Algorithms For Democratic Decision-Making

Jamie Tucker-Foltz • Yale University • Spring 2026

Lecture 14: **Fair Division 2: Divisible Goods**

The Cake-Cutting model

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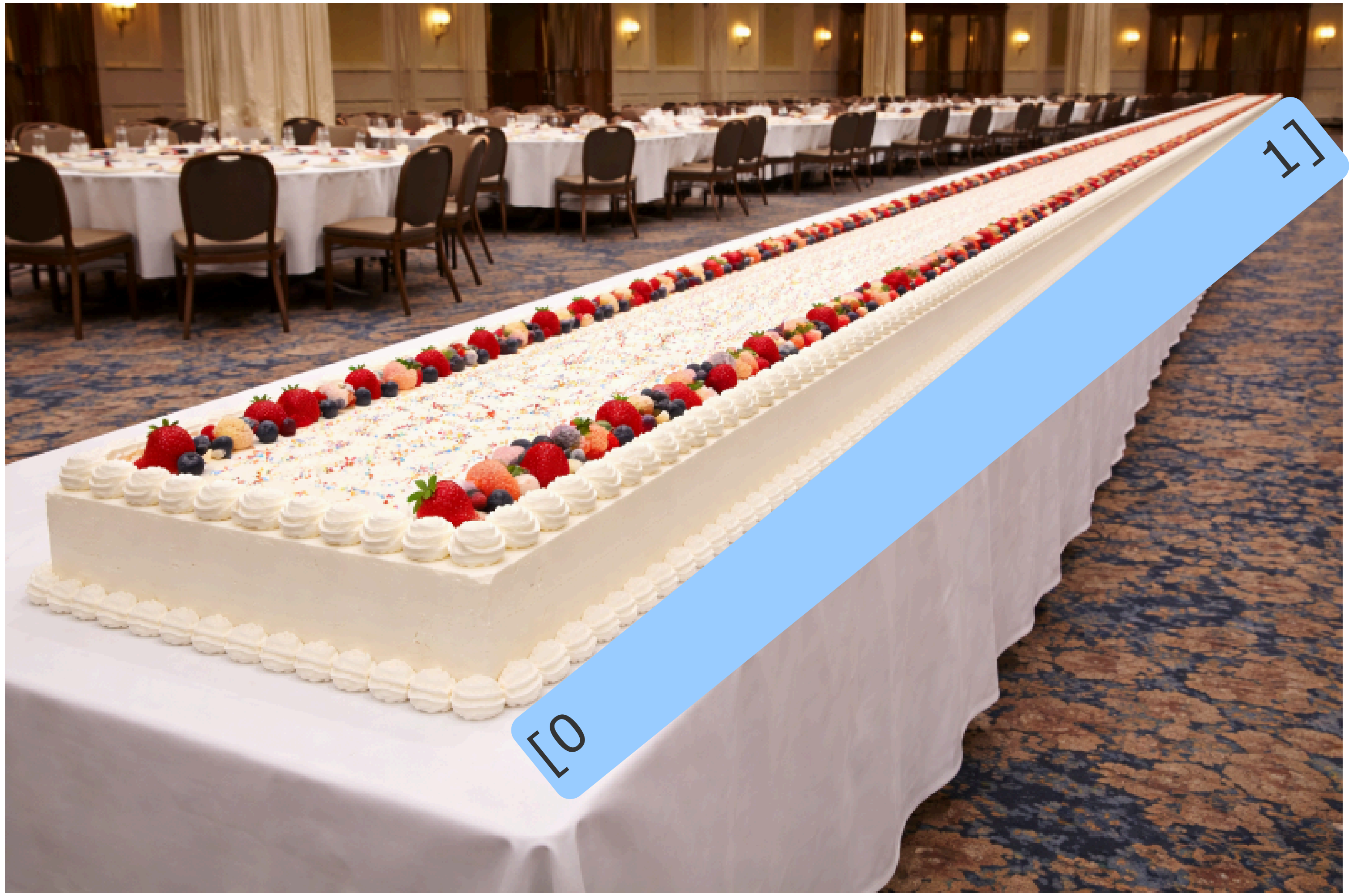

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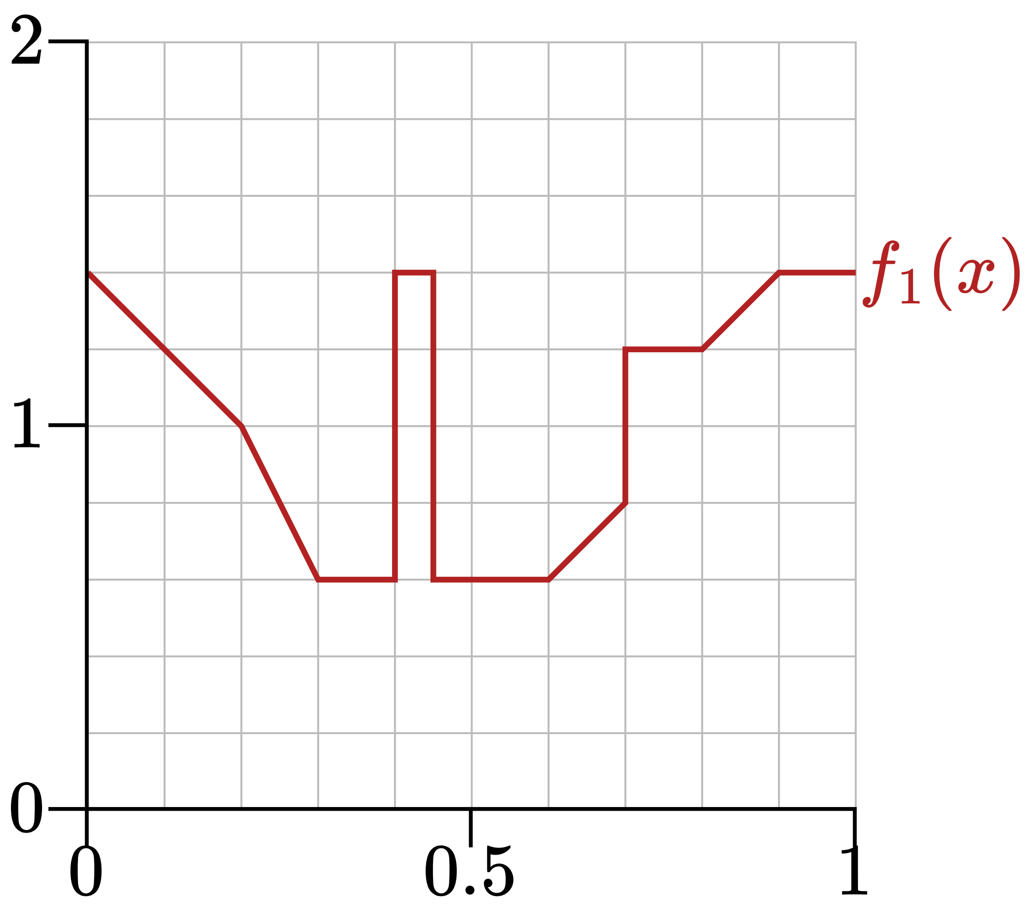

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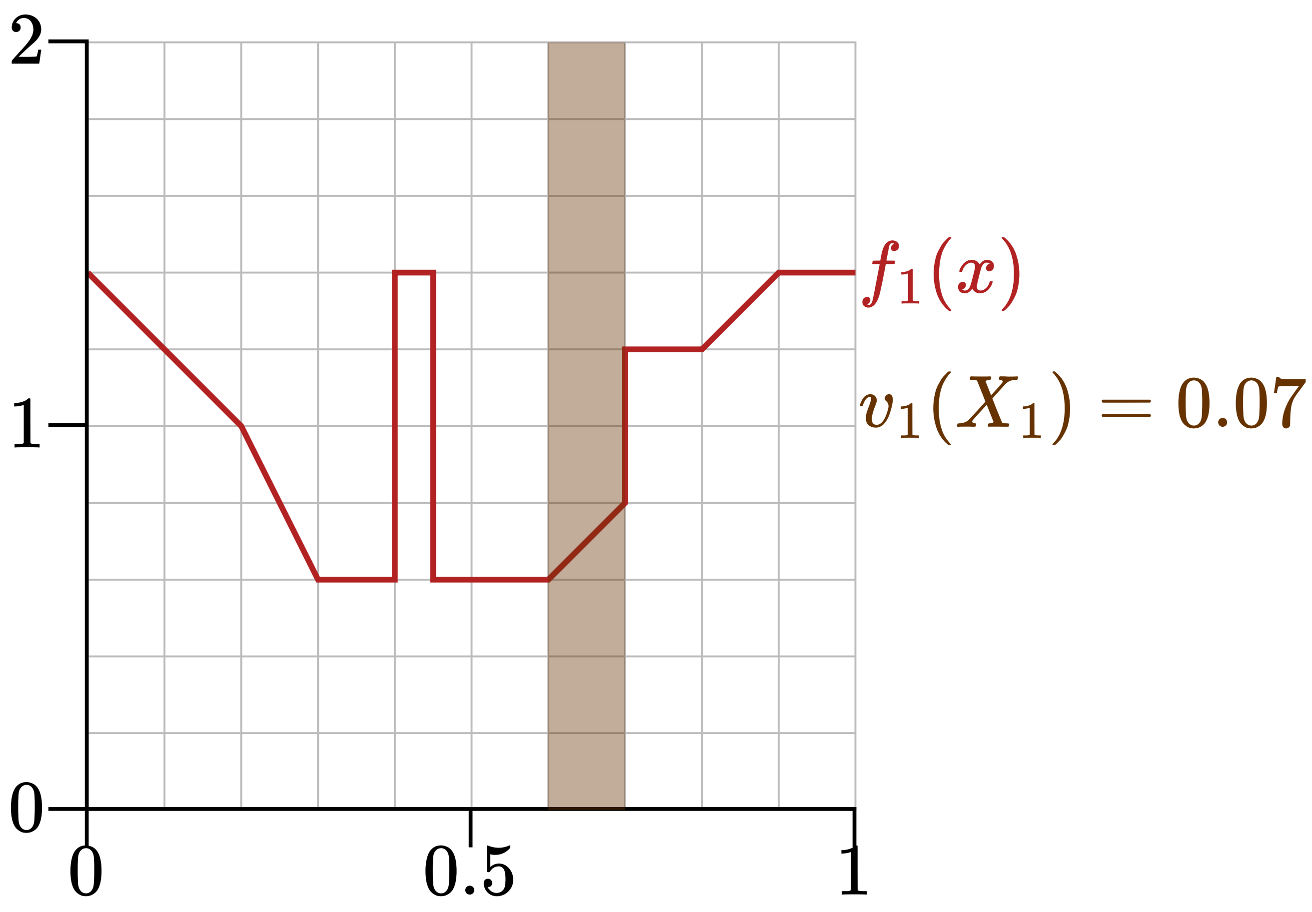

