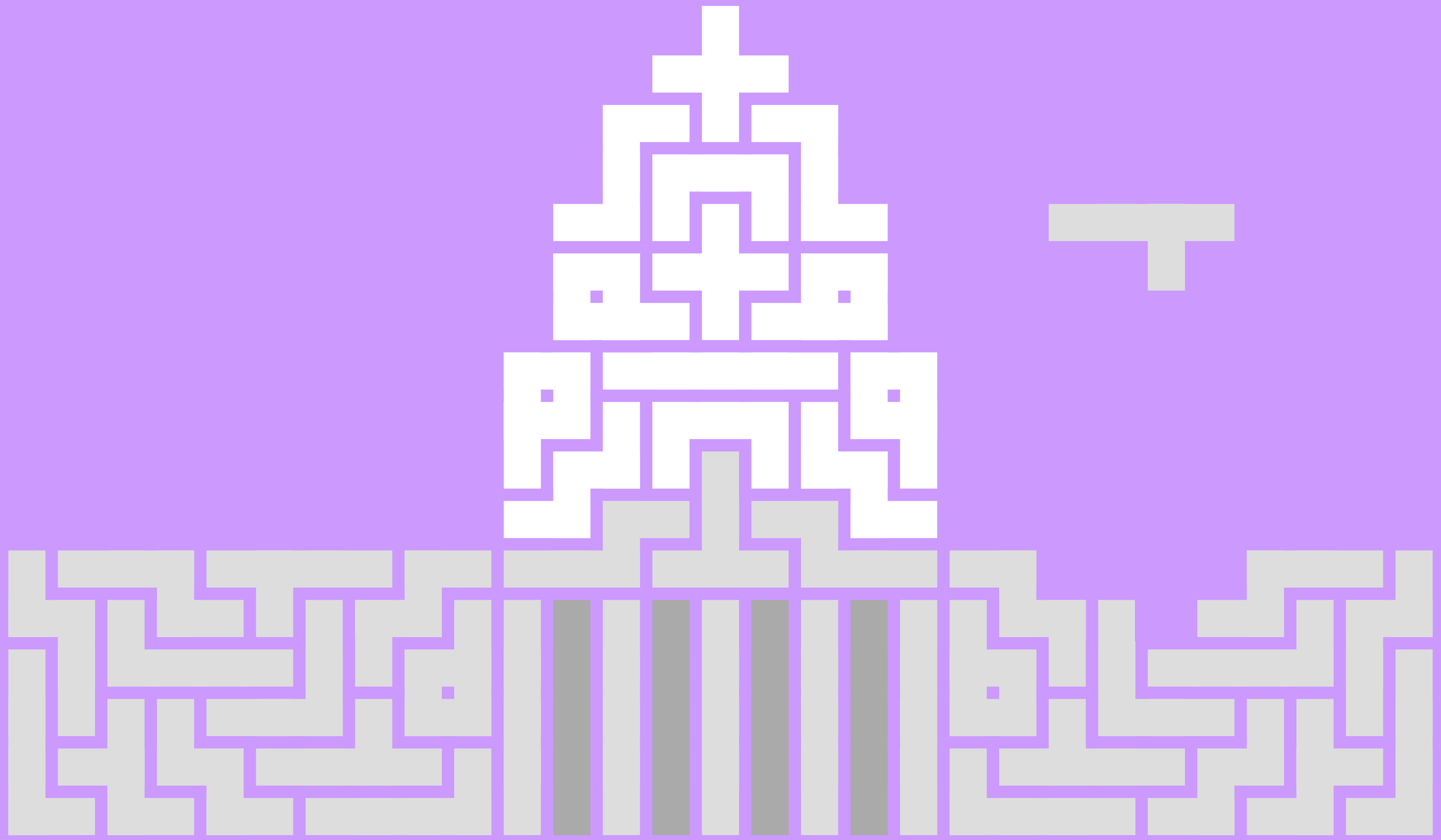


Algorithms For Democratic Decision-Making

Jamie Tucker-Foltz • Yale University • Spring 2026

Lecture 14: **Fair Division 3: Assignment Problems**

Announcements

Project proposals due tonight. Please submit a PDF describing:

- The main research question(s)
- Your proposed approach, i.e., how you model the problem, what existing frameworks from other papers you want to use.
- Possible theoretical result(s) you hope to prove and any ideas about how to prove them
- [Optional] Any data/experiments/empirical validation that might be relevant

There is no word/page limit. This is your chance to get feedback. The more detailed you are, the more I can advise!

Announcements

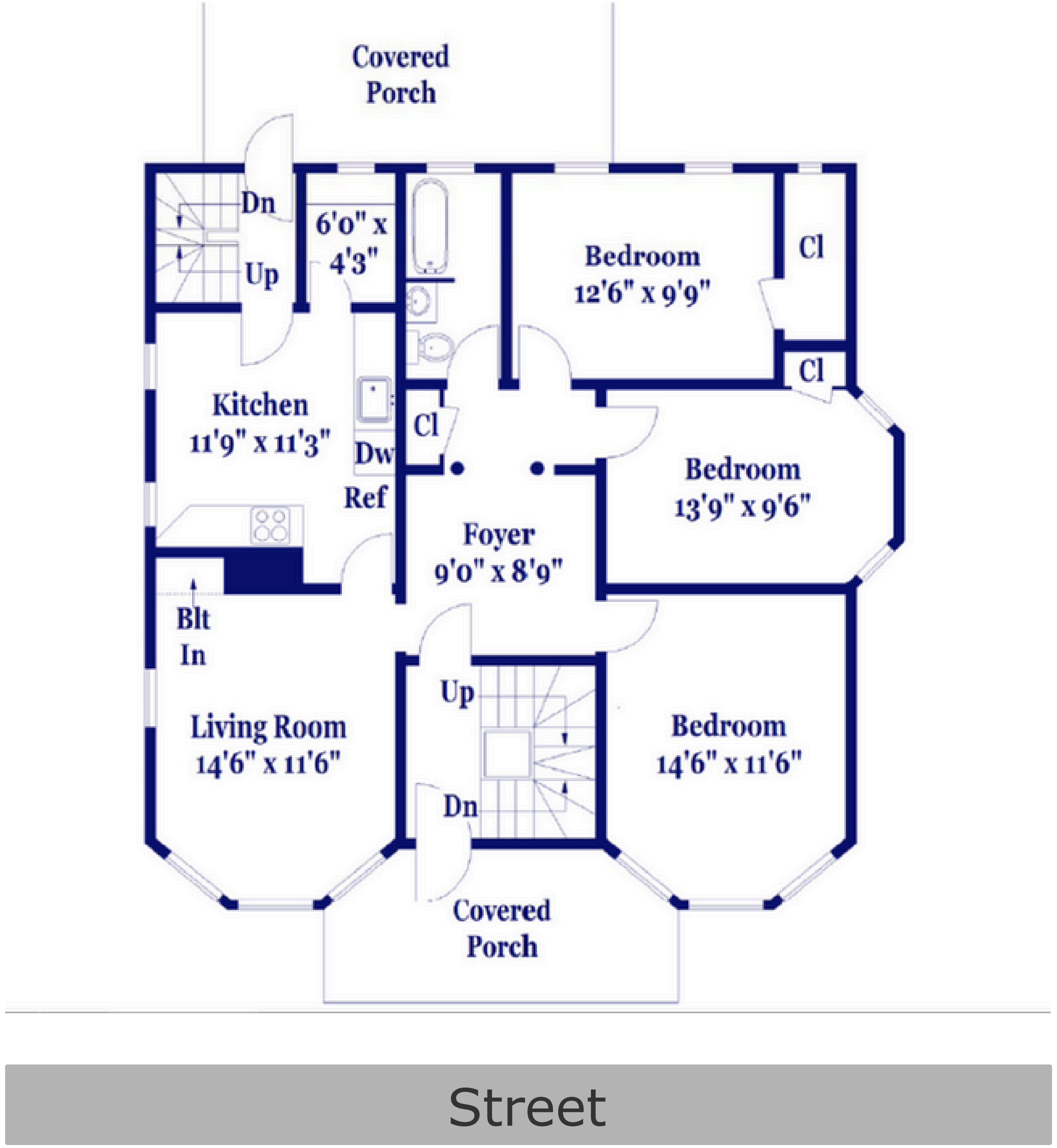
Project proposals due tonight. Please submit a PDF describing:

- The main research question(s)
- Your proposed approach, i.e., how you model the problem, what existing frameworks from other papers you want to use
- Possible theoretical result(s) you hope to prove and any ideas about how to prove them
- [Optional] Any data/experiments/empirical validation that might be relevant

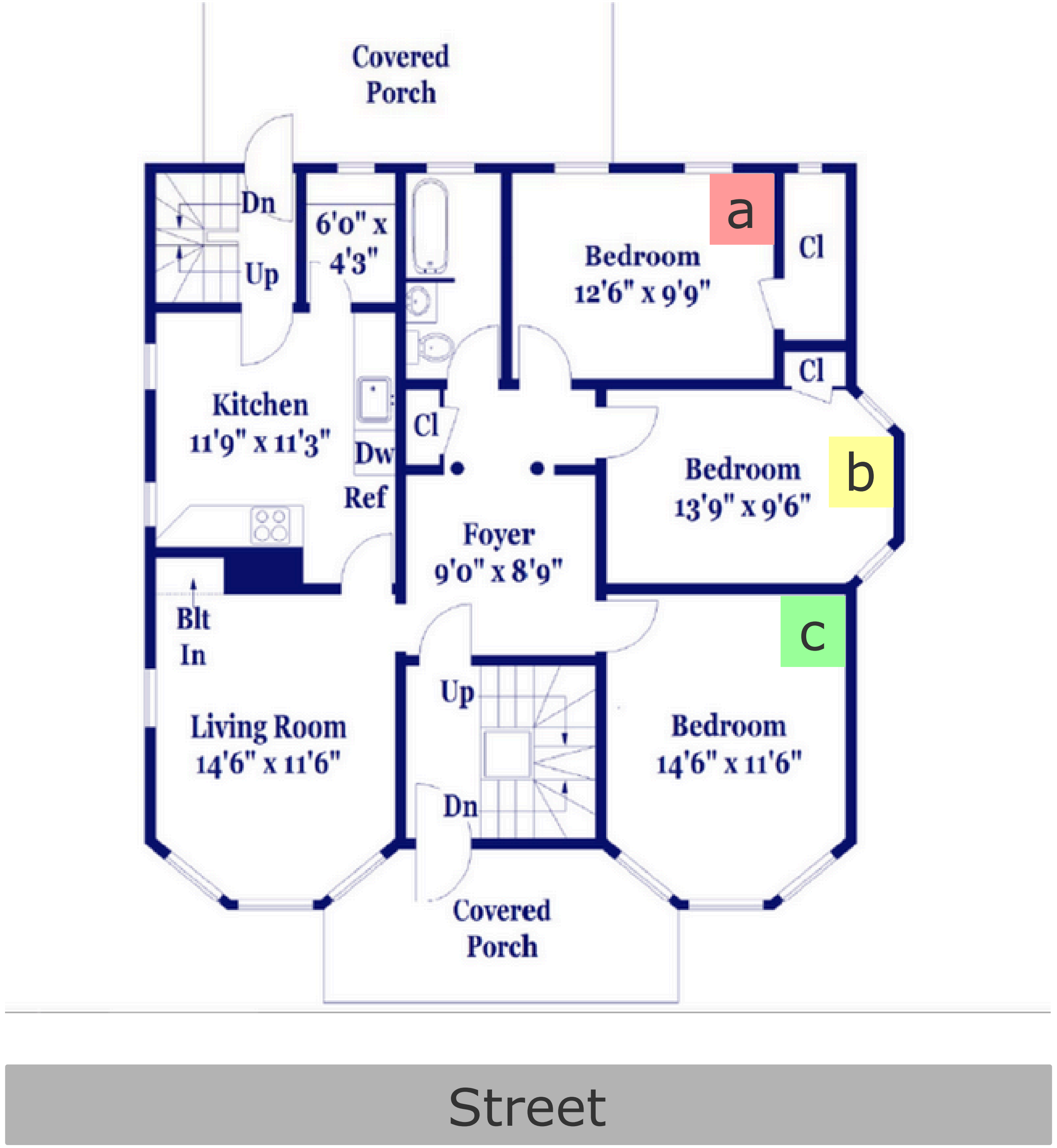
There is no word/page limit. This is your chance to get feedback. The more detailed you are, the more I can advise!

No class on Wednesday. I will be meeting individually with each project group in my office. Check Canvas for the schedule and arrive a few minutes before your time slot.

