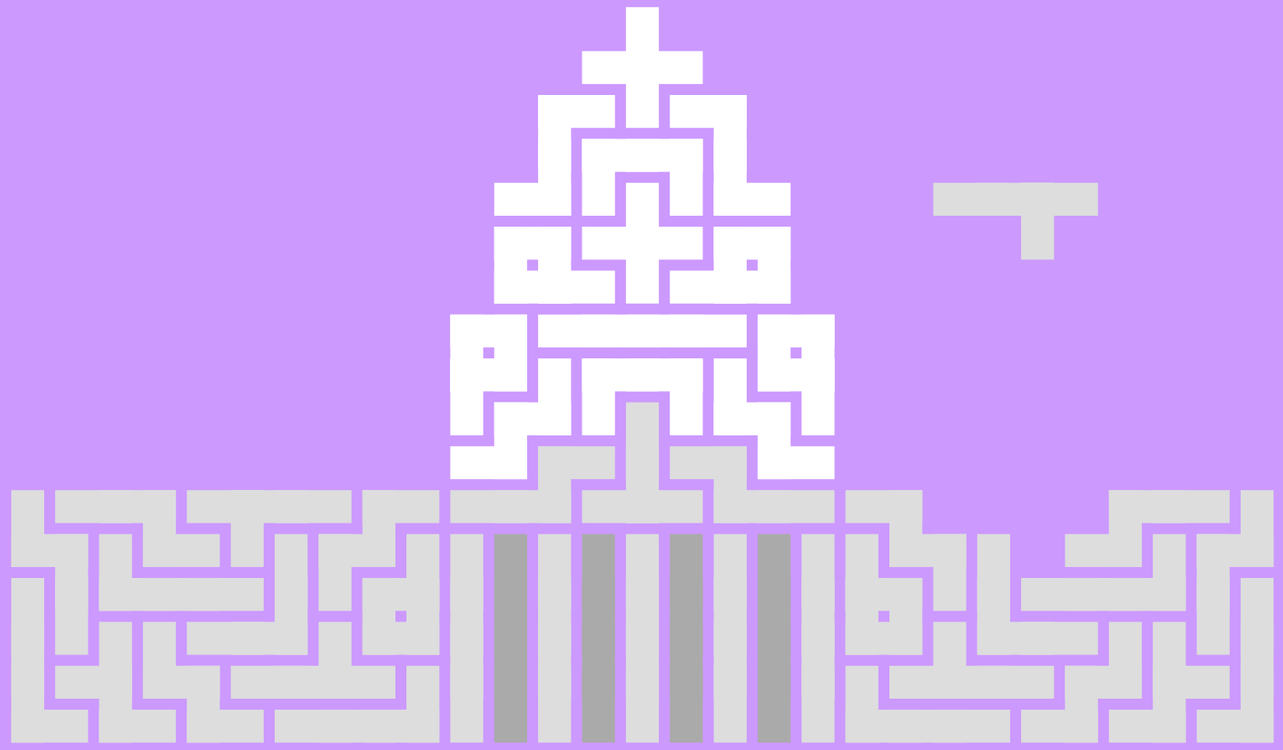

Algorithms For Democratic Decision-Making

Jamie Tucker-Foltz • Yale University • Spring 2026

Lecture 13: **Fair Division 1: Indivisible Goods**

Introduction

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http://spliddit.org/

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Agents take turns picking their favorite item remaining.