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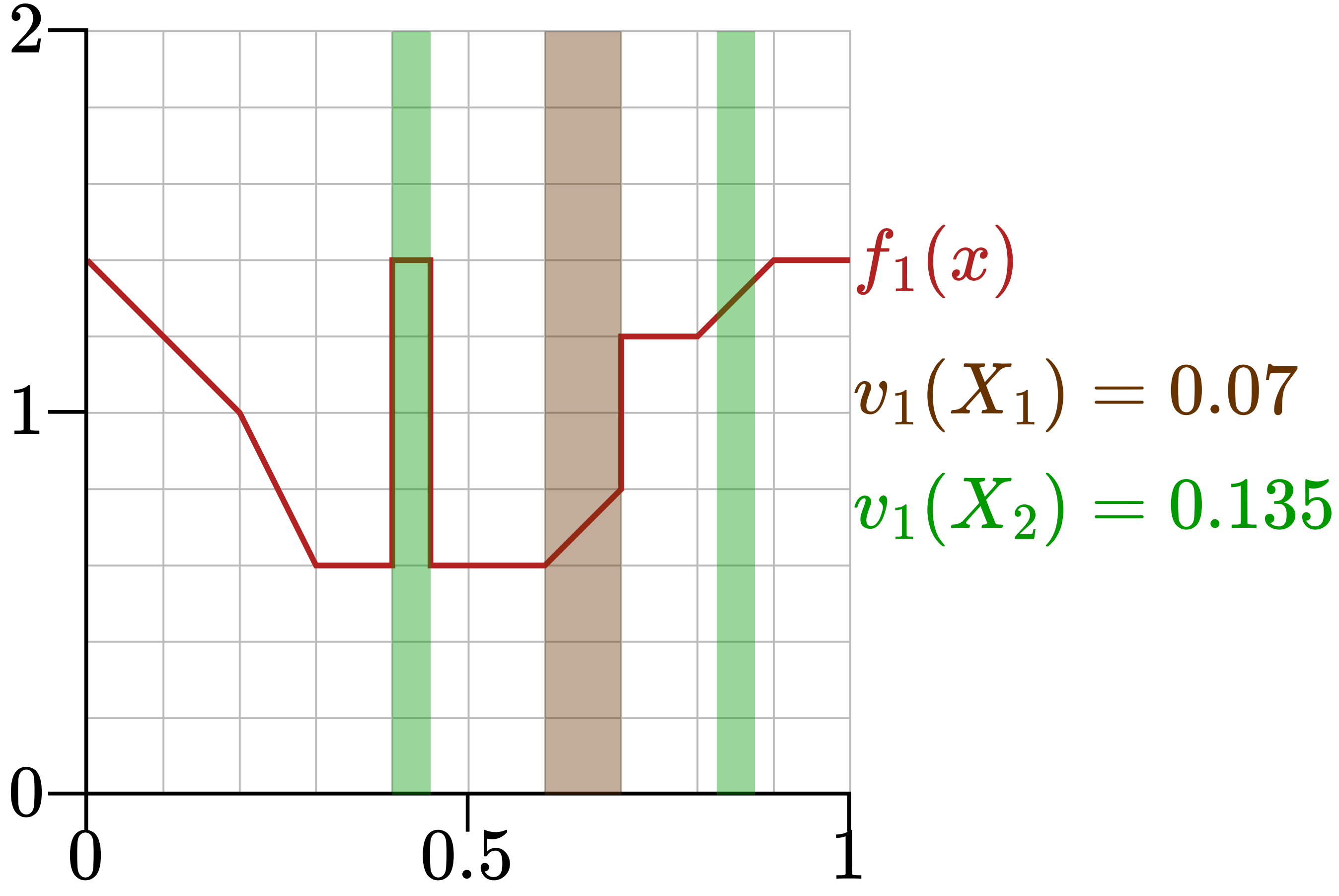

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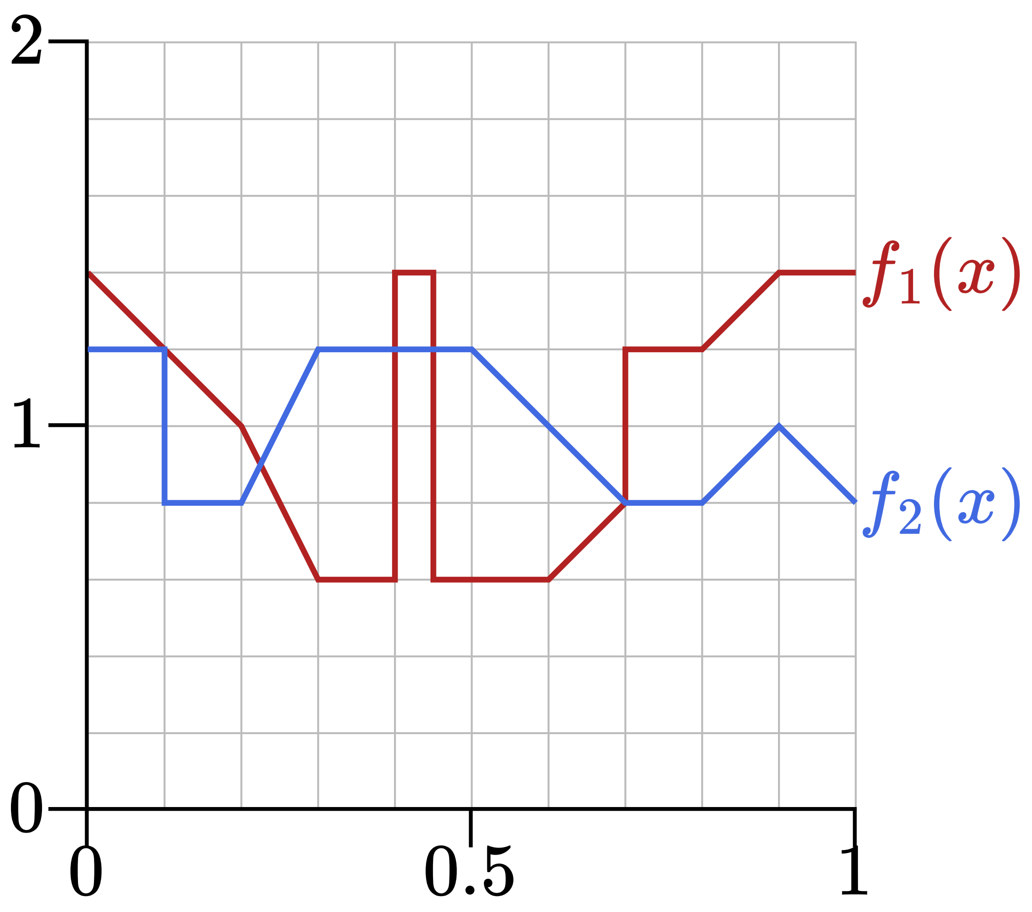

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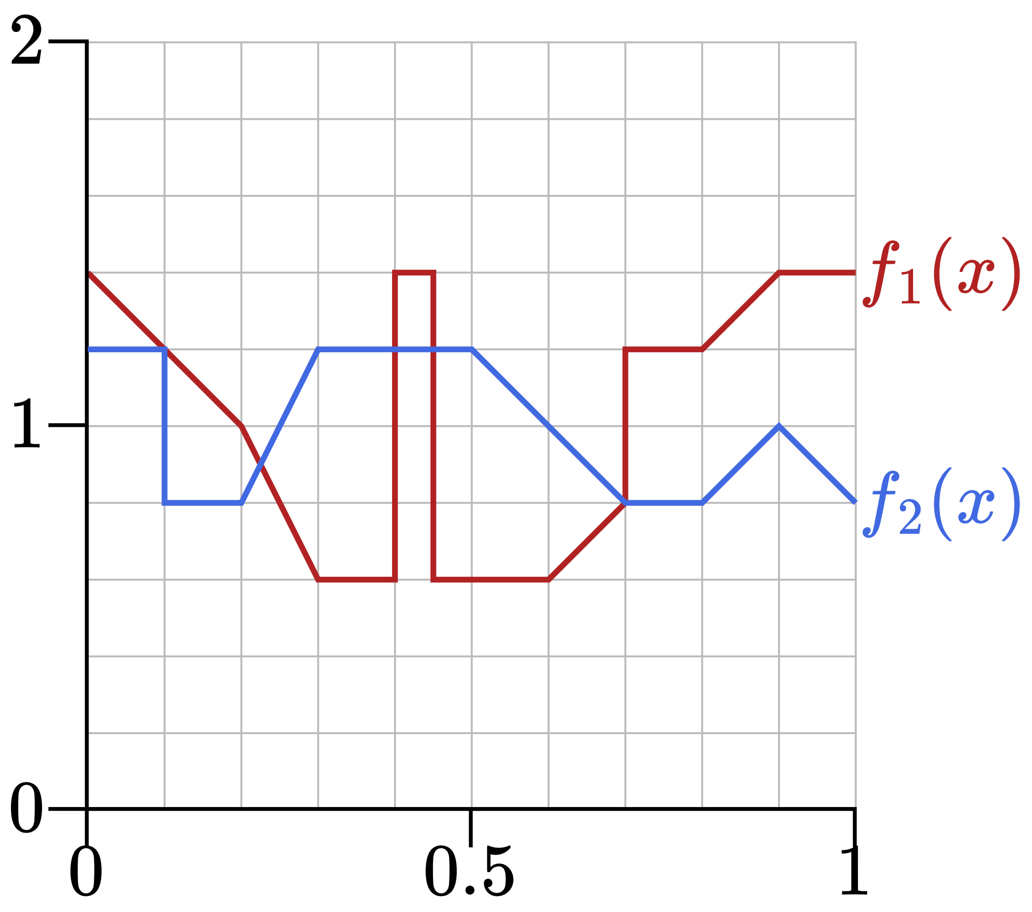
For n agents, an allocation is a partition of $[0,1]$ into $A = (A_1, A_2, \ldots, A_n)$, where each A_i is a finite union of intervals. (Intervals are closed, and can overlap at endpoints.)