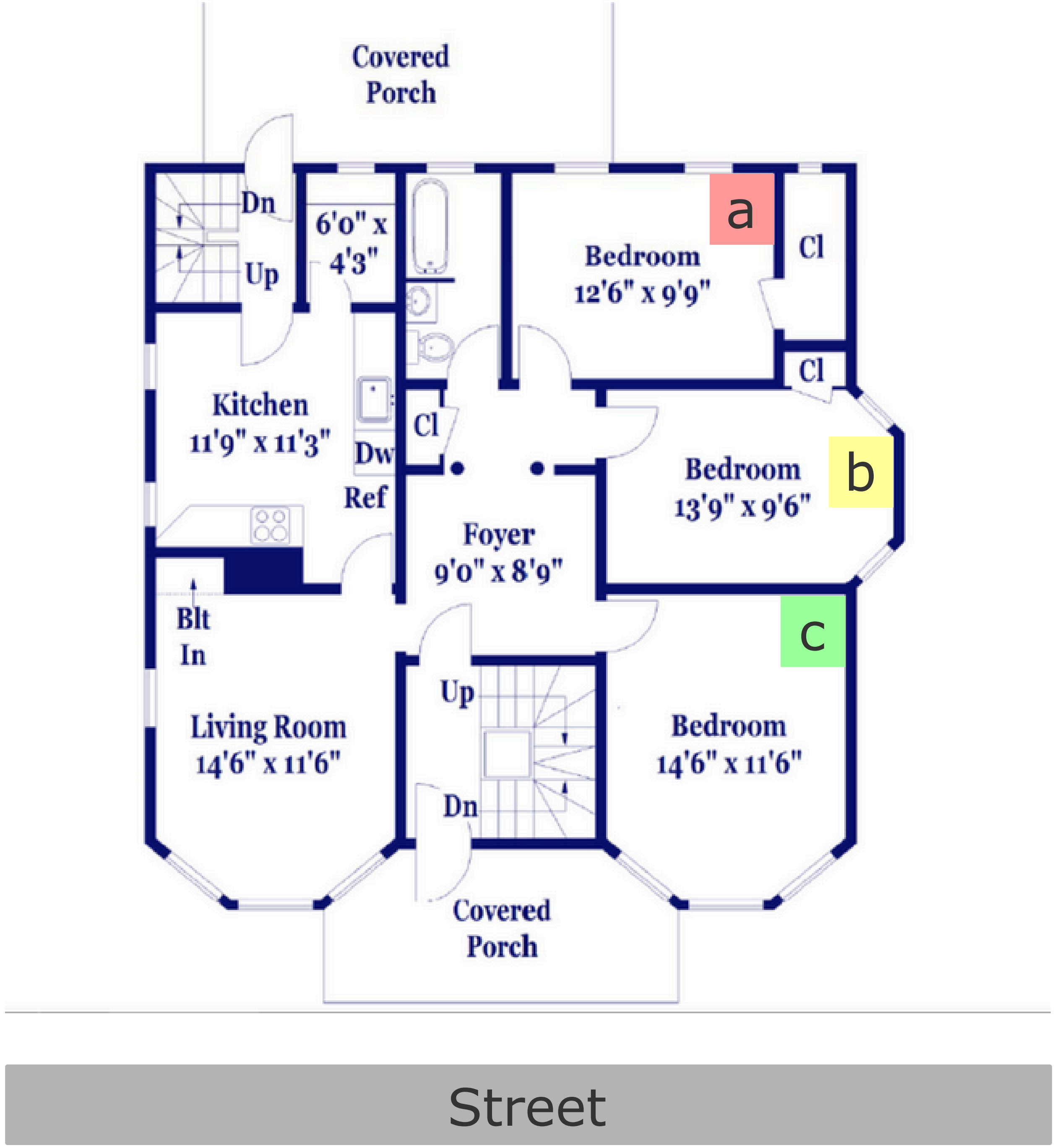
My apartment during grad school



My apartment during grad school

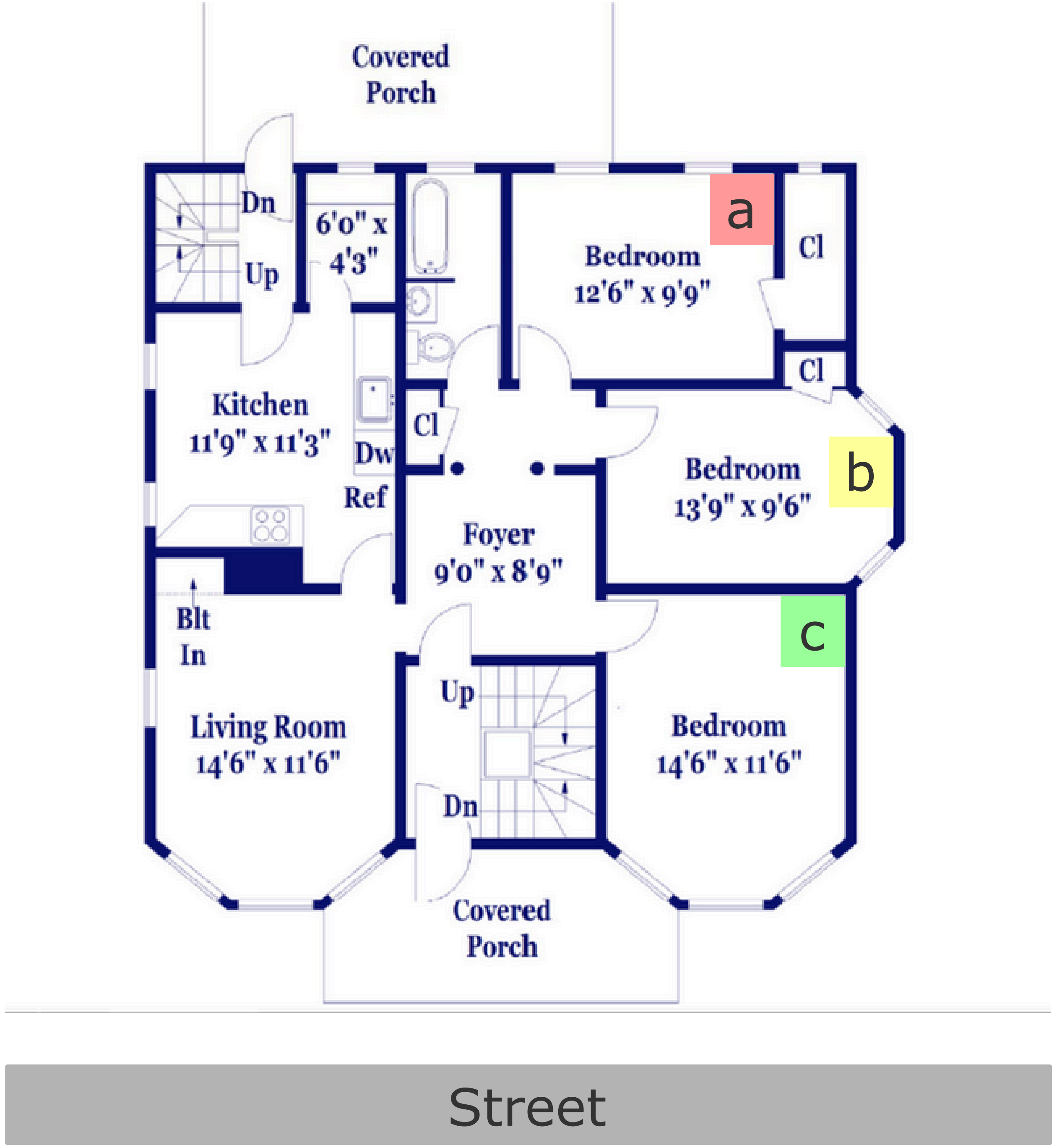


My apartment during grad school



Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

My apartment during grad school

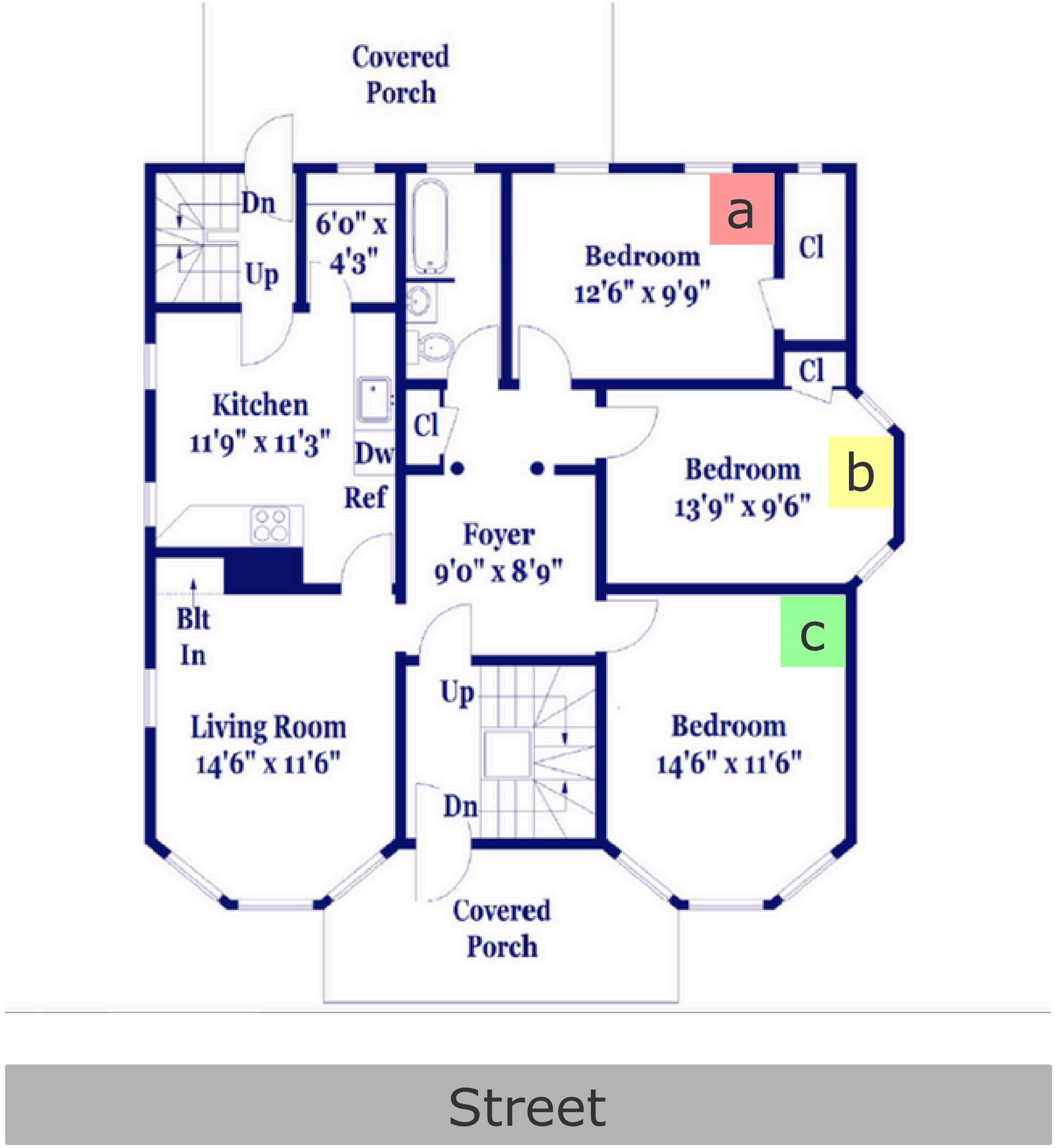


Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

Our decision-making process:

- Of the six possible assignments, only three were Pareto efficient: abc, bac, bca