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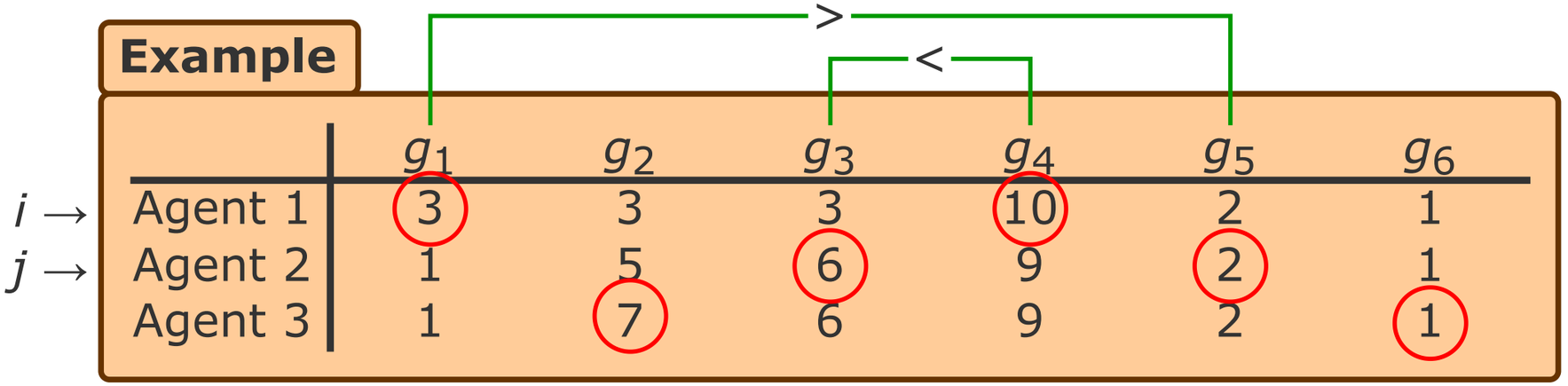

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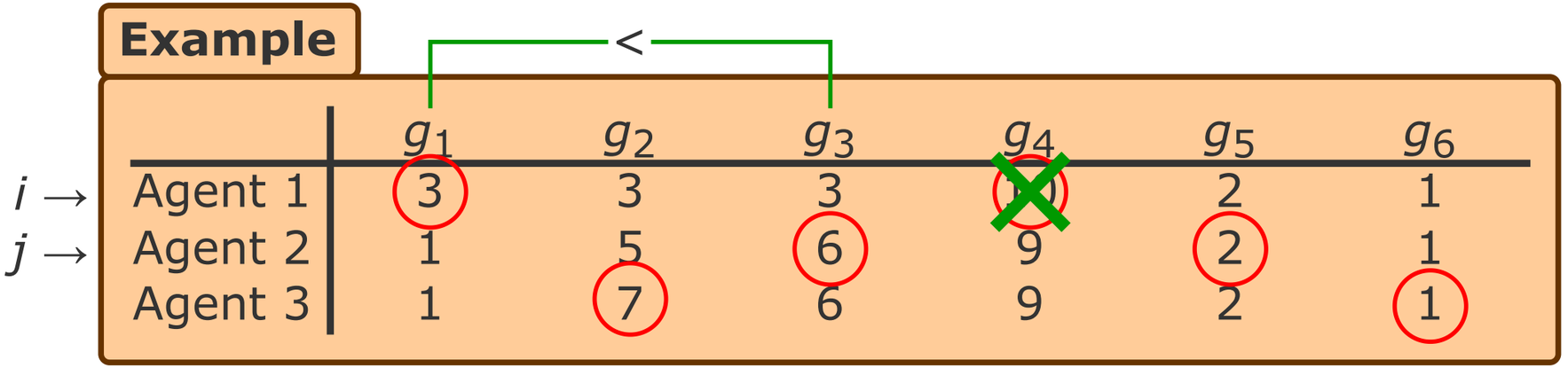

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Proof. Consider an arbitrary pair of agents, $i < j$.

Observe that i does not envy j because i's k^{th} good is worth more to i than j's k^{th} good.