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An allocation A is *envy-free* if for all $i, j \in [n]$,

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▸ For n = 2 agents, what is the relationship between envy-freeness and proportionality?

- **EF $\implies$ Prop**
- **Prop $\implies$ EF**
- **Incomparable**
- **Equivalent**

Respond at:

pollev.com/jtuckerfoltz255 or

bit.ly/jtfpoll or

text jtuckerfoltz255 to 37607

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Lemma (Alon, West, 1986)

A consensus halving always exists using at most n cuts.

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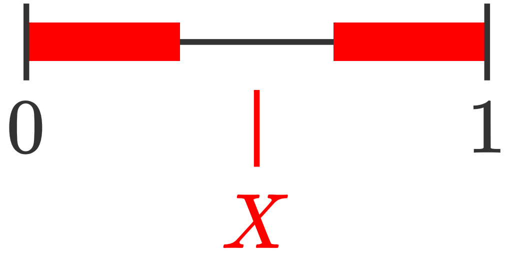

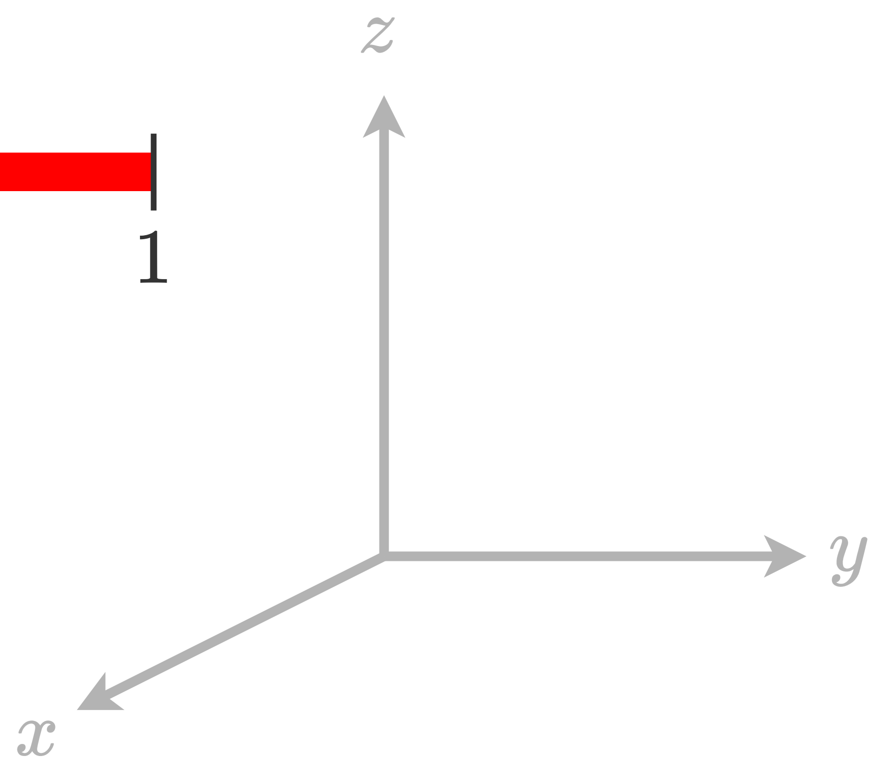

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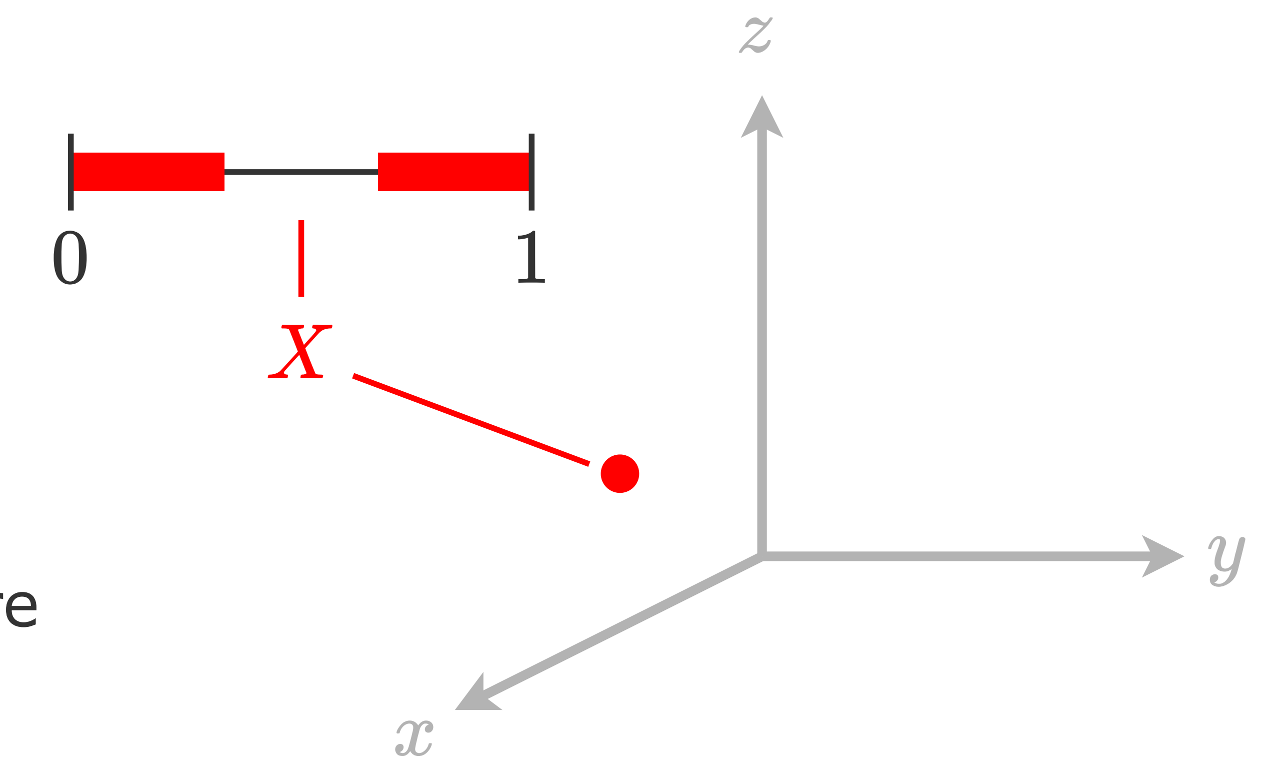

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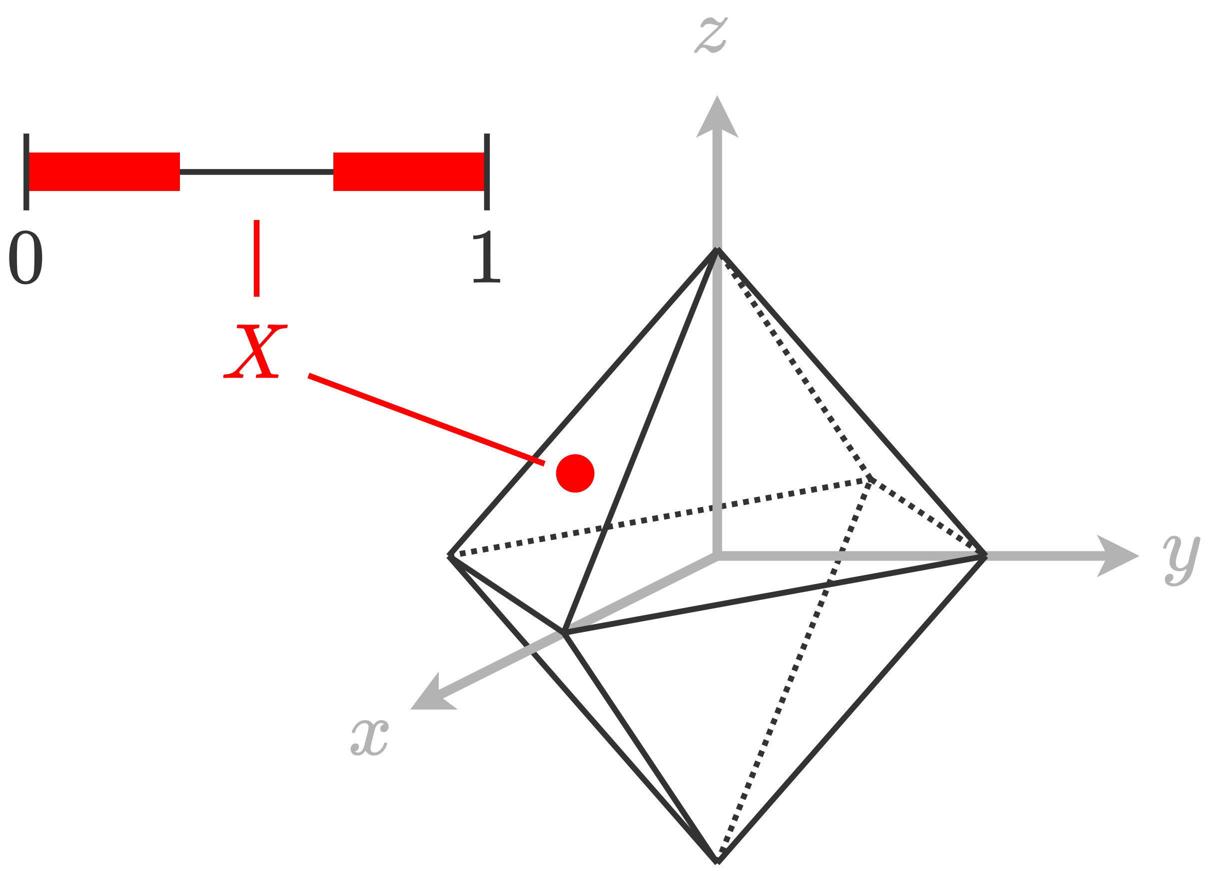