My apartment during grad school

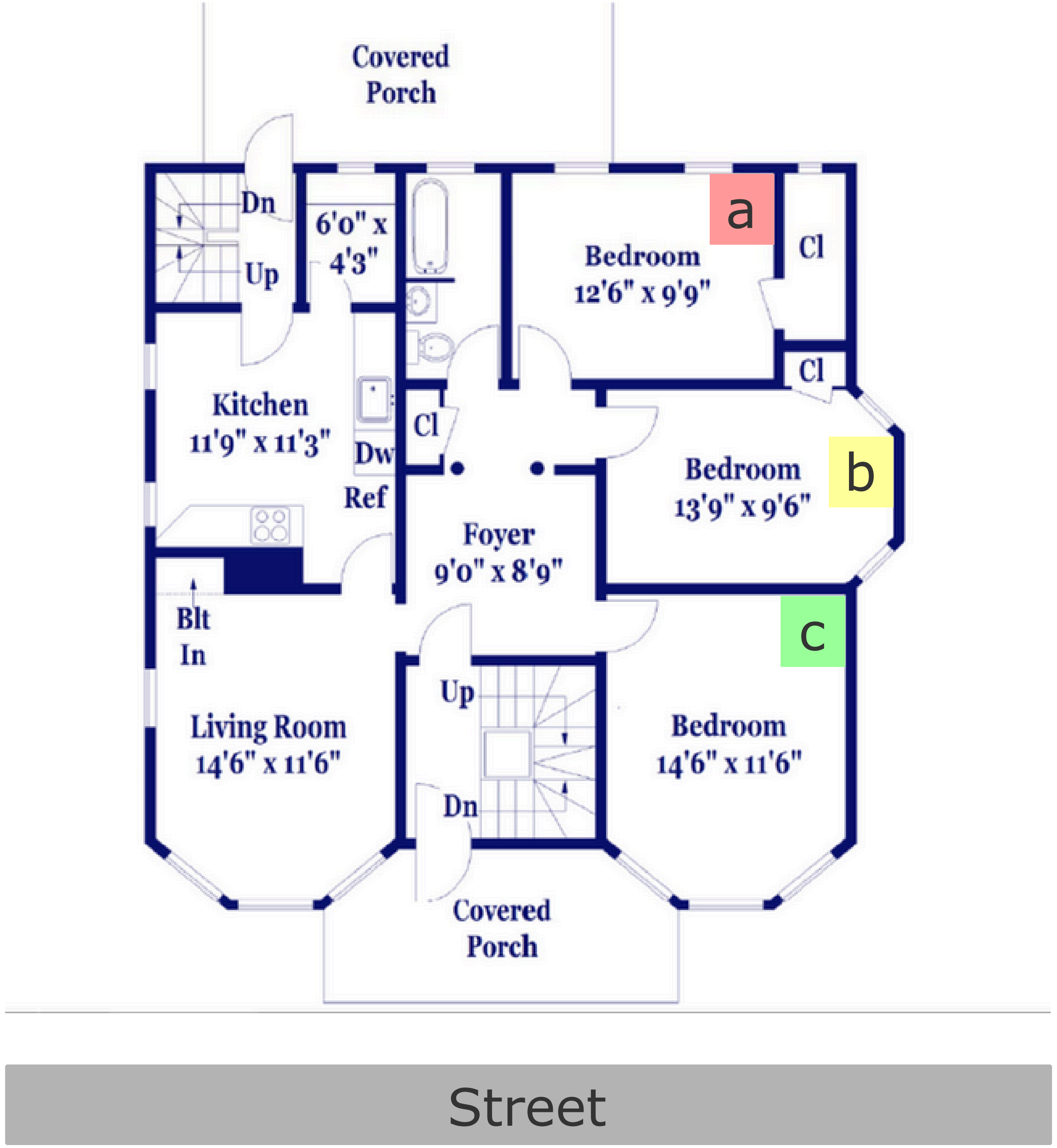


Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

Our decision-making process:

- Of the six possible assignments, only three were Pareto efficient: abc, bac, bca
- Of those, only two ensured no one got their least-favorite room: abc, bca

My apartment during grad school

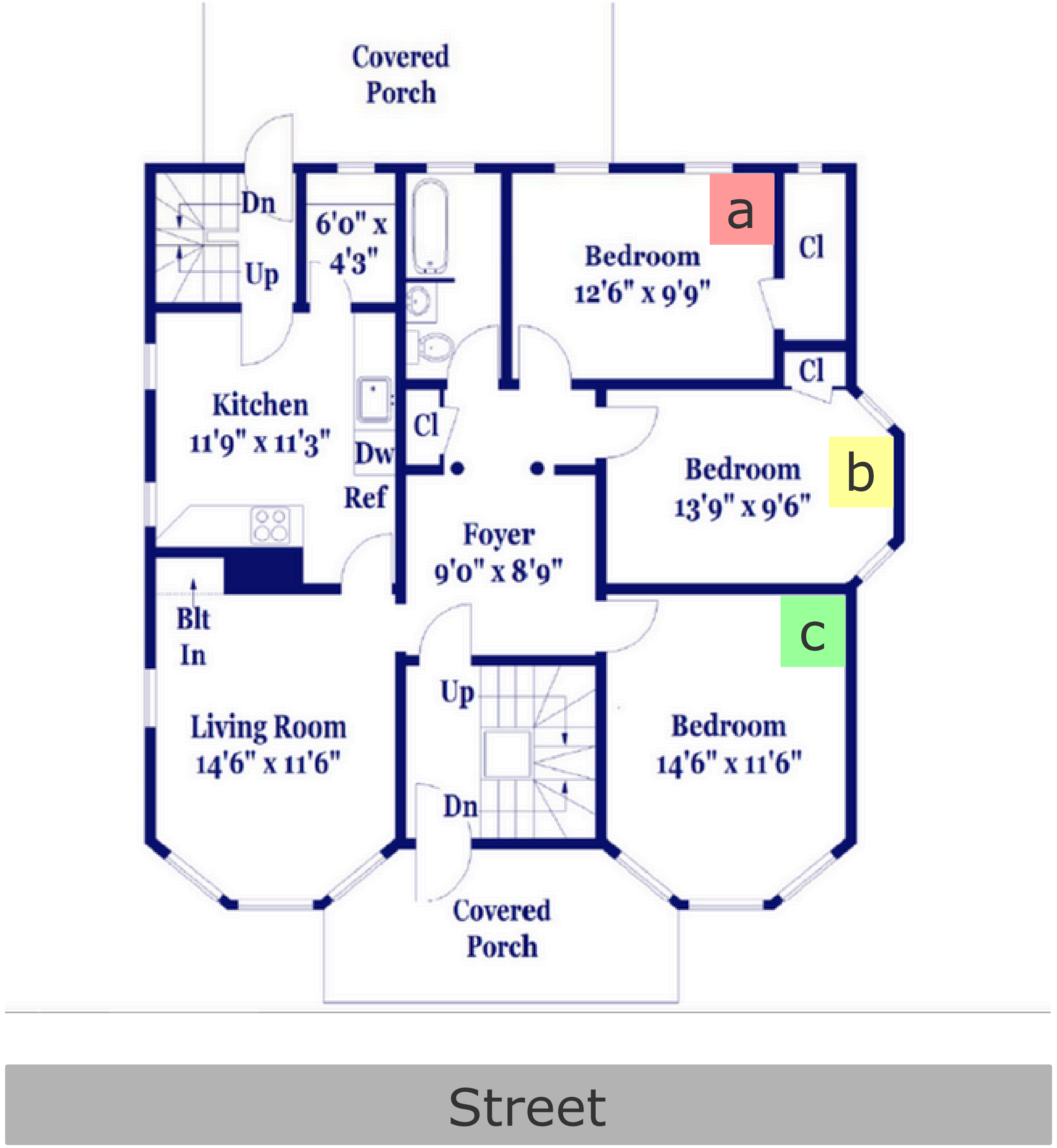


Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

Our decision-making process:

- Of the six possible assignments, only three were Pareto efficient: abc, bac, bca
- Of those, only two ensured no one got their least-favorite room: abc, bca
- Of those, it seemed more fair for only one player to get their second choice instead of two, so we went with abc

My apartment during grad school



Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

Our decision-making process:

- Of the six possible assignments, only three were Pareto efficient: abc, bac, bca
- Of those, only two ensured no one got their least-favorite room: abc, bca
- Of those, it seemed more fair for only one player to get their second choice instead of two, so we went with abc
- And we decided to divide rent equally

The rent division problem

Can we get an envy-free room assignment by dividing rent unequally?

The rent division problem

Can we get an envy-free room assignment by dividing rent unequally?

Theorem (Su, 1999)

An envy-free room assignment and rent division always exists under the following assumptions on preferences:

The rent division problem

Can we get an envy-free room assignment by dividing rent unequally?

Theorem (Su, 1999)

An envy-free room assignment and rent division always exists under the following assumptions on preferences:

(1) Each player prefers paying zero rent to paying nonzero rent, regardless of room.

The rent division problem

Can we get an envy-free room assignment by dividing rent unequally?

Theorem (Su, 1999)

An envy-free room assignment and rent division always exists under the following assumptions on preferences:

(1) Each player prefers paying zero rent to paying nonzero rent, regardless of room.

(2) The set of rent divisions for which a given player prefers a given room is closed.

The rent division problem

Can we get an envy-free room assignment by dividing rent unequally?

Theorem (Su, 1999)

An envy-free room assignment and rent division always exists under the following assumptions on preferences:

(1) Each player prefers paying zero rent to paying nonzero rent, regardless of room.

(2) The set of rent divisions for which a given player prefers a given room is closed.

Proof for n = 3. We will show that, for each player i, there exists an infinite sequence of rent divisions $r_i^1, r_i^2, r_i^3, \cdots \in \mathbb{R}^n_{\geq 0}$ such that:

The rent division problem

Can we get an envy-free room assignment by dividing rent unequally?

Theorem (Su, 1999)

An envy-free room assignment and rent division always exists under the following assumptions on preferences:

(1) Each player prefers paying zero rent to paying nonzero rent, regardless of room.

(2) The set of rent divisions for which a given player prefers a given room is closed.

Proof for n = 3. We will show that, for each player i, there exists an infinite sequence of rent divisions $r_i^1, r_i^2, r_i^3, \cdots \in \mathbb{R}^n_{\geq 0}$ such that:

(A) The limits $\lim_{t\to\infty} r_i^t$ exist and are equal for each player i.

The rent division problem

Can we get an envy-free room assignment by dividing rent unequally?

Theorem (Su, 1999)

An envy-free room assignment and rent division always exists under the following assumptions on preferences:

(1) Each player prefers paying zero rent to paying nonzero rent, regardless of room.

(2) The set of rent divisions for which a given player prefers a given room is closed.

Proof for n = 3. We will show that, for each player i, there exists an infinite sequence of rent divisions $r_i^1, r_i^2, r_i^3, \cdots \in \mathbb{R}^n_{\geq 0}$ such that:

(A) The limits $\lim_{t\to\infty} r_i^t$ exist and are equal for each player i.

(B) For all $i, j \in [n]$, the room i prefers at r_i^t is different from the room j prefers at r_j^t.

The rent division problem

Can we get an envy-free room assignment by dividing rent unequally?

Theorem (Su, 1999)

An envy-free room assignment and rent division always exists under the following assumptions on preferences:

(1) Each player prefers paying zero rent to paying nonzero rent, regardless of room.

(2) The set of rent divisions for which a given player prefers a given room is closed.

Proof for n = 3. We will show that, for each player i, there exists an infinite sequence of rent divisions $r_i^1, r_i^2, r_i^3, \cdots \in \mathbb{R}^n_{\geq 0}$ such that:

(A) The limits $\lim_{t\to\infty} r_i^t$ exist and are equal for each player i.

(B) For all $i, j \in [n]$, the room i prefers at r_i^t is different from the room j prefers at r_j^t.

By taking a subsequence, we can ensure that each player prefers the same room across different indices t.

The rent division problem

Can we get an envy-free room assignment by dividing rent unequally?

Theorem (Su, 1999)

An envy-free room assignment and rent division always exists under the following assumptions on preferences:

(1) Each player prefers paying zero rent to paying nonzero rent, regardless of room.

(2) The set of rent divisions for which a given player prefers a given room is closed.

Proof for n = 3. We will show that, for each player i, there exists an infinite sequence of rent divisions $r_i^1, r_i^2, r_i^3, \cdots \in \mathbb{R}^n_{\geq 0}$ such that:

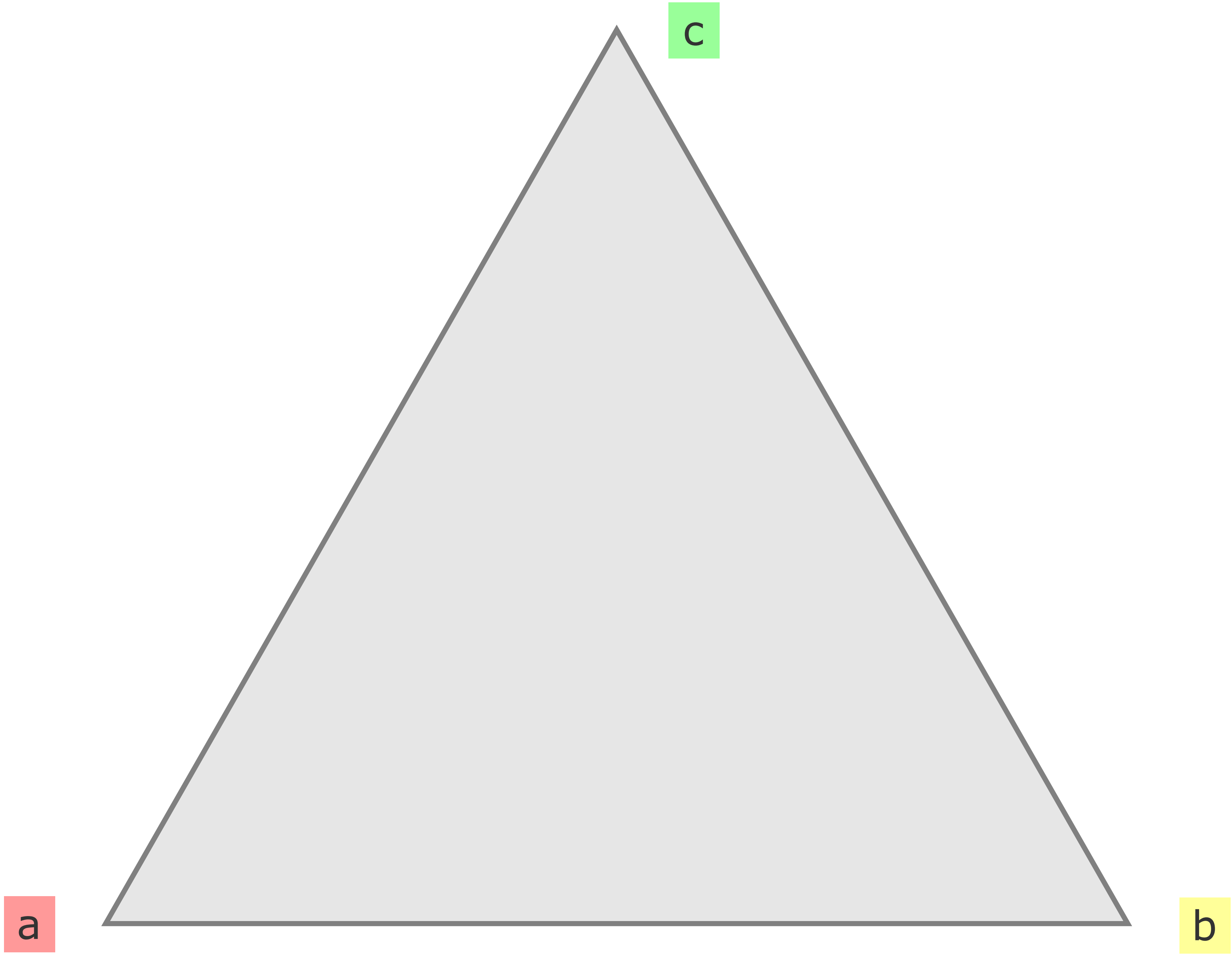
(A) The limits $\lim_{t\to\infty} r_i^t$ exist and are equal for each player i.