And j does not envy i after removing the first good i picked, because j's k^{th} good is worth more to j than i's $(k + 1)^{\text{th}}$ good. ■

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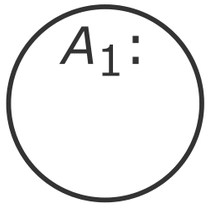

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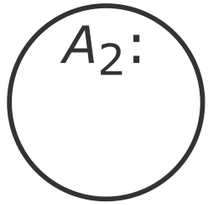

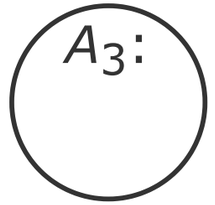

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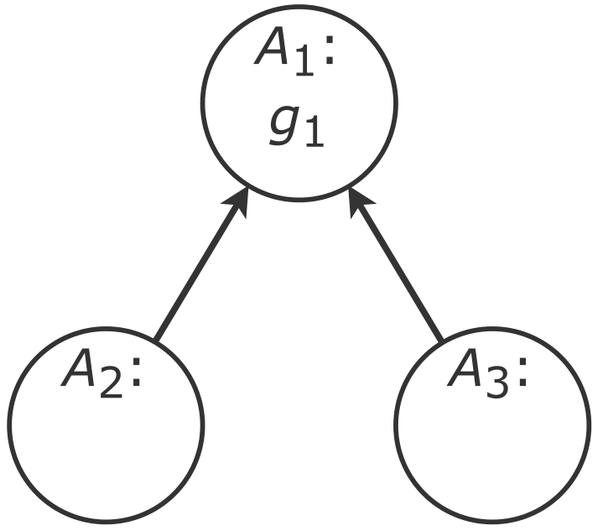

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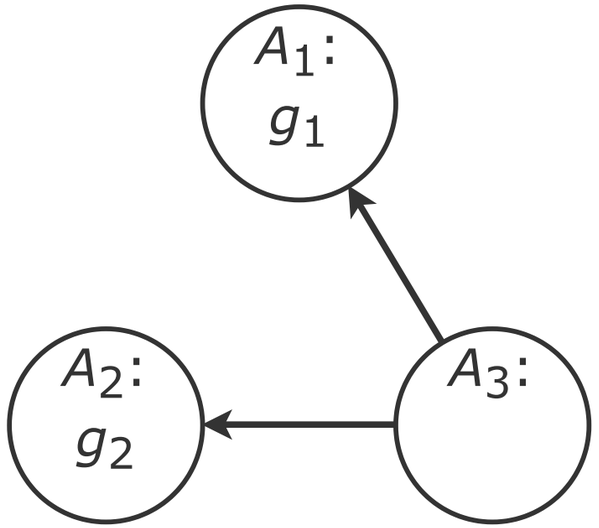

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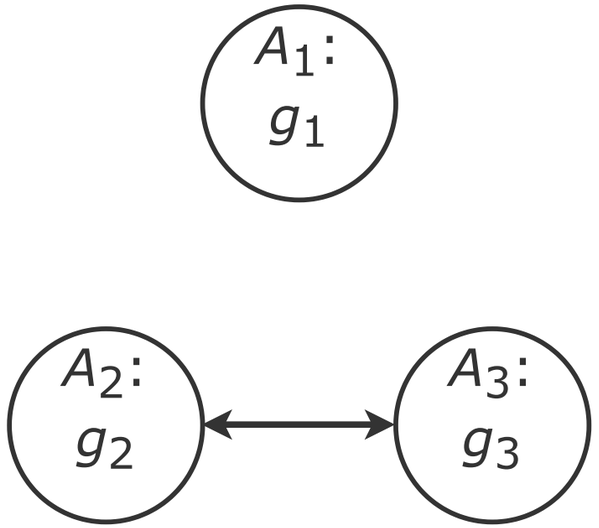