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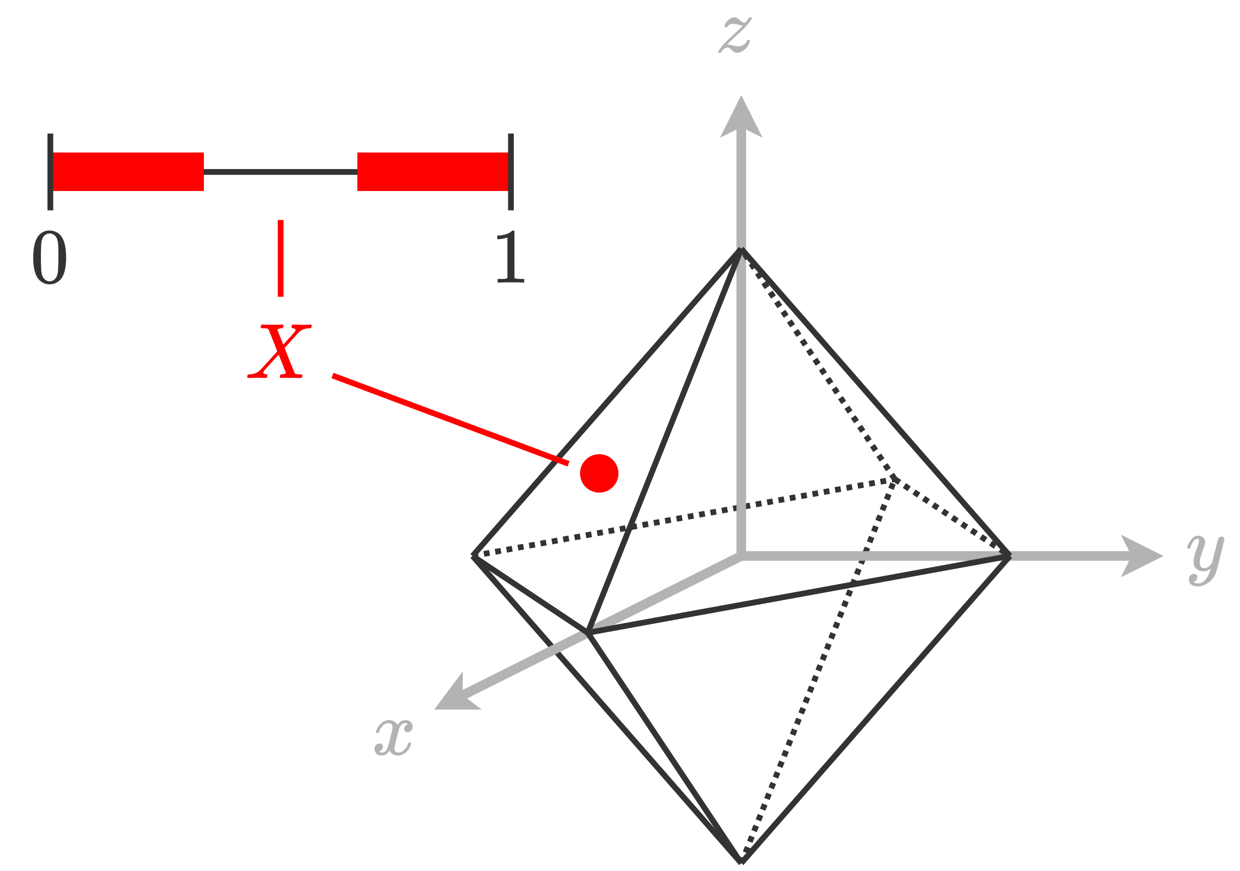

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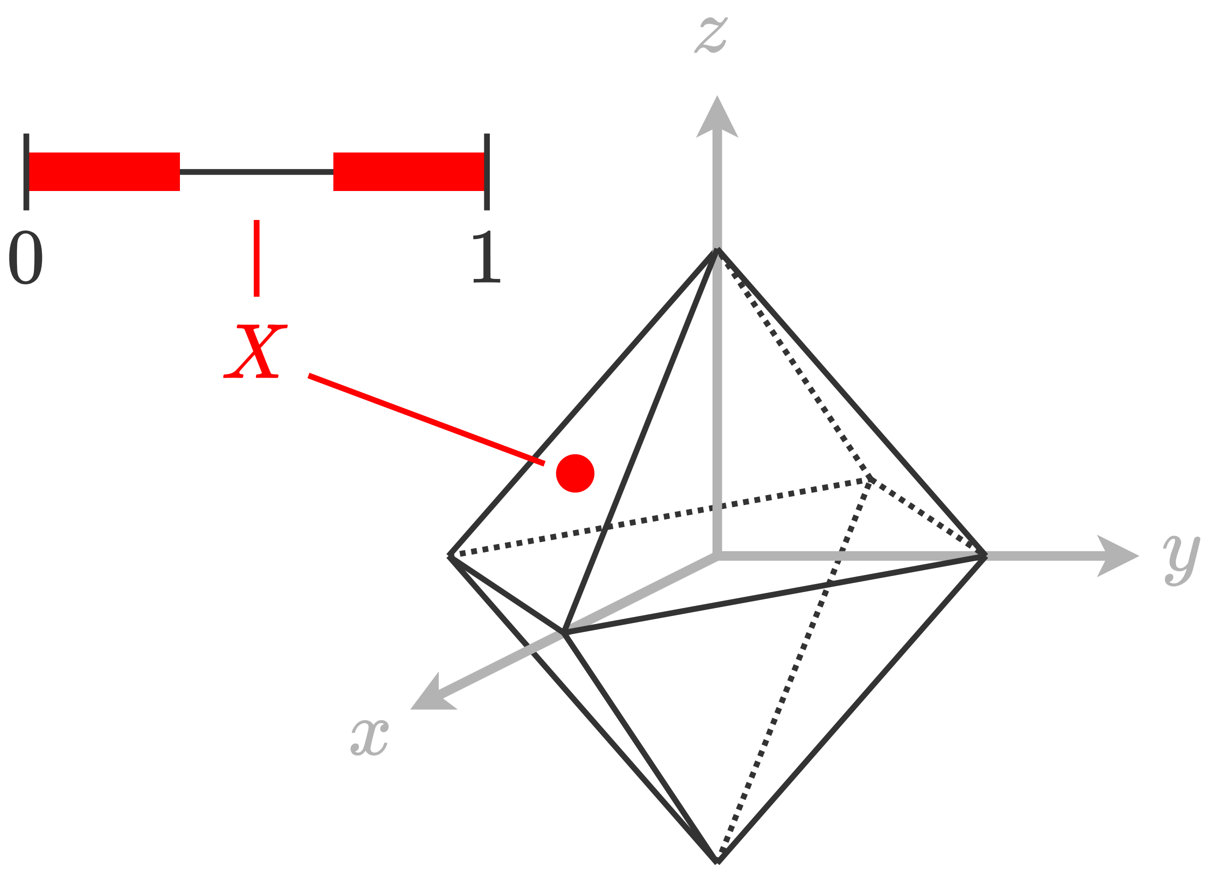

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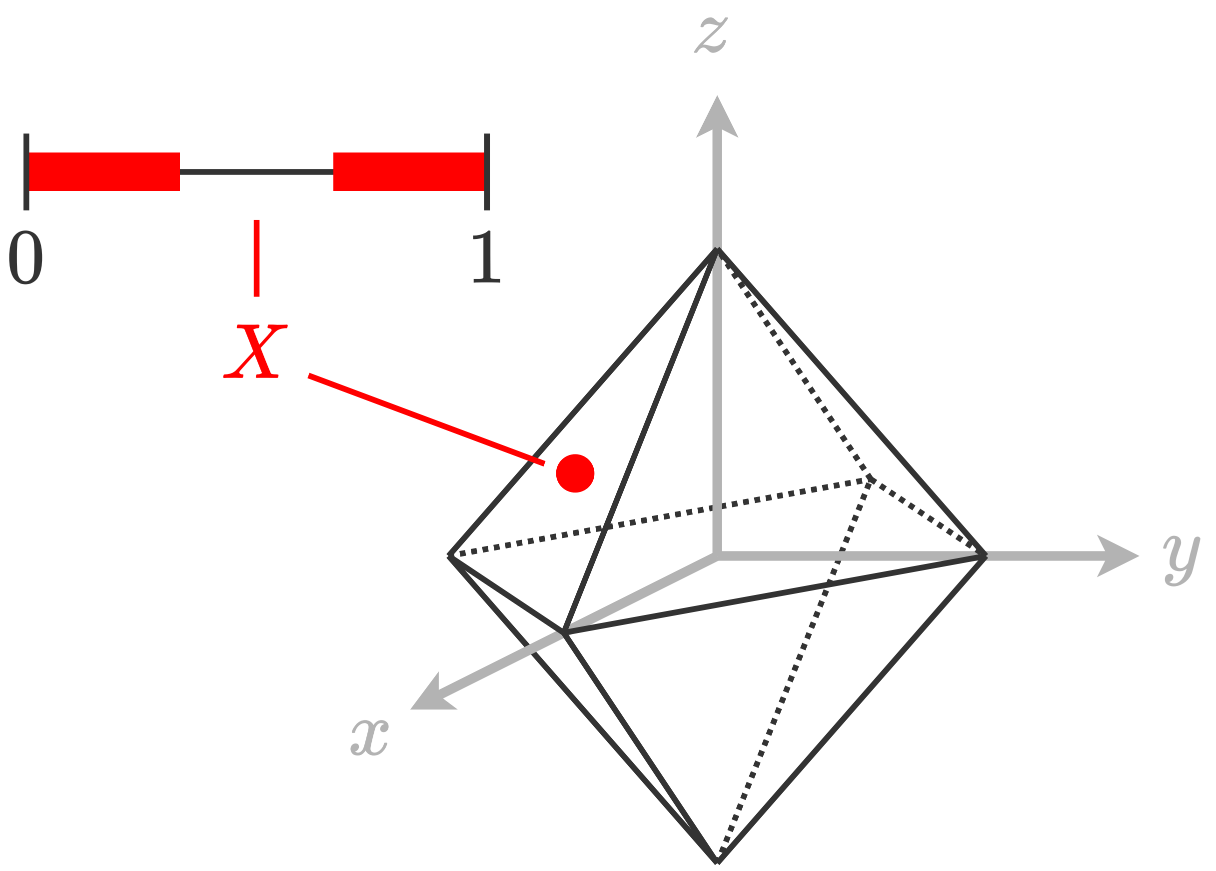