(B) For all $i, j \in [n]$, the room i prefers at r_i^t is different from the room j prefers at r_j^t.

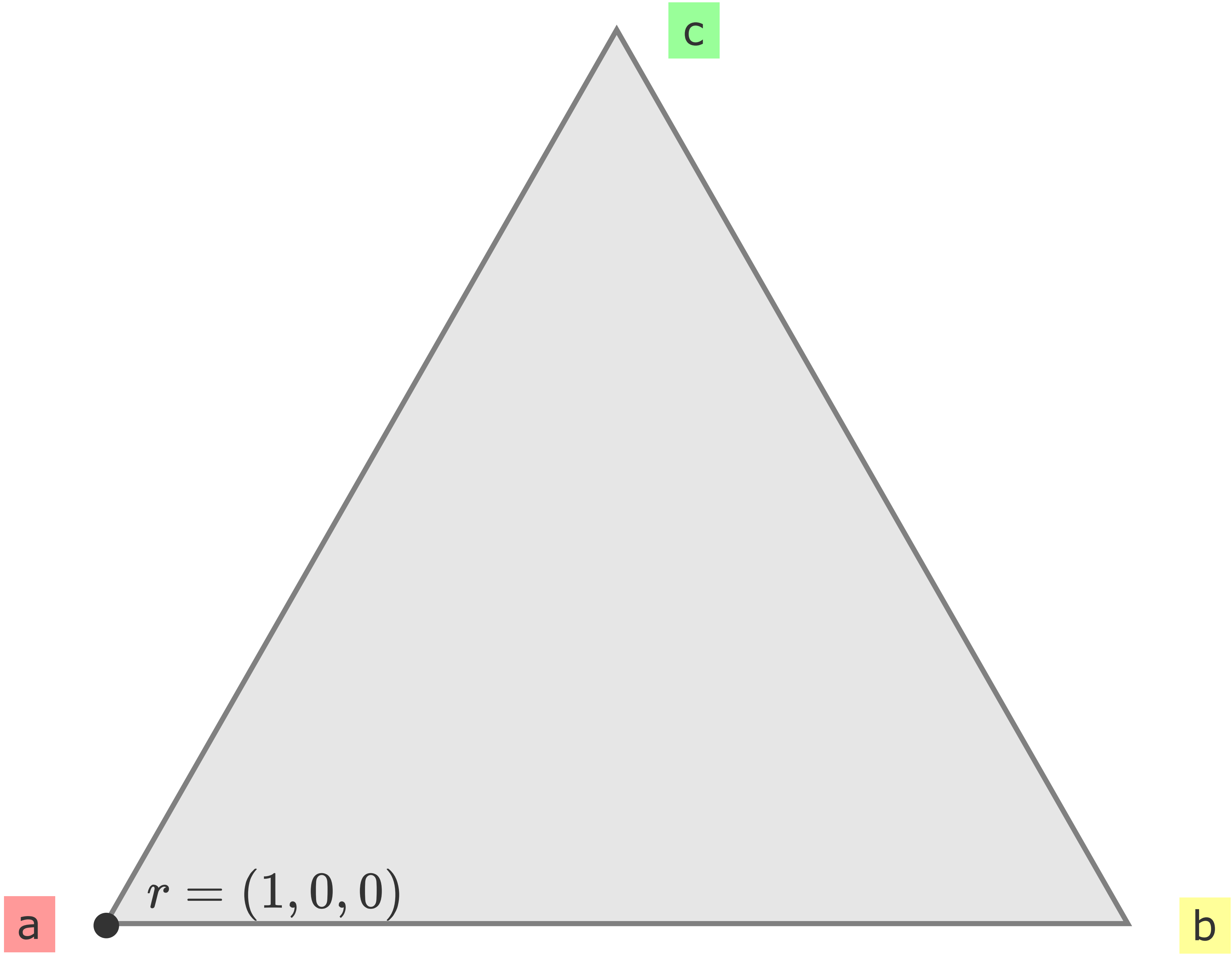
By taking a subsequence, we can ensure that each player prefers the same room across different indices t.

Then (2) $\implies$ the limiting rent division is envy-free under that assignment.

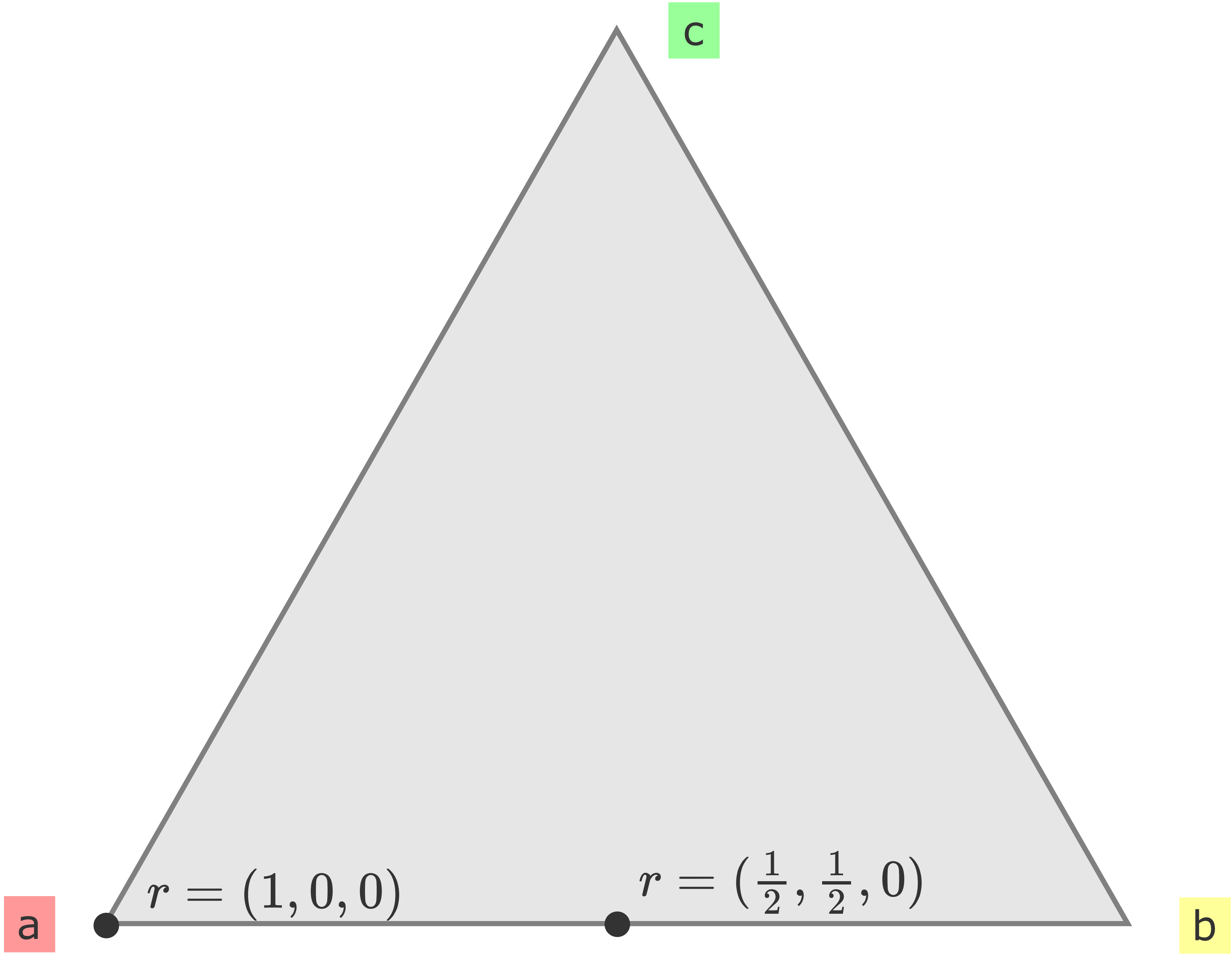
Proof via Sperner's Lemma (n = 3)



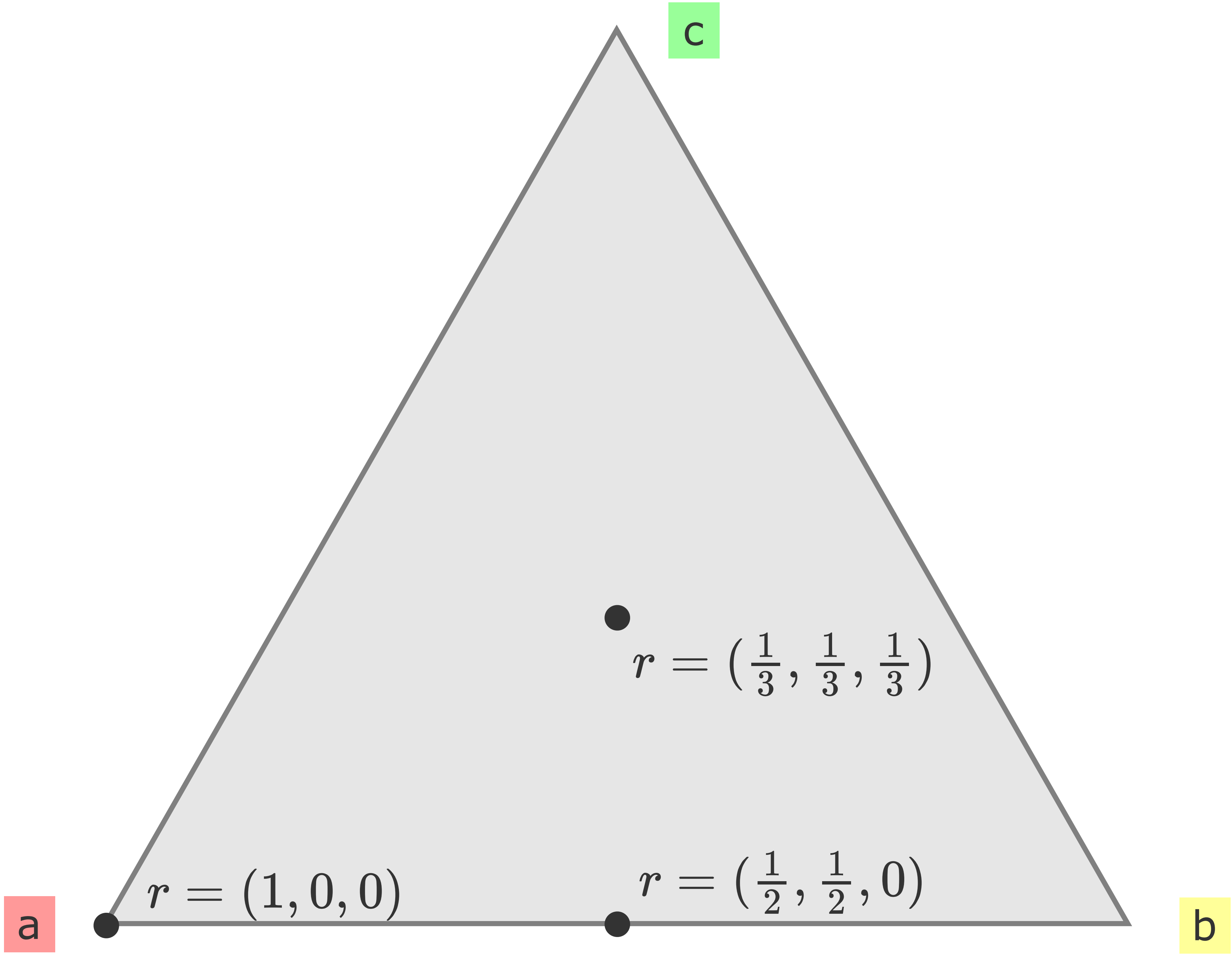
Proof via Sperner's Lemma ($n = 3$)