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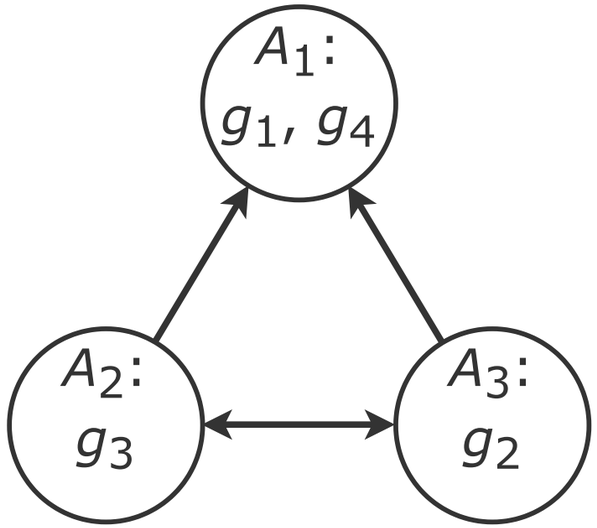

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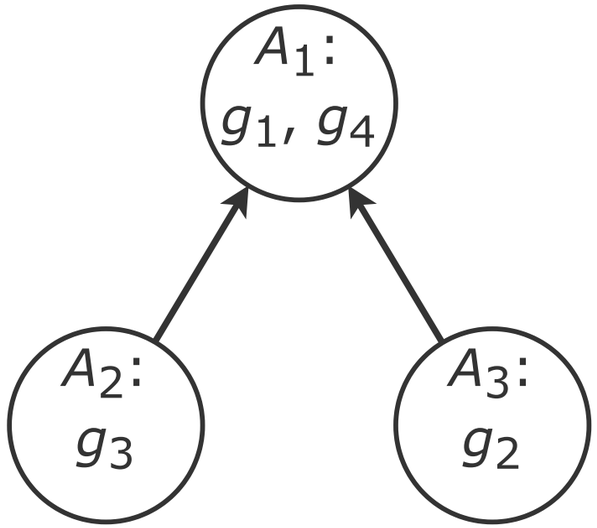

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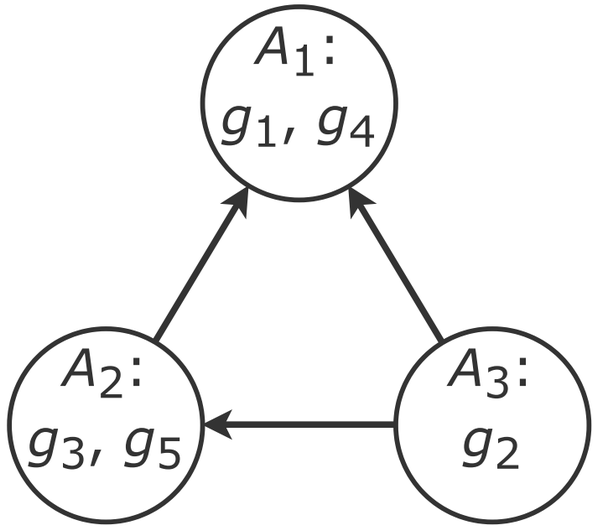

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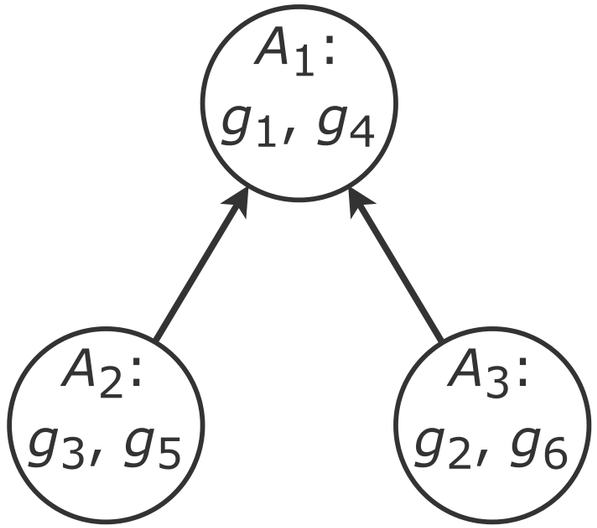