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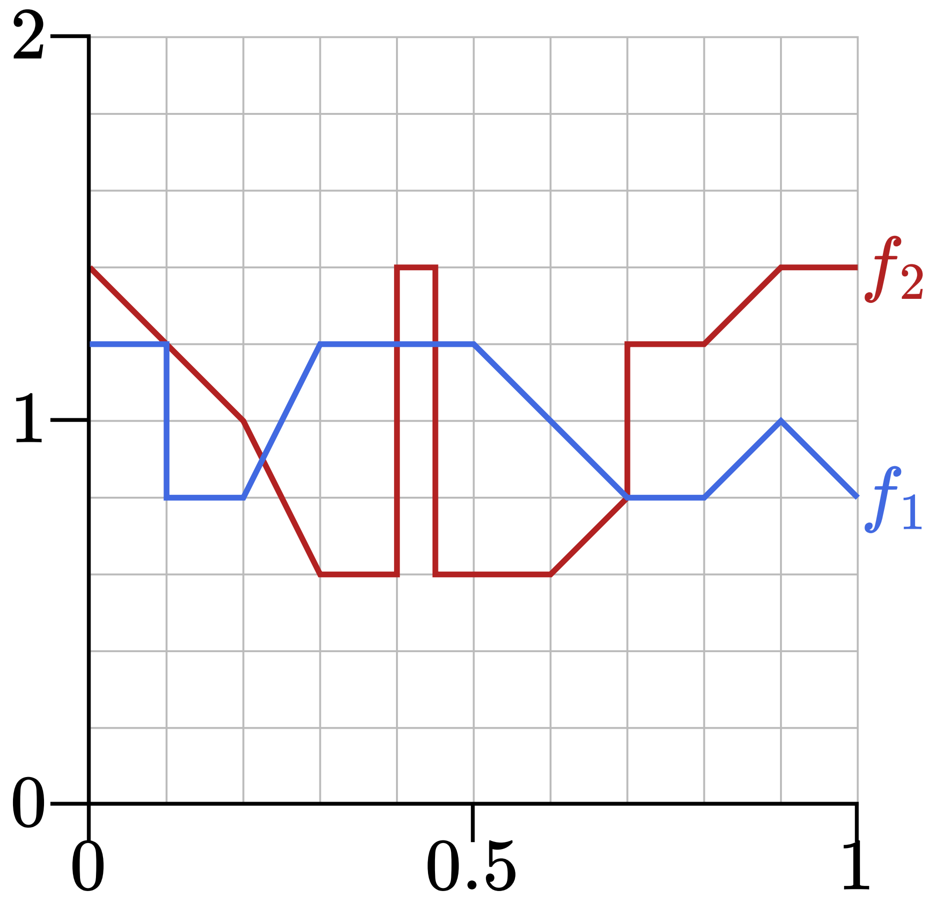

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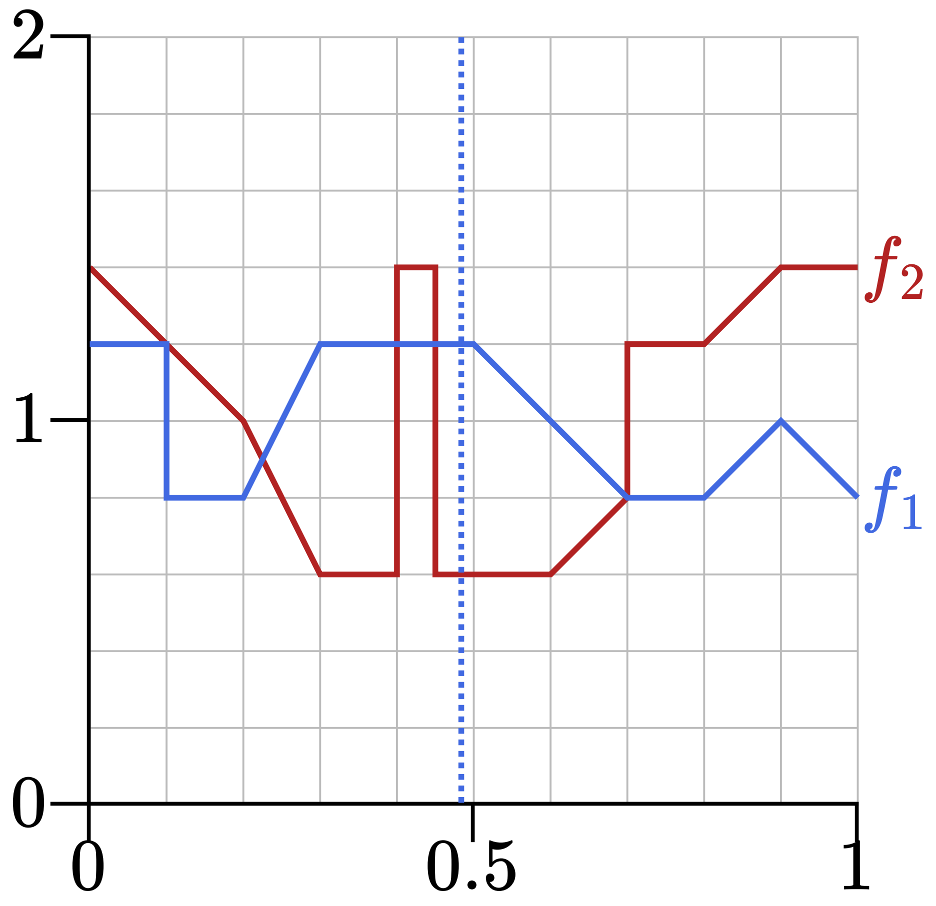

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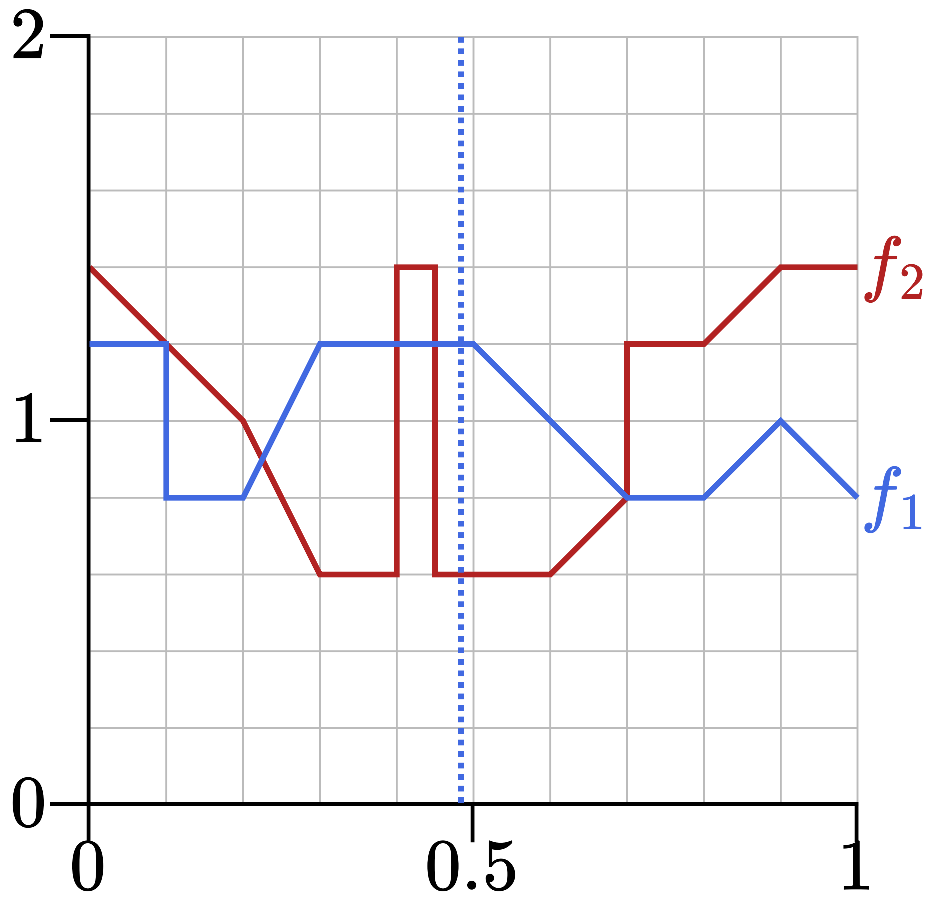

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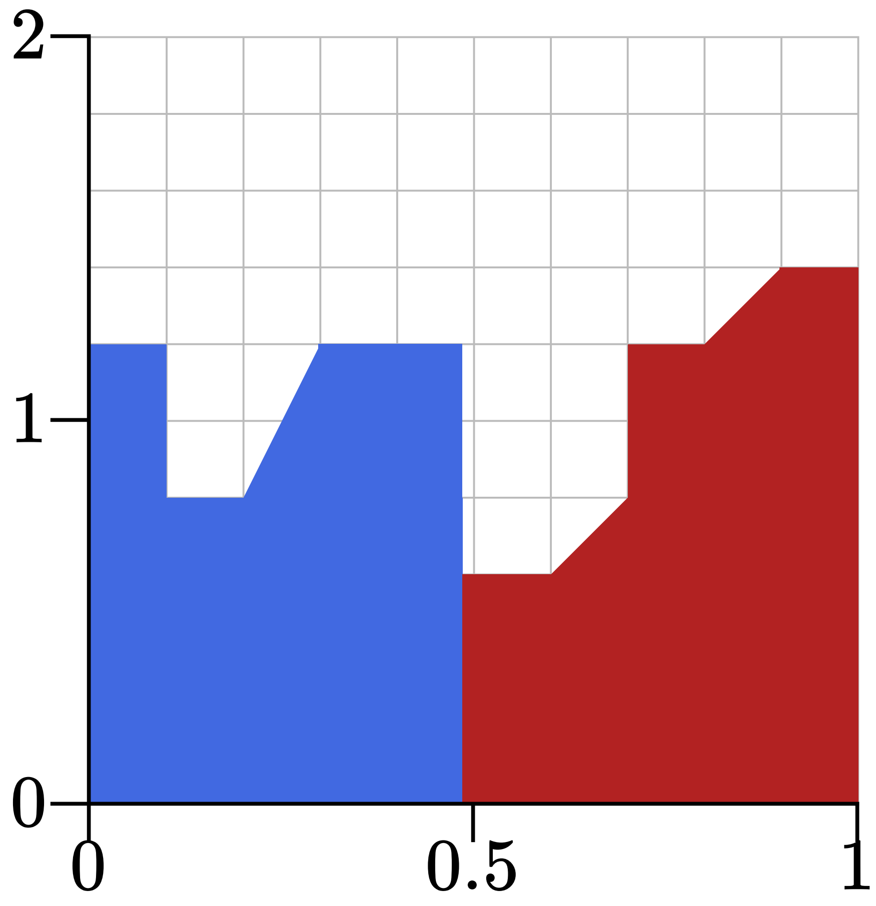