Proof via Sperner's Lemma (n = 3)

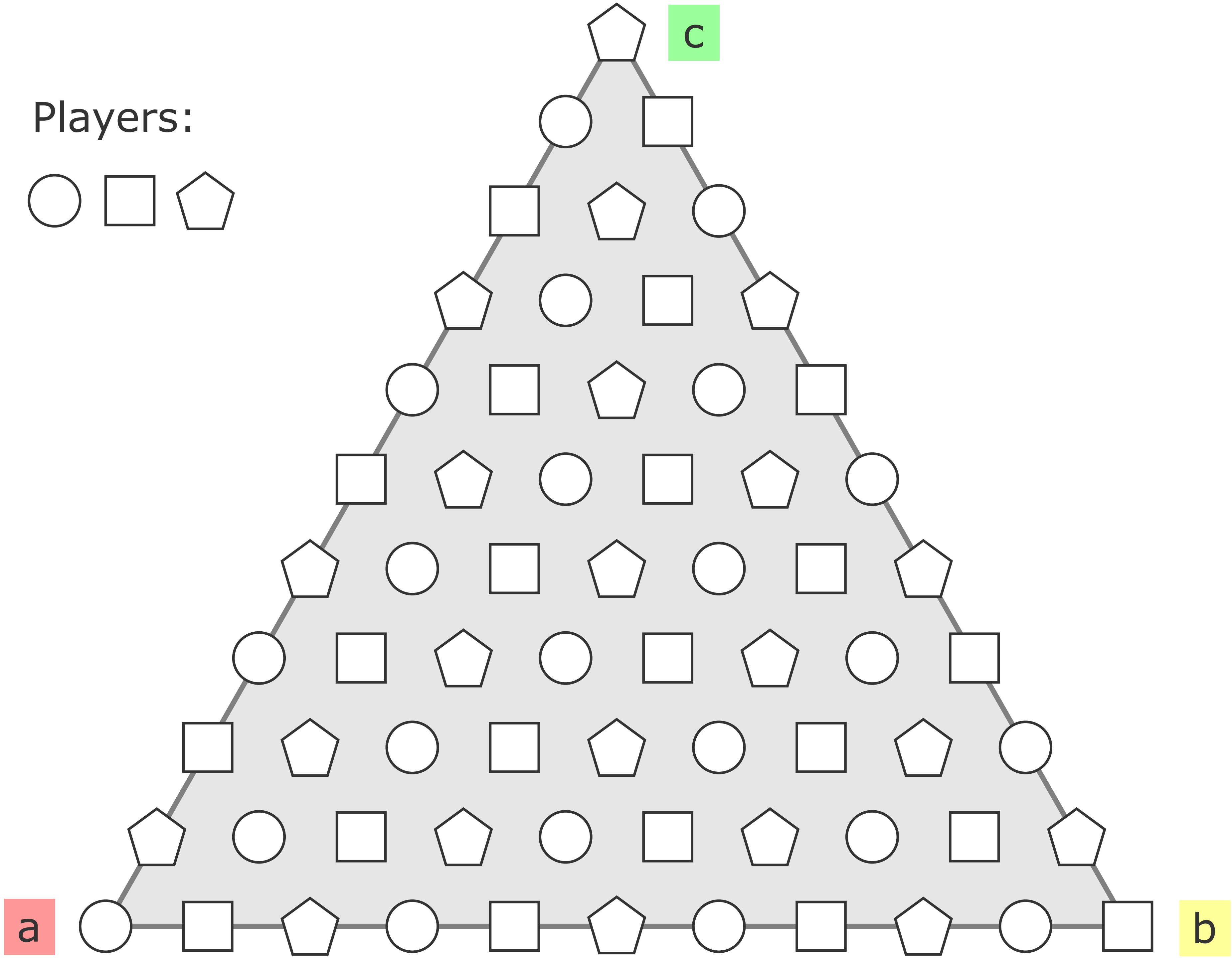


Proof via Sperner's Lemma (n = 3)



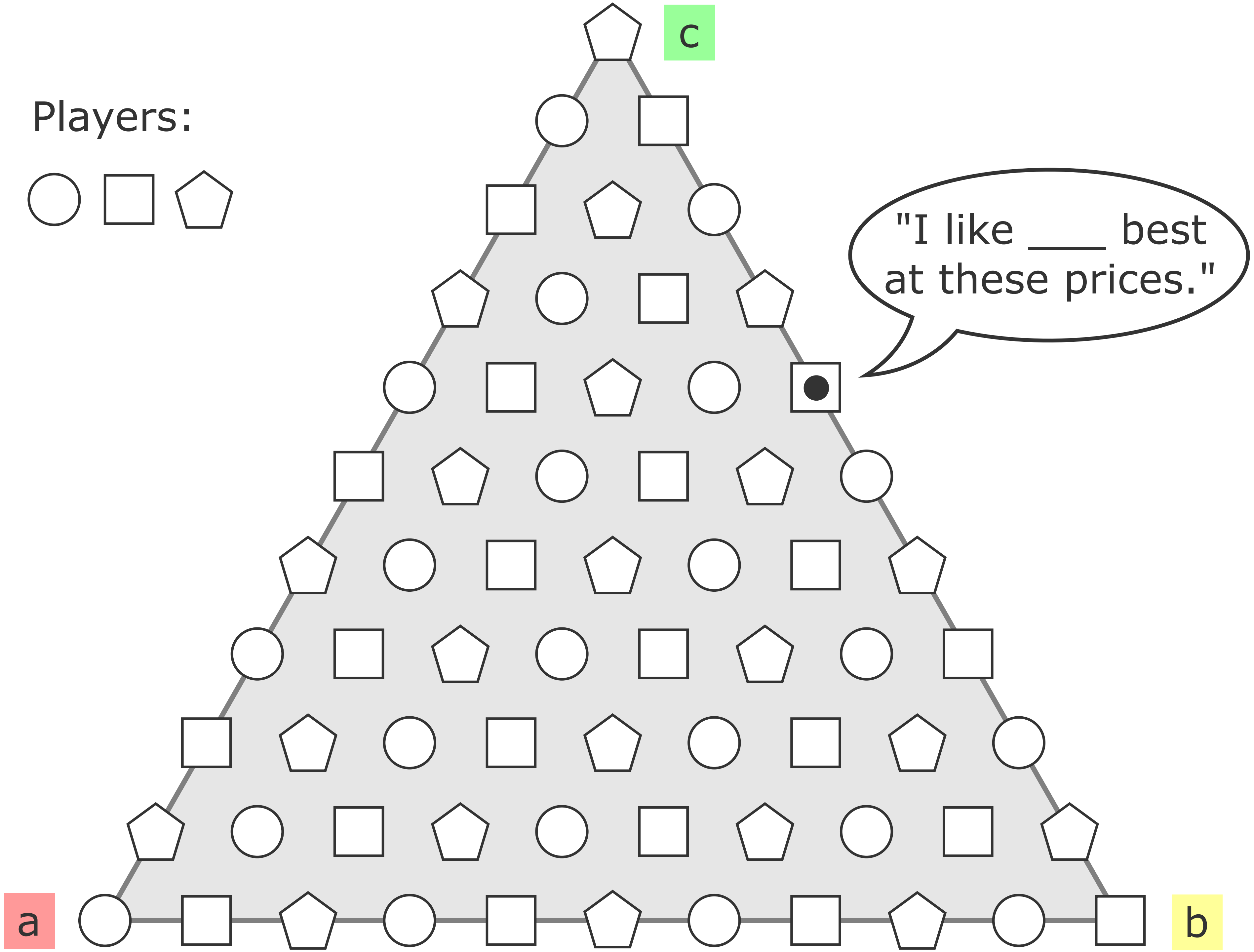
Proof via Sperner's Lemma (n = 3)

Players:



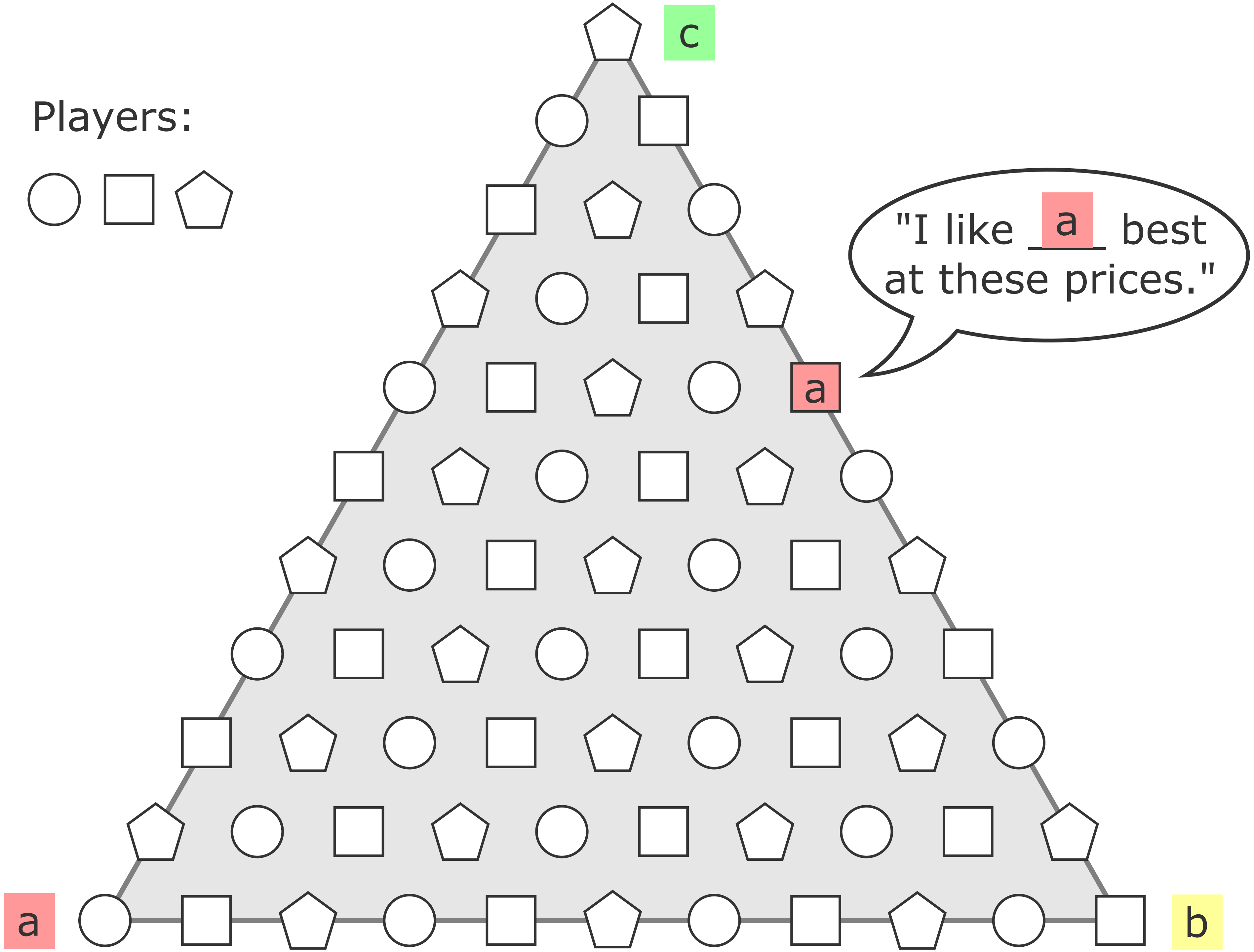
Proof via Sperner's Lemma (n = 3)

Players:



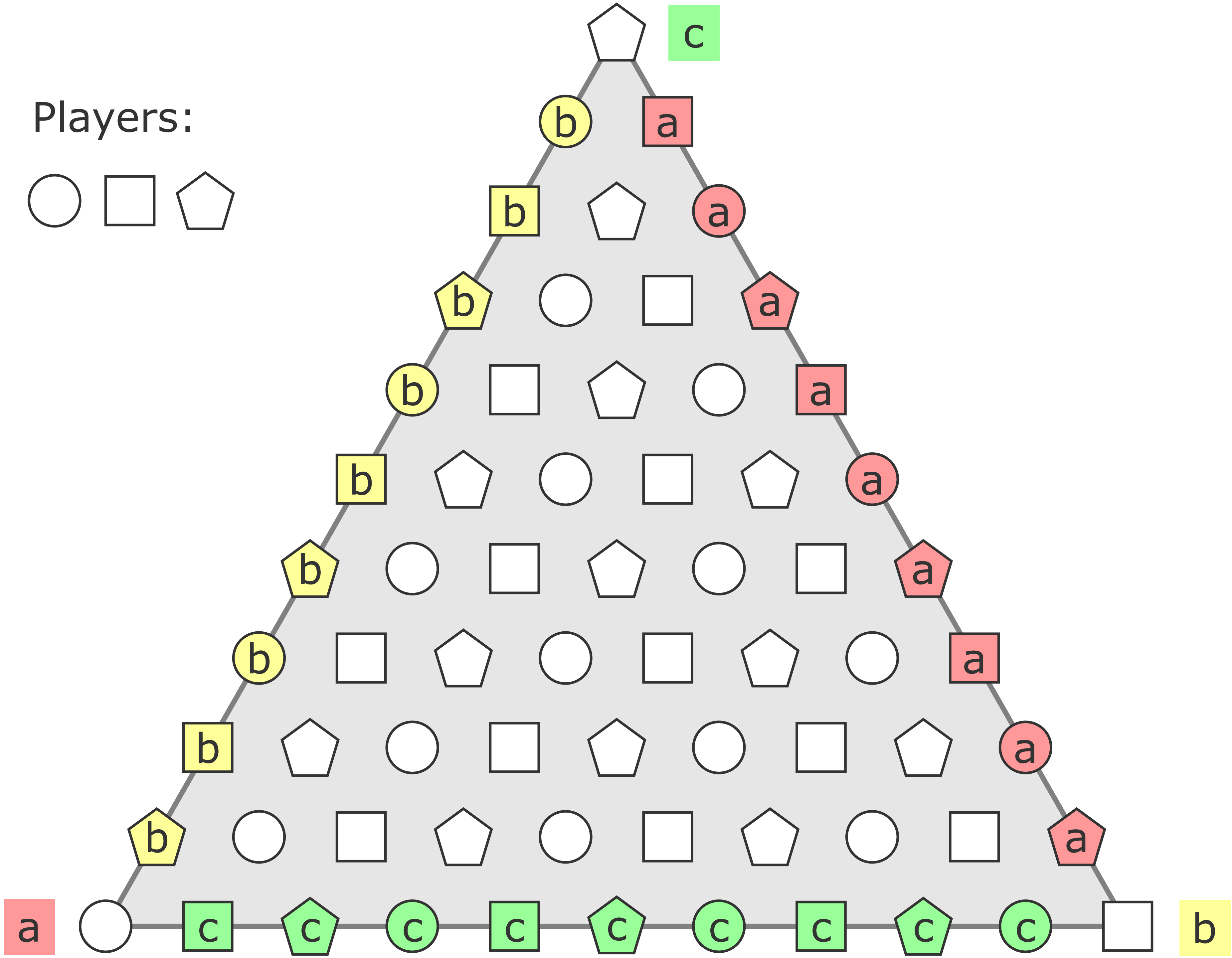
Proof via Sperner's Lemma (n = 3)

Players:



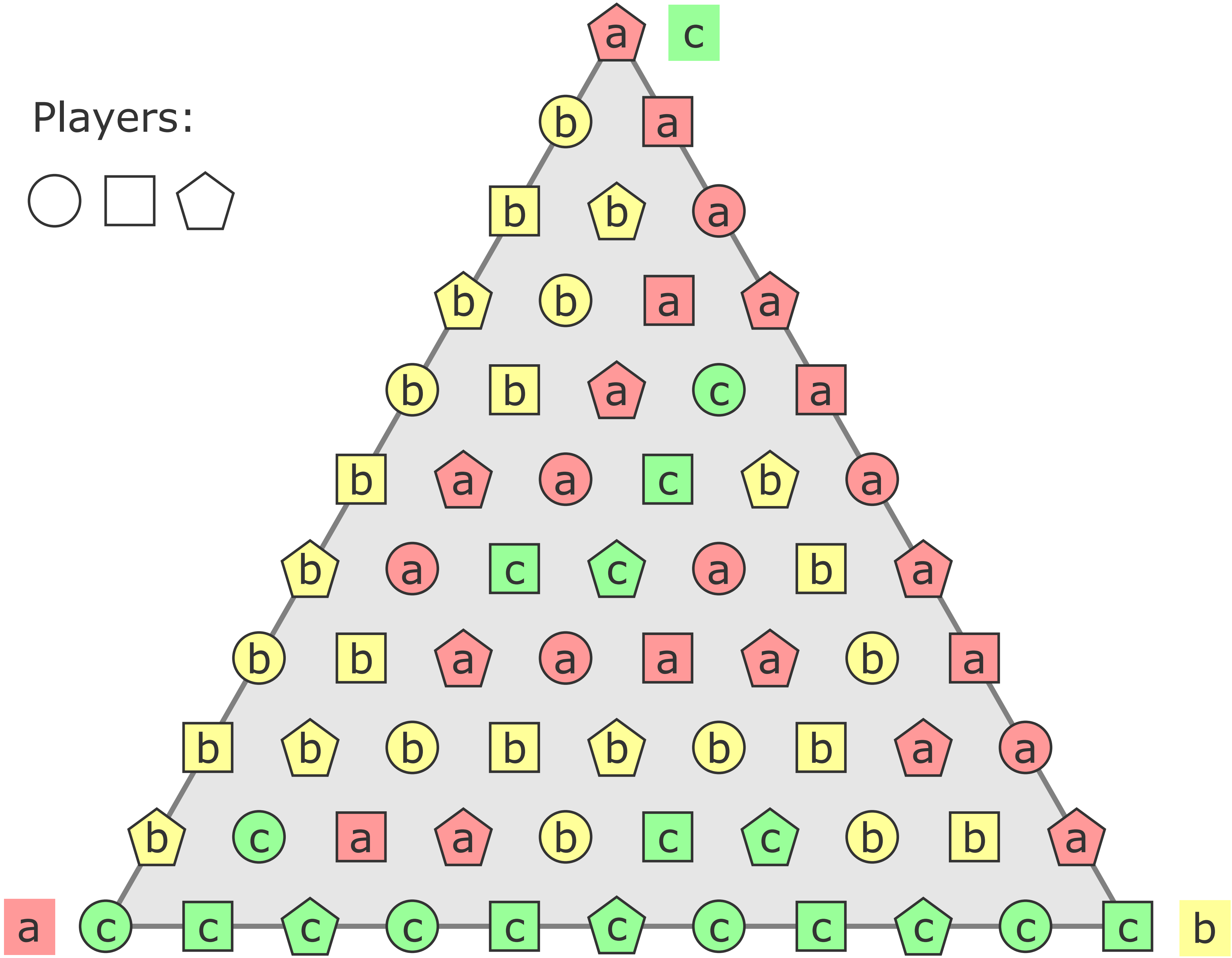
Proof via Sperner's Lemma (n = 3)

Players:



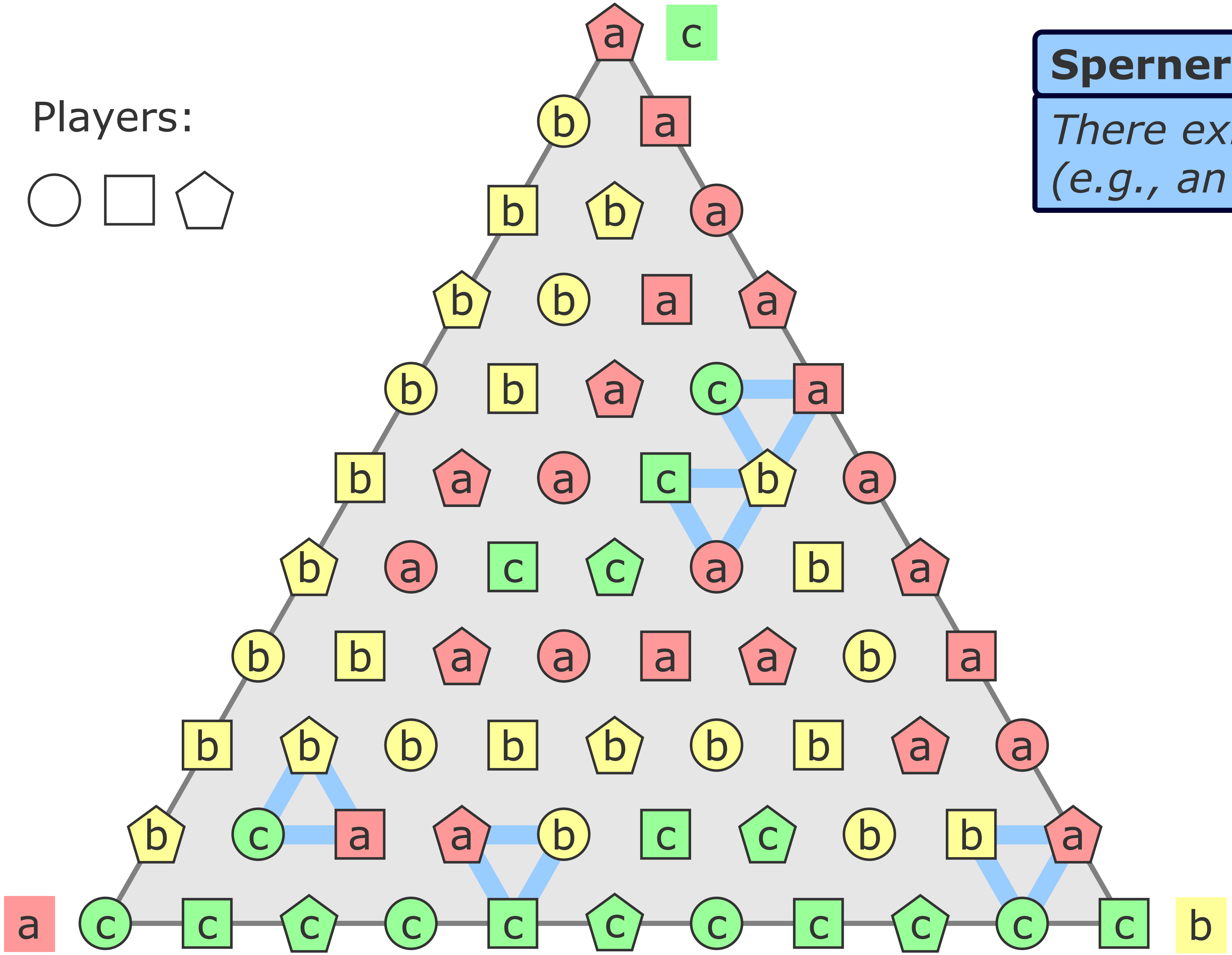
Proof via Sperner's Lemma (n = 3)

Players:



Proof via Sperner's Lemma (n = 3)

Players:

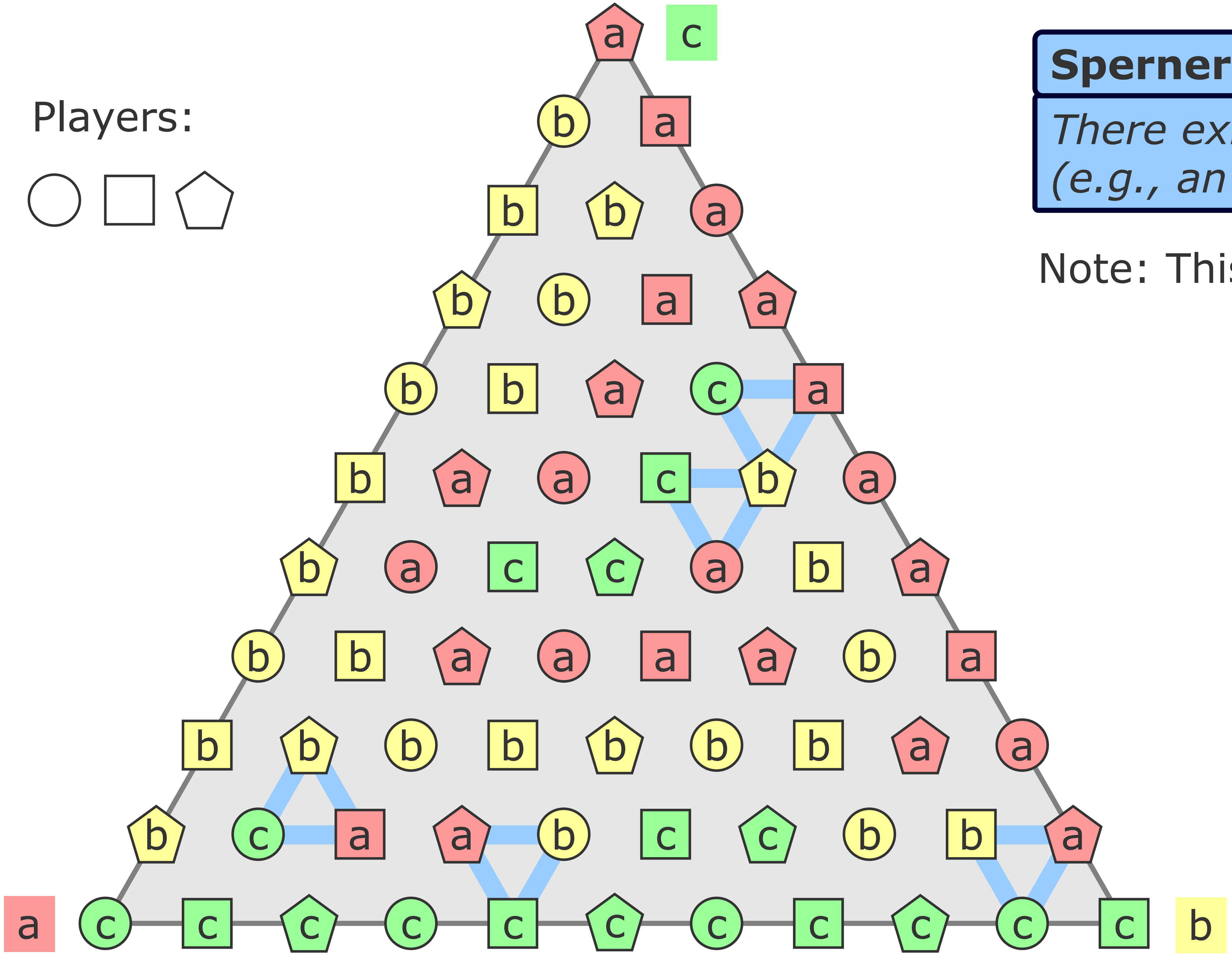


Sperner's Lemma

There exists a simplex with all n labels (e.g., an abc triangle).