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✘ Not EF1

Algorithm 3: Max Nash Welfare

Select the allocation A maximizing the *Nash welfare* of A:

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Example

	g_1	g_2	g_3	g_4	g_5	g_6
Agent 1	3	3	3	10	2	1
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	g_1	g_2	g_3	g_4	g_5	g_6
Agent 1	(3)	3	3	(10)	2	1
Agent 2	1	5	(6)	9	(2)	1
Agent 3	1	(7)	6	9	2	(1)

$\mathrm{NW}(A) =$
$(3 + 10)(6 + 2)(7 + 1)$
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Theorem (Caragiannis, Kurokawa, Moulin, Procaccia, Shah, Wang, 2004)

The allocation maximizing Nash welfare satisfies EF1.

Proof that MNW satisfies EF1

Proof. Suppose i envies j by more than one good. We'll show that we can increase the Nash welfare by transferring some good g^* from A_j to A_i.

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	CR:		*7/5*	*6/6*			

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Proof that MNW satisfies EF1, continued

g^* maximizes $\dfrac{v_i(g^*)}{v_j(g^*)}$ over A_j

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g^* maximizes $\dfrac{v_i(g^*)}{v_j(g^*)}$ over $A_j \implies \dfrac{v_i(g^*)}{v_j(g^*)} \geq \dfrac{v_i(A_j)}{v_j(A_j)}$

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Recall that an allocation is *Pareto efficient* if it is not possible to make all agents better off (weakly for all agents and strictly for at least one).

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Respond at:

pollev.com/jtuckerfoltz255 or

bit.ly/jtfpoll or

text jtuckerfoltz255 to 37607

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	g_1	g_2	g_3	g_4	g_5
Agent 1	4	3	1	2	2
Agent 2	4	1	3	2	2

Pareto efficiency

Recall that an allocation is *Pareto efficient* if it is not possible to make all agents better off (weakly for all agents and strictly for at least one).

For the special case of $n = 2$, which of the following rules always yields a Pareto efficient allocation?

- ✗ **Round Robin**
- ✗ **Envy Cycle Elimination**
- ✓ **Max Nash Welfare**

Proof for MNW.
A Pareto improvement would strictly increase one term in the Nash welfare product, and weakly increase all others, giving an overall strictly higher Nash welfare. ■

RR Counterexample

	g_1	g_2	g_3	g_4
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Open Question: Is there a polynomial-time algorithm to find a Pareto efficient and EF1 allocation?

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EFX

An allocation A is *envy-free up to one good (EF1)* if $\forall\, i, j \in [n], \exists\, g \in A_j$, $v_i(A_i) \geq v_i(A_j \setminus \{g\})$

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For $0 \leq \alpha \leq 1$, an allocation A is α-*EFX* if $\forall\, i, j \in [n]$, $\forall\, g \in A_j$,
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EFX is tricky

Recall that RR, ECE, and MNW all produced the following allocation:

Example

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There always exists a 0.618-EFX allocation, which can be computed in polynomial time.

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$\phi = 1.618\ldots$
is the *Golden Ratio*:
$\frac{1}{\phi} = \phi - 1 = 0.618\ldots$

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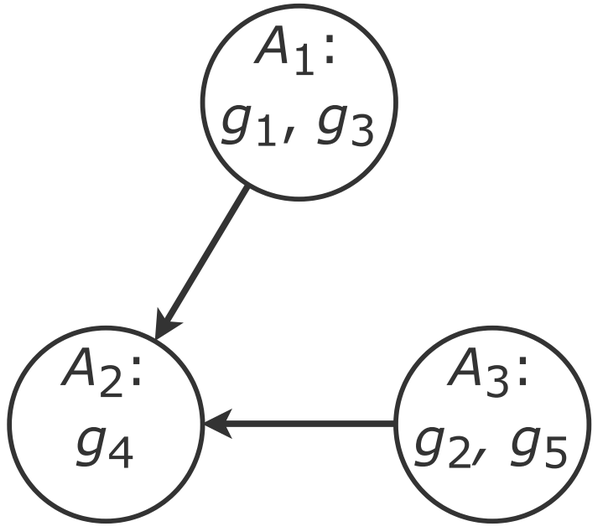