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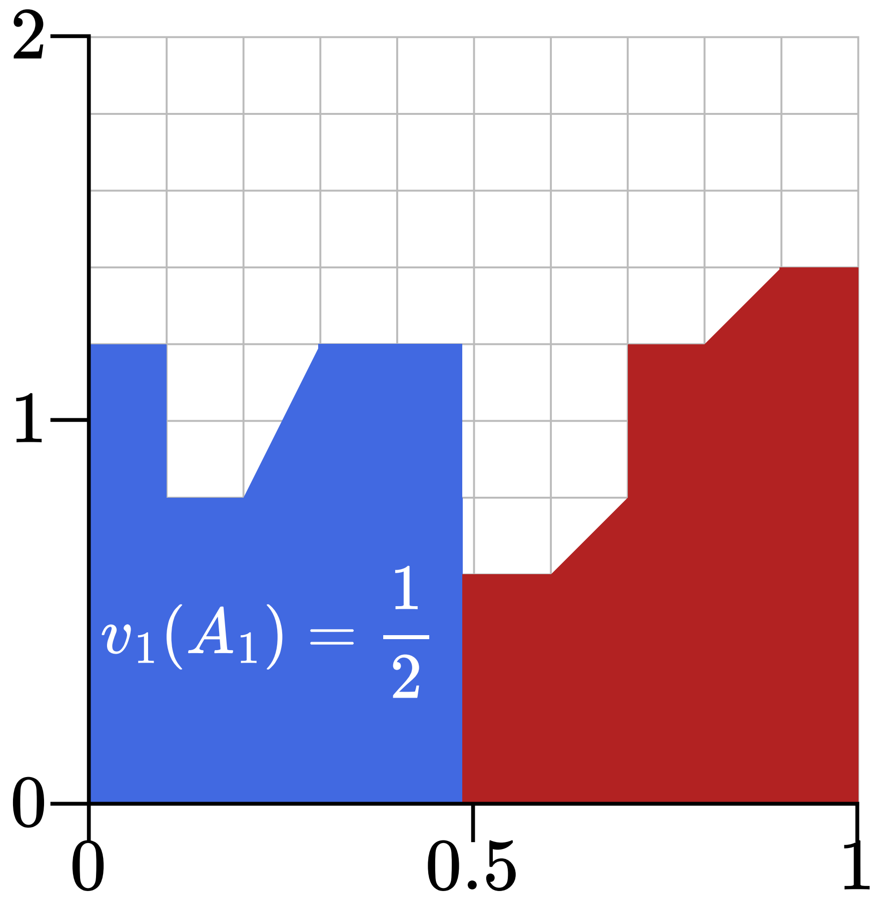

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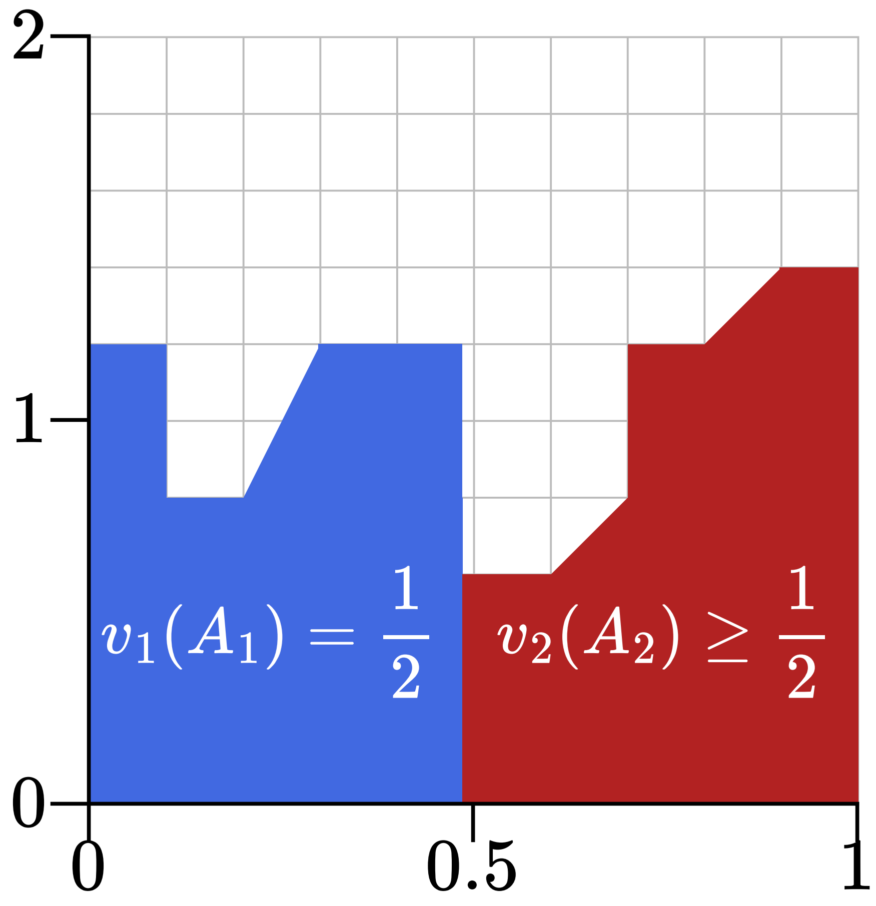

Query complexity of Proportionality

Theorem (Dubbins, Spanier, 1961)

The query complexity of proportionality is $O(n^2)$.

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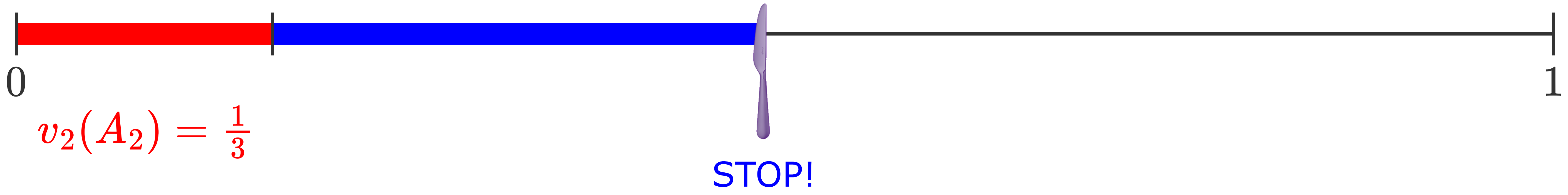

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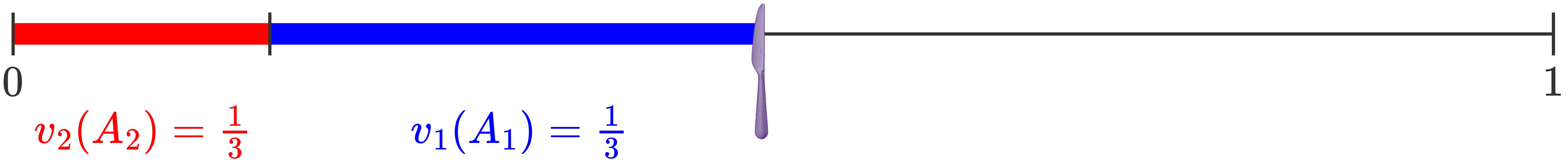

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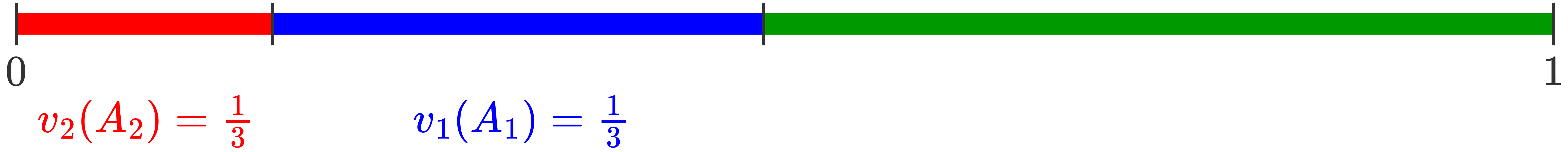

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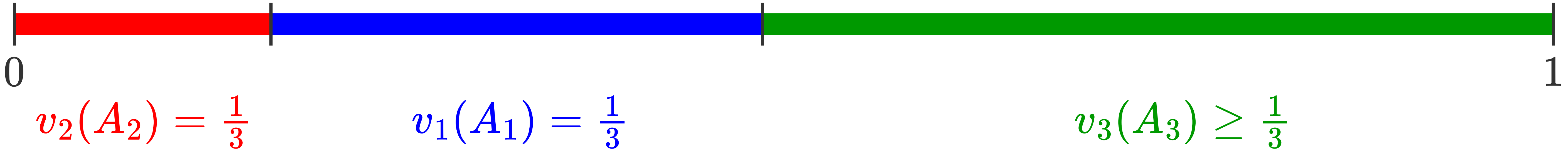

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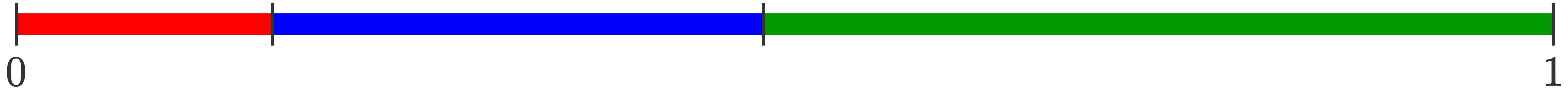

$v_2(A_2) = \frac{1}{3}$ $v_1(A_1) = \frac{1}{3}$ $v_3(A_3) \geq \frac{1}{3}$

Theorem

The query complexity of proportionality is $\Theta(n \log n)$.