Proof via Sperner's Lemma (n = 3)

Players:



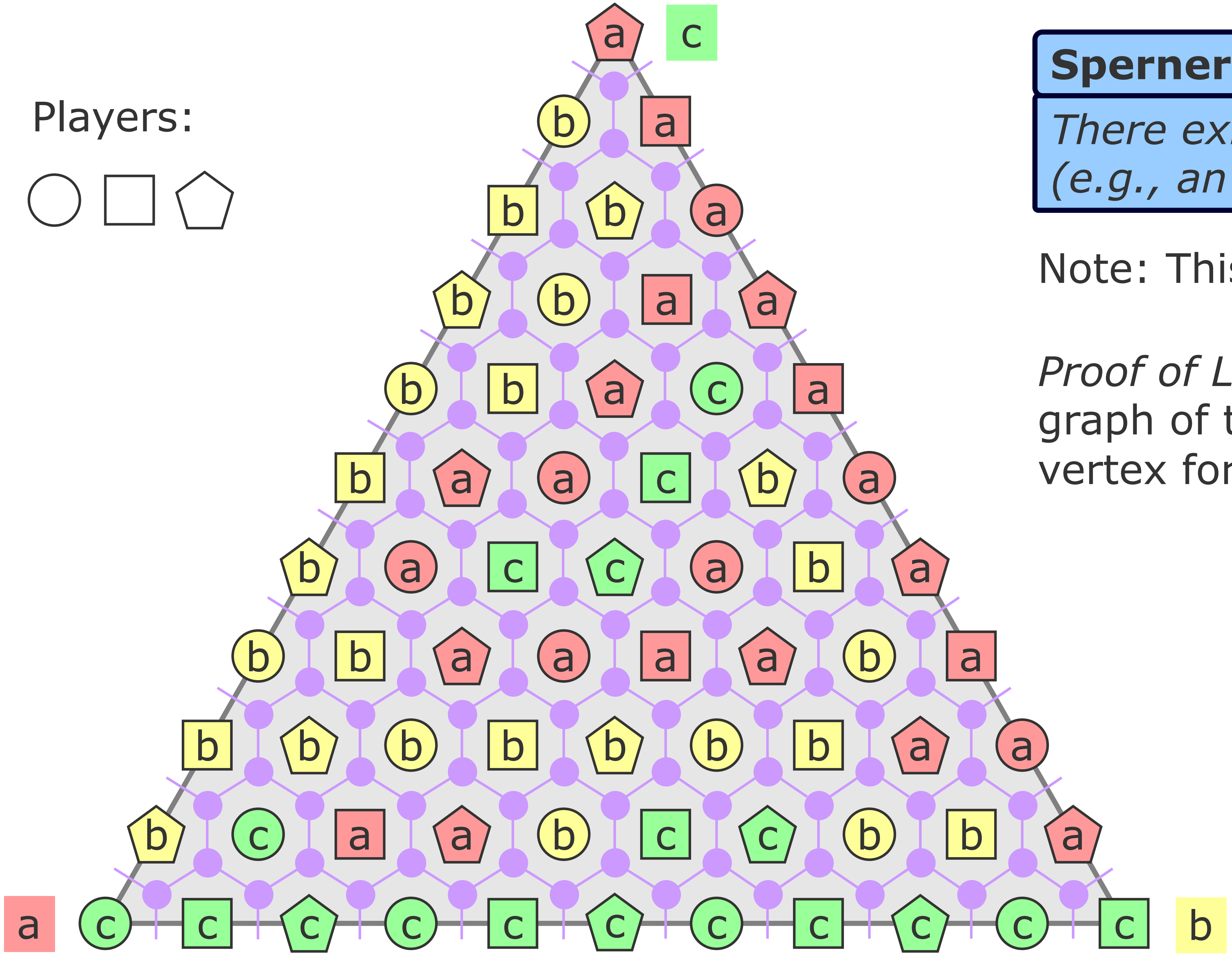
Sperner's Lemma

There exists a simplex with all n labels (e.g., an abc triangle).

Note: This proves the previous claim!

Proof via Sperner's Lemma (n = 3)

Players:



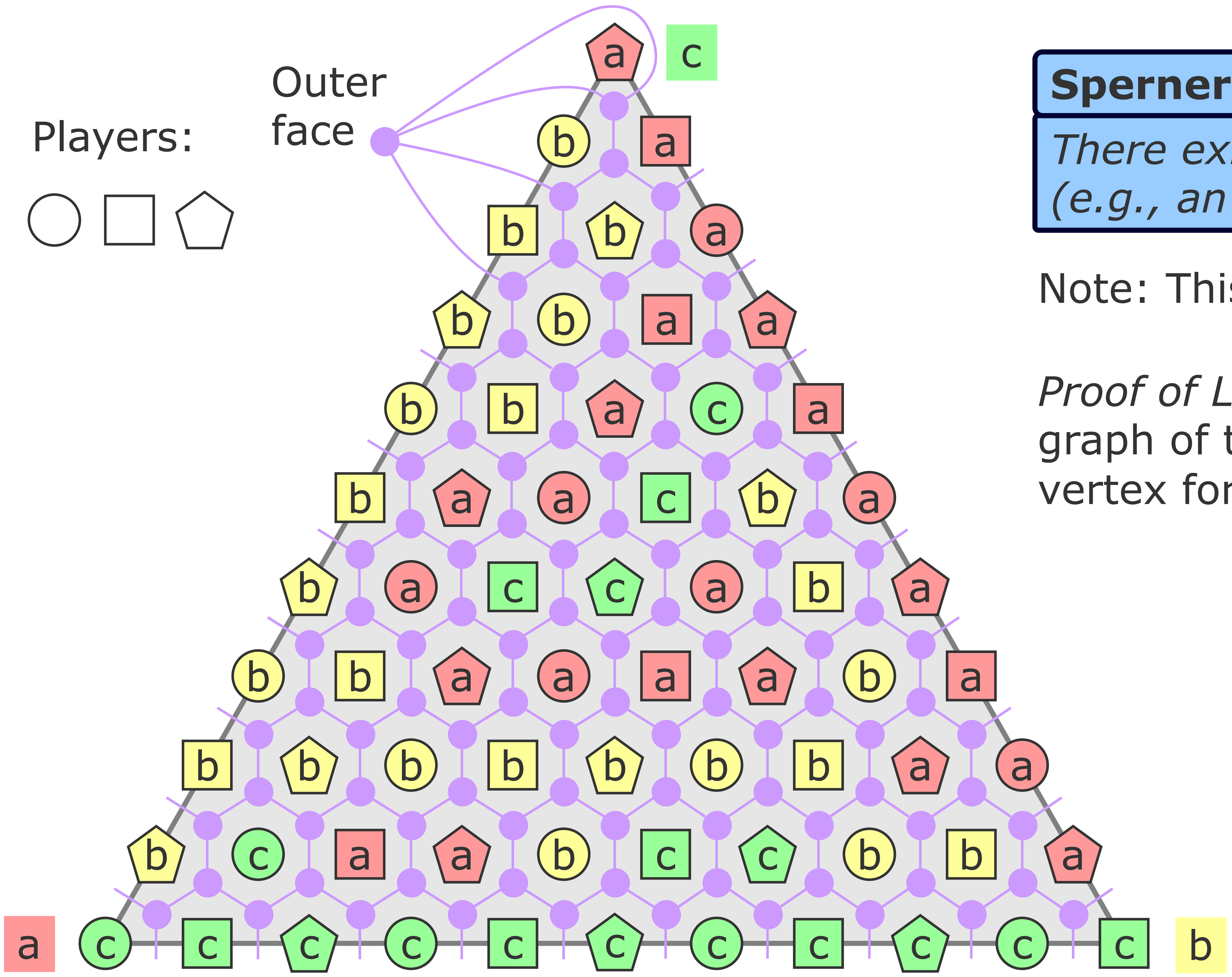
Sperner's Lemma

There exists a simplex with all n labels (e.g., an abc triangle).

Note: This proves the previous claim!

Proof of Lemma. Consider the dual graph of the triangulation, with one vertex for each triangular room.

Proof via Sperner's Lemma (n = 3)



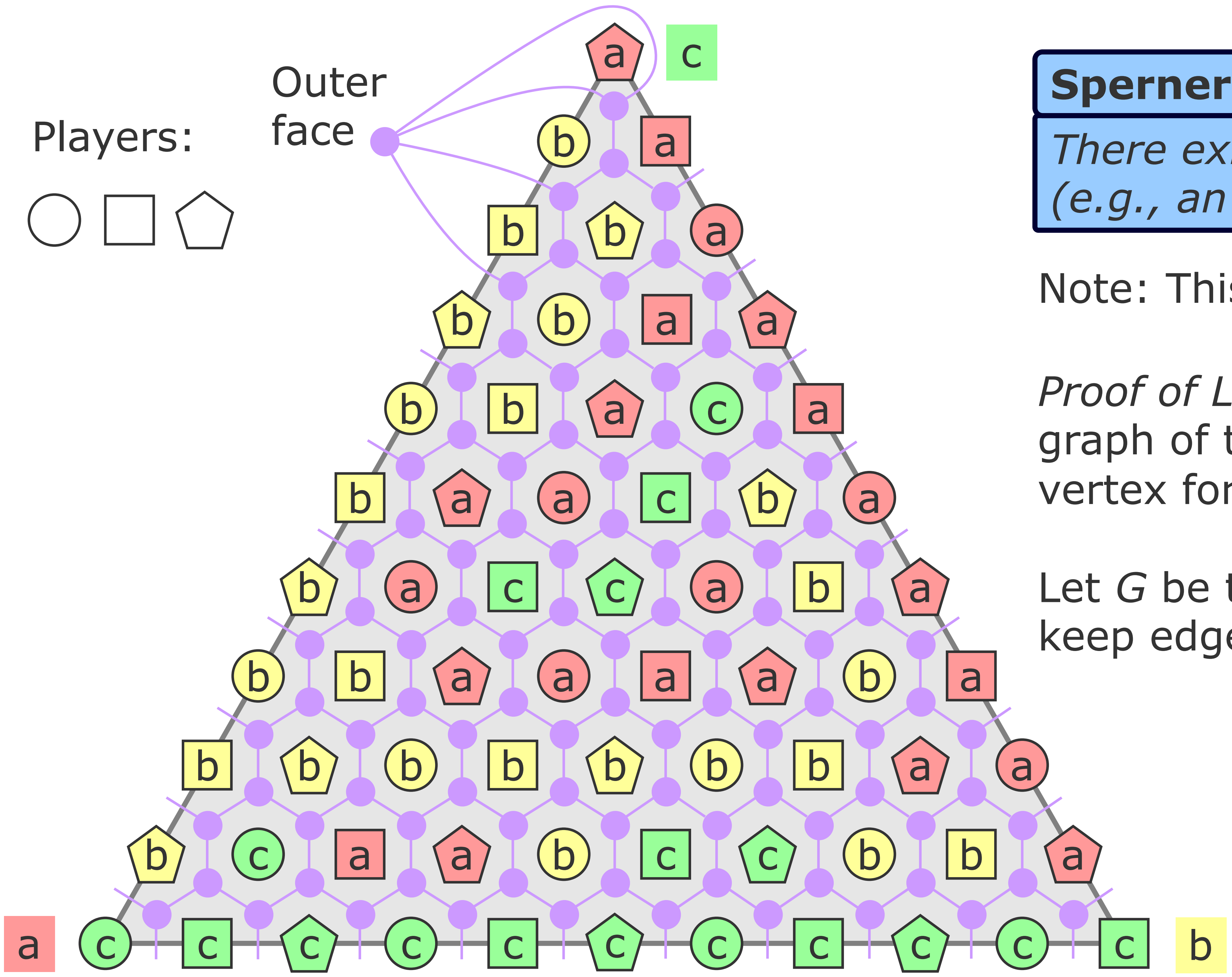
Sperner's Lemma

There exists a simplex with all n labels (e.g., an abc triangle).