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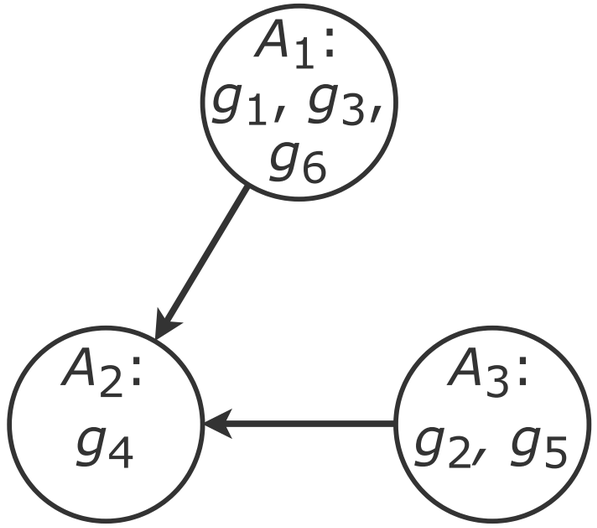

Proof of $(\phi - 1)$-EFX, part 1

Lemma 1

Before beginning ECE, the partial allocation is $(\phi - 1)$-EFX.

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Proof. We don't have to worry about anyone envying agents in L since they each only get one good. Thus the only cases we have to worry about are:

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For an agent not in L, observe that both of their goods are preferred to any unallocated goods. Thus, they value their own bundle higher by a factor of at least $2 > \phi$. ■

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Proof of Theorem. We now claim that Lemmas 1 and 2 continue to hold after every iteration of ECE. For Lemma 2, this is obvious, since utilities are increasing throughout ECE.

Proof of $(\phi - 1)$-EFX, part 3

Suppose *A* is a $(\phi - 1)$-EFX partial allocation, then we perform an iteration of ECE to obtain a new partial allocation *A'*. There are two cases:

Case 1: An envy cycle gets rotated.

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$$\begin{aligned} v_i(A'_j) &= v_i(A_j) + v_i(g) \\ &\leq v_i(A_i) + \frac{v_i(A_i)}{\phi} \end{aligned}$$

Proof of $(\phi - 1)$-EFX, part 3

Suppose *A* is a $(\phi - 1)$-EFX partial allocation, then we perform an iteration of ECE to obtain a new partial allocation *A′*. There are two cases:

Case 1: An envy cycle gets rotated.
This doesn't make anyone newly envious by the same reason as in the ECE EF1 proof.

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$$\begin{aligned} v_i(A'_j) &= v_i(A_j) + v_i(g) \\ &\leq v_i(A_i) + \frac{v_i(A_i)}{\phi} \\ &= \left(1 + \frac{1}{\phi}\right) v_i(A'_i) \end{aligned}$$

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So here we actually get $(\phi - 1)$-EF, which implies $(\phi - 1)$-EFX. ■

Algorithm 5: MNW and ECE hybrid

Step 1: Compute the partial allocation maximizing Nash welfare (product of utilities) with respect to the following utility functions:

$$u_i(A_i) := \begin{cases} \phi \cdot v_i(A_i) & \text{if } |A_i| = 1 \\ v_i(A_i) & \text{if } |A_i| = 2 \\ 0 & \text{if } |A_i| \geq 3 \end{cases}$$

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The same proof approach shows that this procedure also finds a $(\phi - 1)$-EFX allocation! If the conclusions of Lemmas 1 and 2 do not hold, then you can increase Nash welfare.