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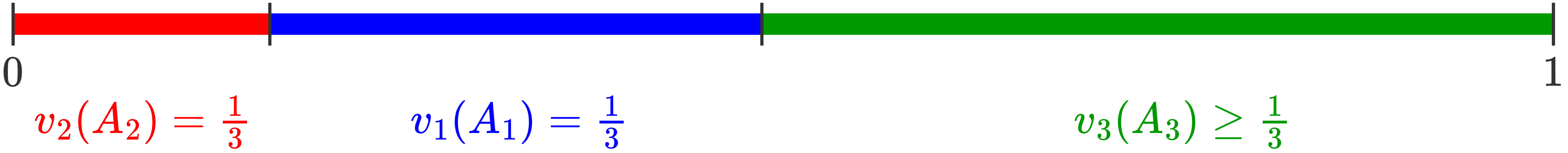

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Main idea behind algorithm (Even, Paz, 1984): Have every agent score the cake where they would divide in half, then cut at the median mark and recurse on both halves.

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n	Num queries of best known algorithm to find an EF allocation
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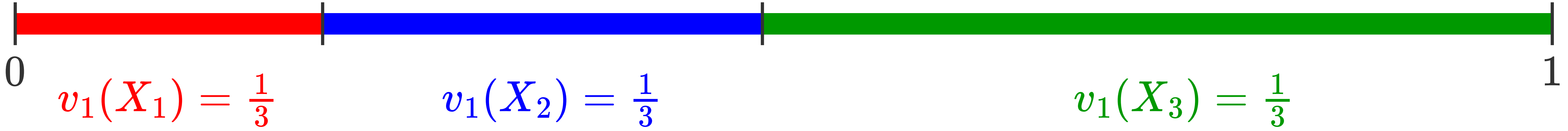

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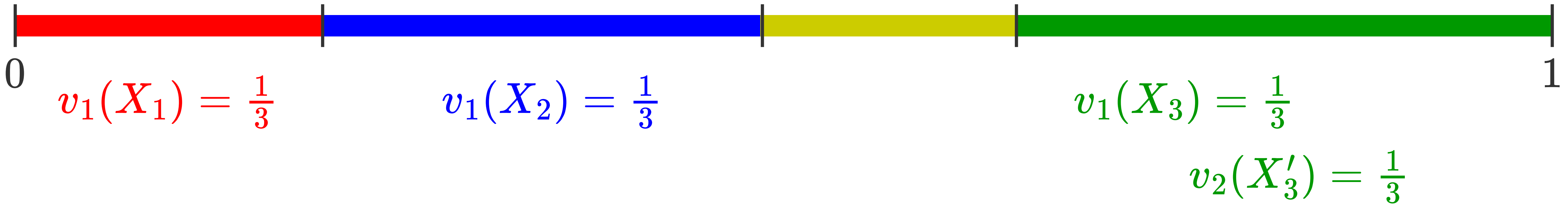

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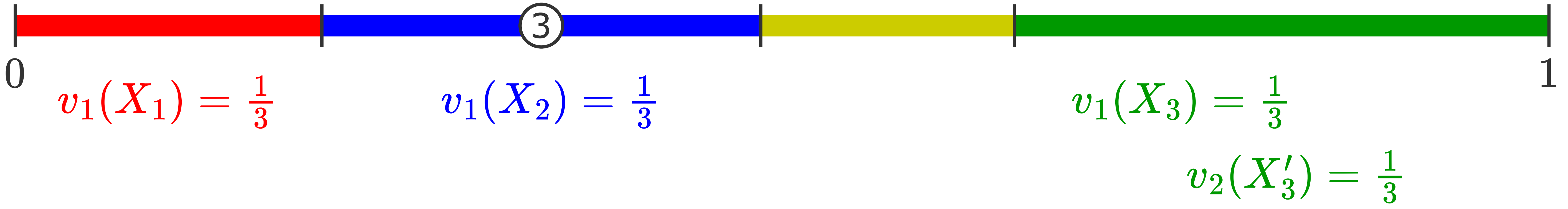

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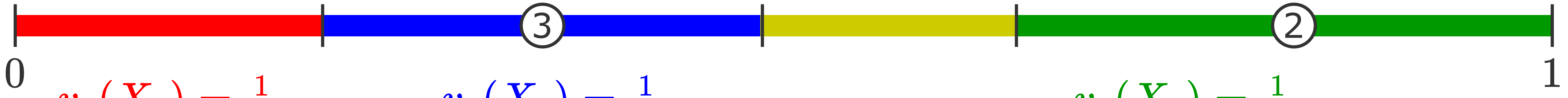

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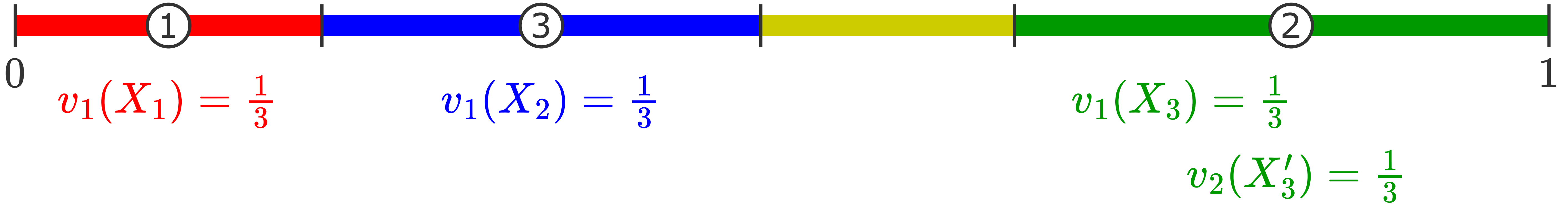

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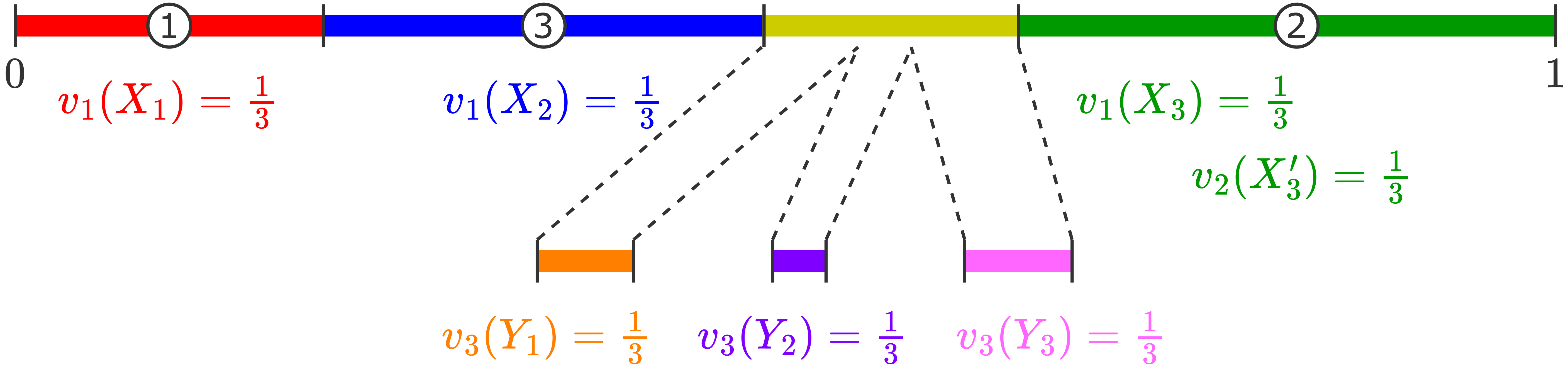

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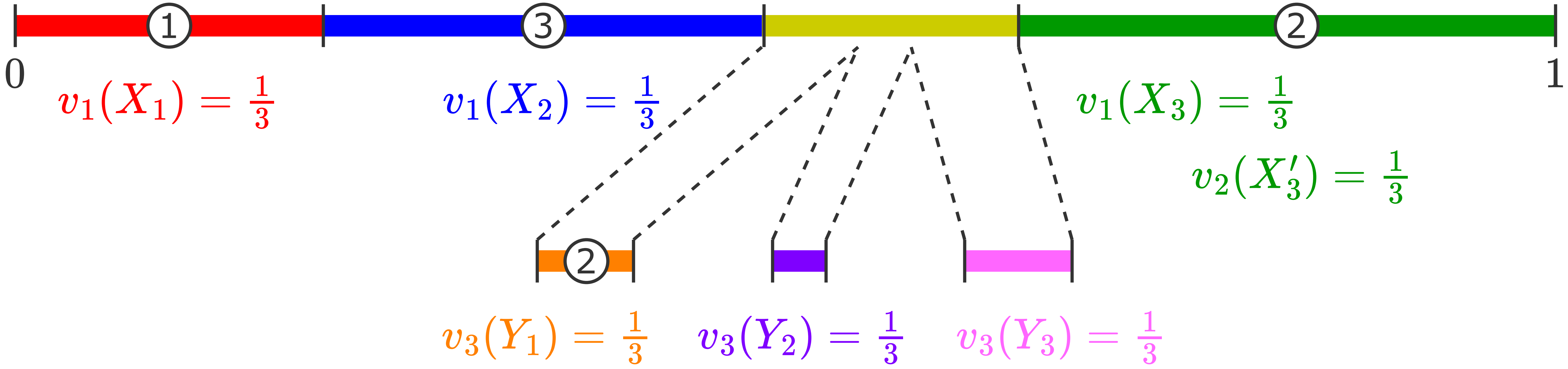

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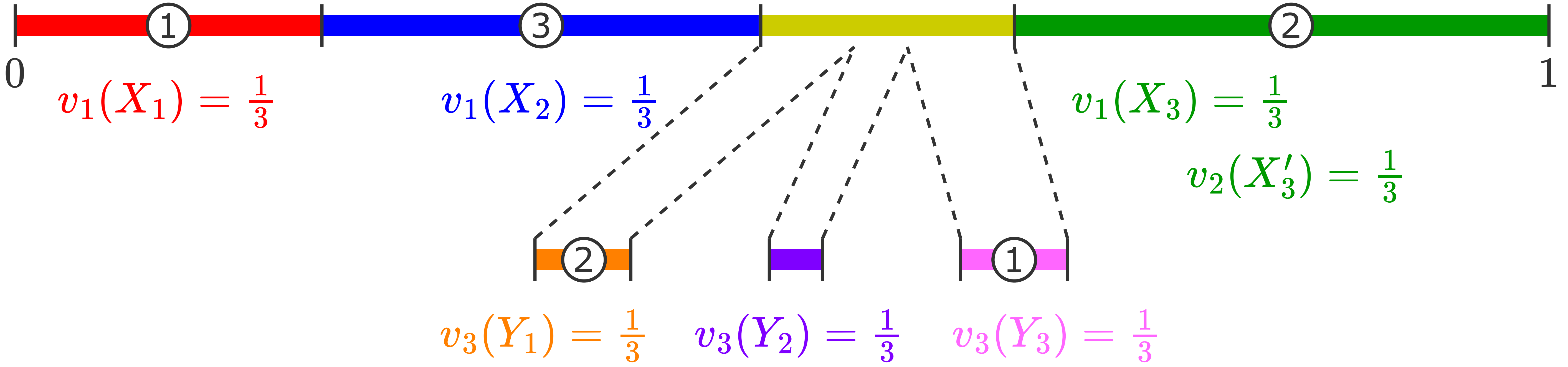

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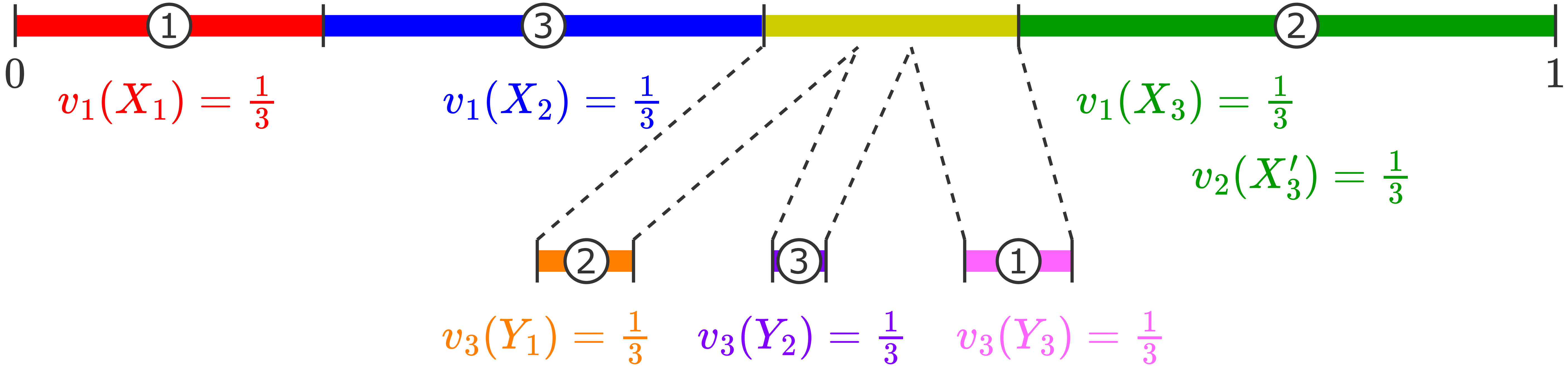

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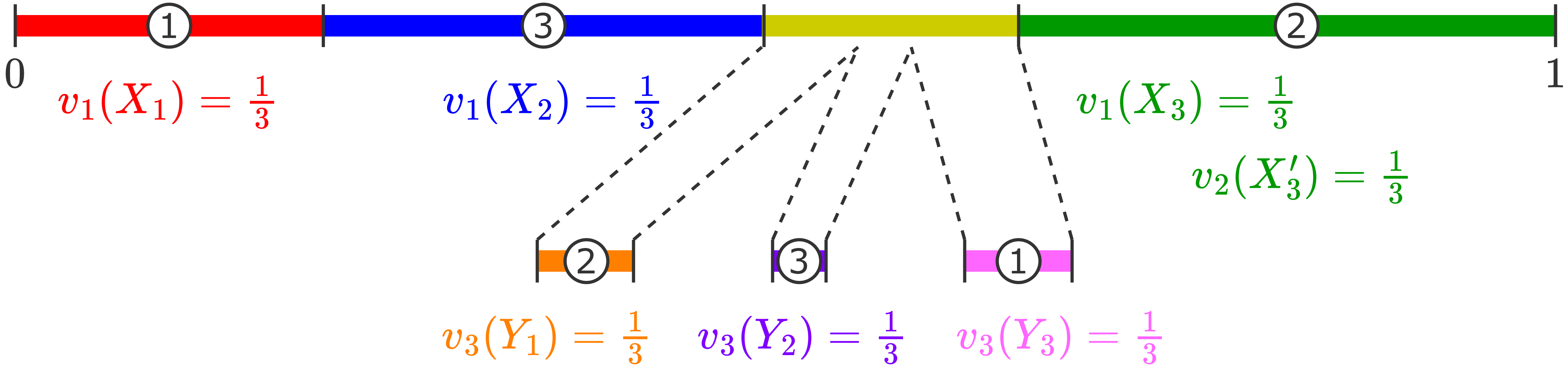

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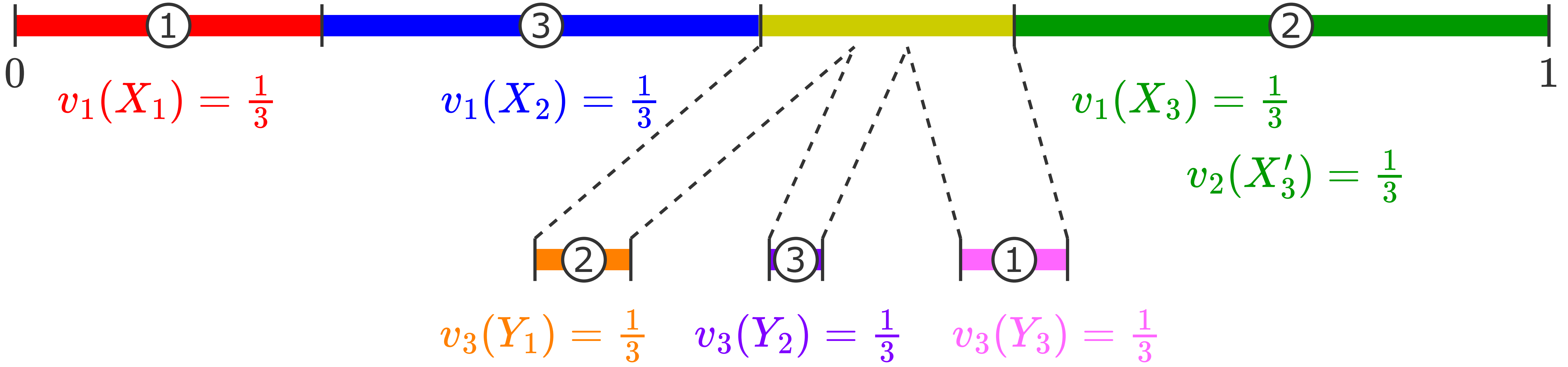

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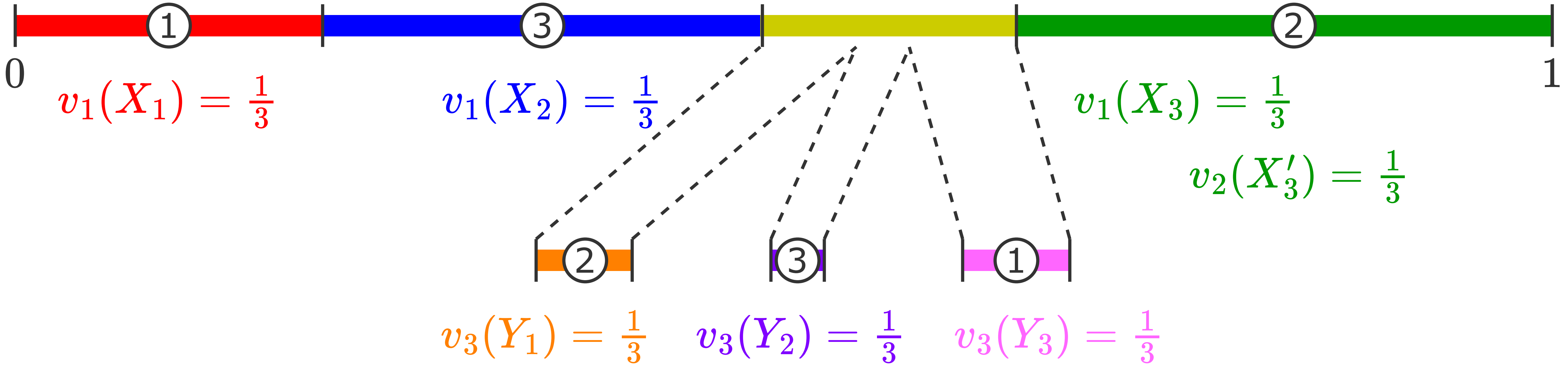
Open Question: Narrow the gap!



Strategy in Divide-and-Choose

Strategic setting (Tucker-Foltz, Zeckhauser, 2024):

- 2 players, n goods
- Divider's and Chooser's value for good i drawn from commonly-known distributions $\mathcal{G}_i^D$ and $\mathcal{G}_i^C$
- Each good is divisible

▸ **When there is no uncertainty (i.e. each distribution is supported on a single value), would you rather be Divider or Chooser?**

Respond at:

pollev.com/jtuckerfoltz255 or

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Strategy: Put 45 best goods in one pile, 55 worst in other.

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Divider versus Chooser utilities

Theorem (Tucker-Foltz, Zeckhauser, 2024)

Suppose both players' values for all n goods are drawn i.i.d. from a common distribution $\mathcal{D}$ with bounded, positive, nontrivial support. Then:

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Divider	$= 0.264$	0.375
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$\mathcal{D} = \mathcal{N}(1, 0.2)$