Note: This proves the previous claim!

Proof of Lemma. Consider the dual graph of the triangulation, with one vertex for each triangular room.

Proof via Sperner's Lemma (n = 3)



Sperner's Lemma

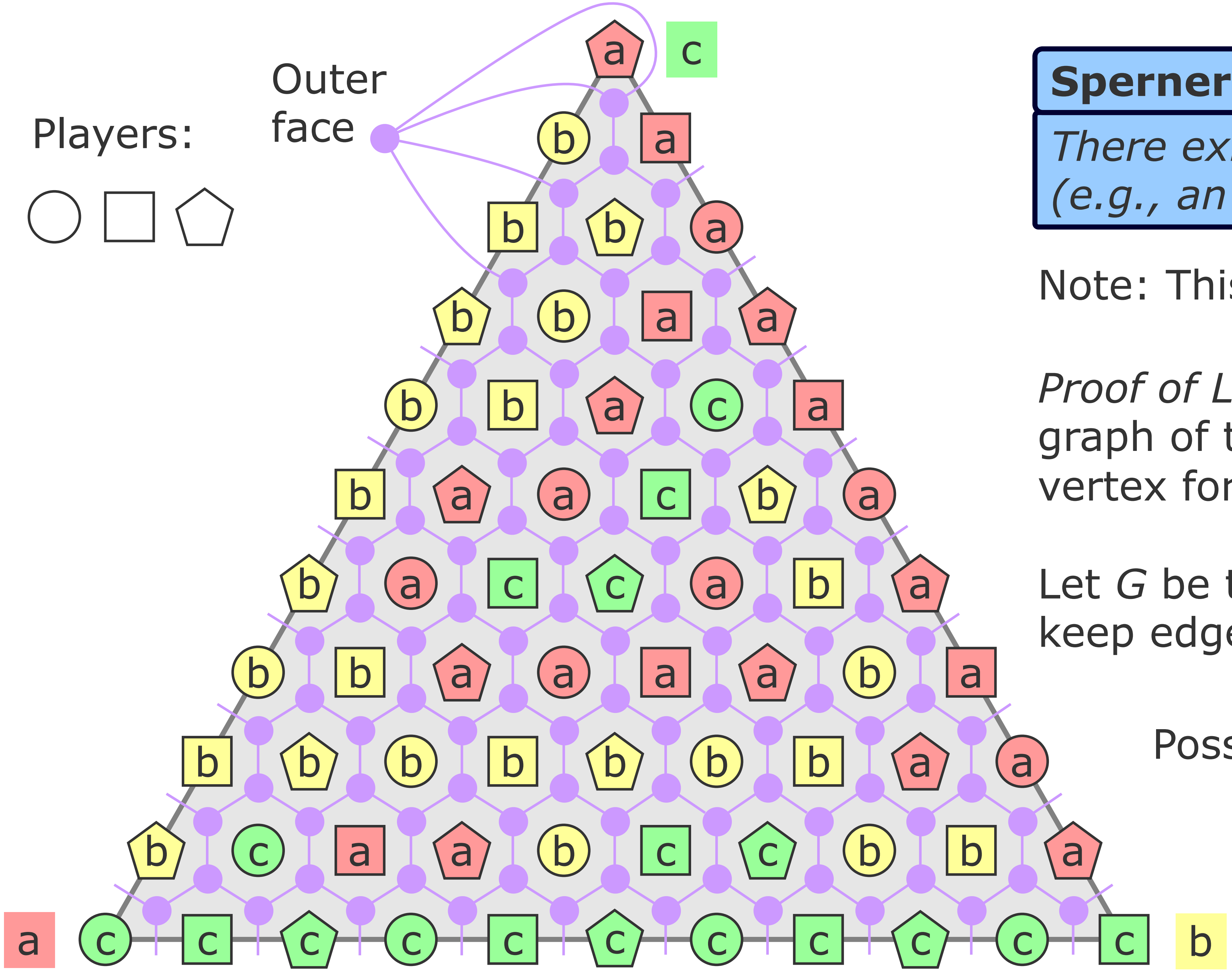
There exists a simplex with all n labels (e.g., an abc triangle).

Note: This proves the previous claim!

Proof of Lemma. Consider the dual graph of the triangulation, with one vertex for each triangular room.

Let *G* be the subgraph where we only keep edges crossing an a b door.

Proof via Sperner's Lemma (n = 3)



Sperner's Lemma

There exists a simplex with all n labels (e.g., an abc triangle).

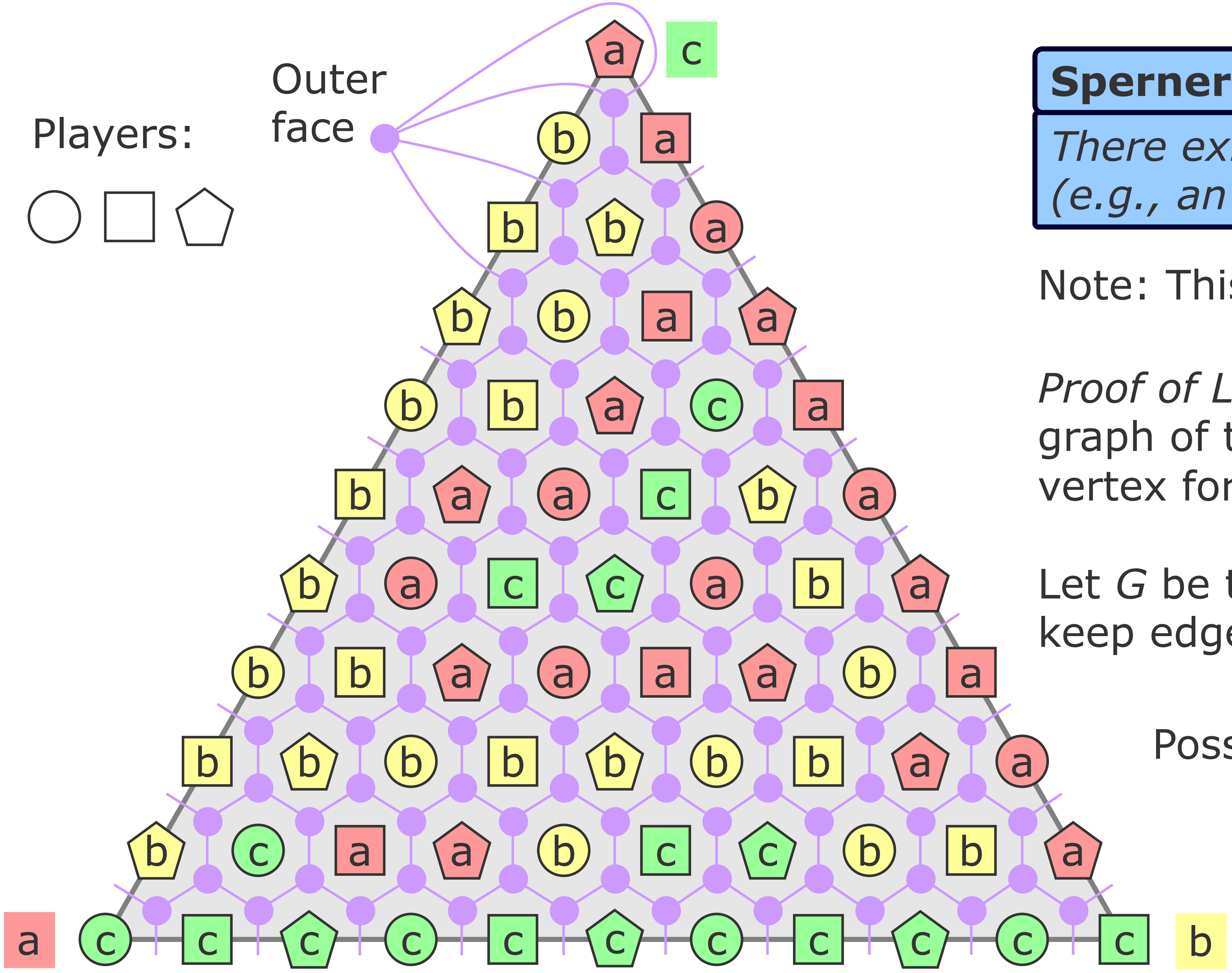
Note: This proves the previous claim!

Proof of Lemma. Consider the dual graph of the triangulation, with one vertex for each triangular room.

Let G be the subgraph where we only keep edges crossing an a b door.

Possible degrees: 0, 1, 2, 3

Proof via Sperner's Lemma (n = 3)



Sperner's Lemma

There exists a simplex with all n labels (e.g., an abc triangle).

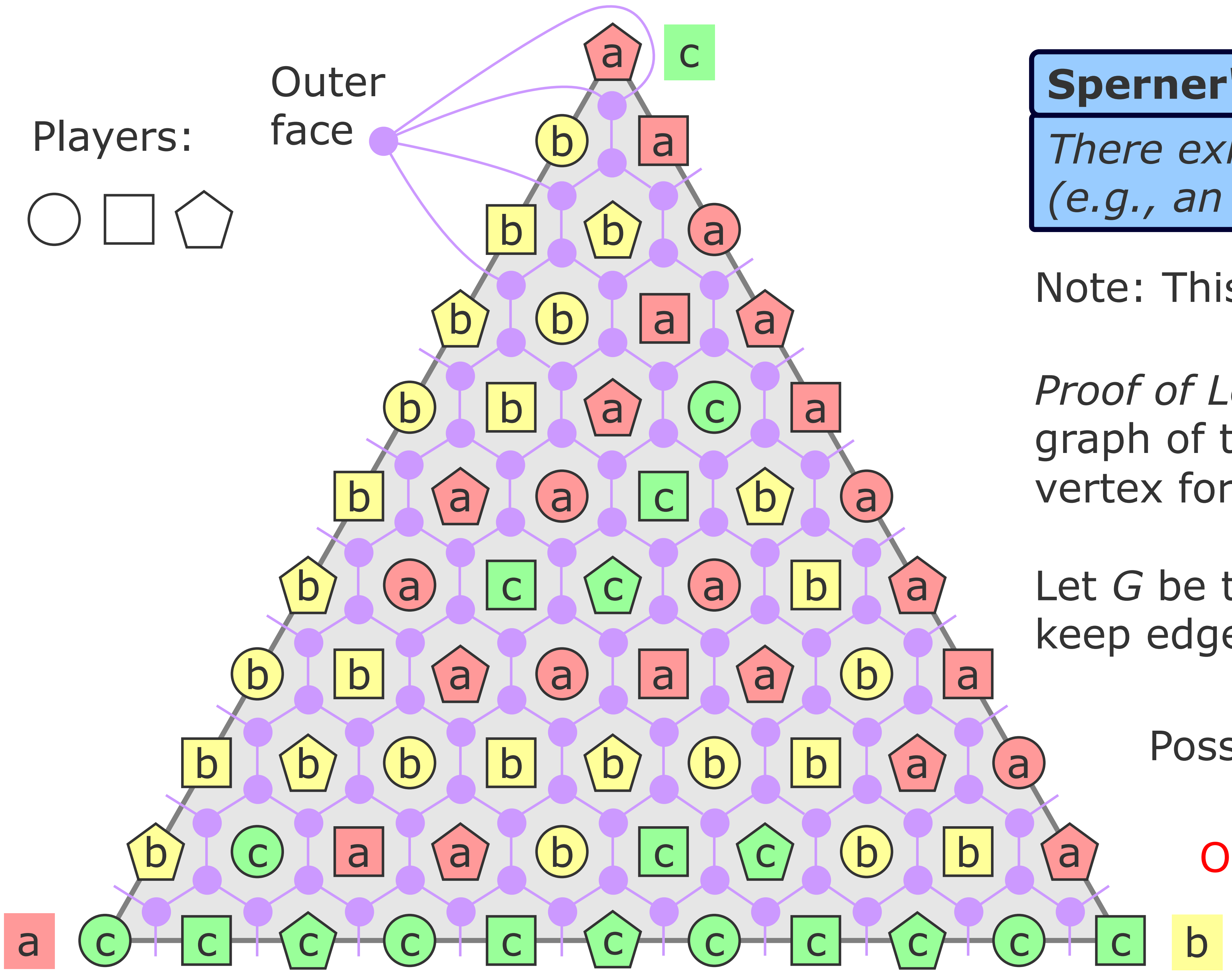
Note: This proves the previous claim!

Proof of Lemma. Consider the dual graph of the triangulation, with one vertex for each triangular room.

Let G be the subgraph where we only keep edges crossing an a b door.

Possible degrees: 0, 1, 2, ~~3~~

Proof via Sperner's Lemma (n = 3)



Sperner's Lemma

There exists a simplex with all n labels (e.g., an abc triangle).

Note: This proves the previous claim!

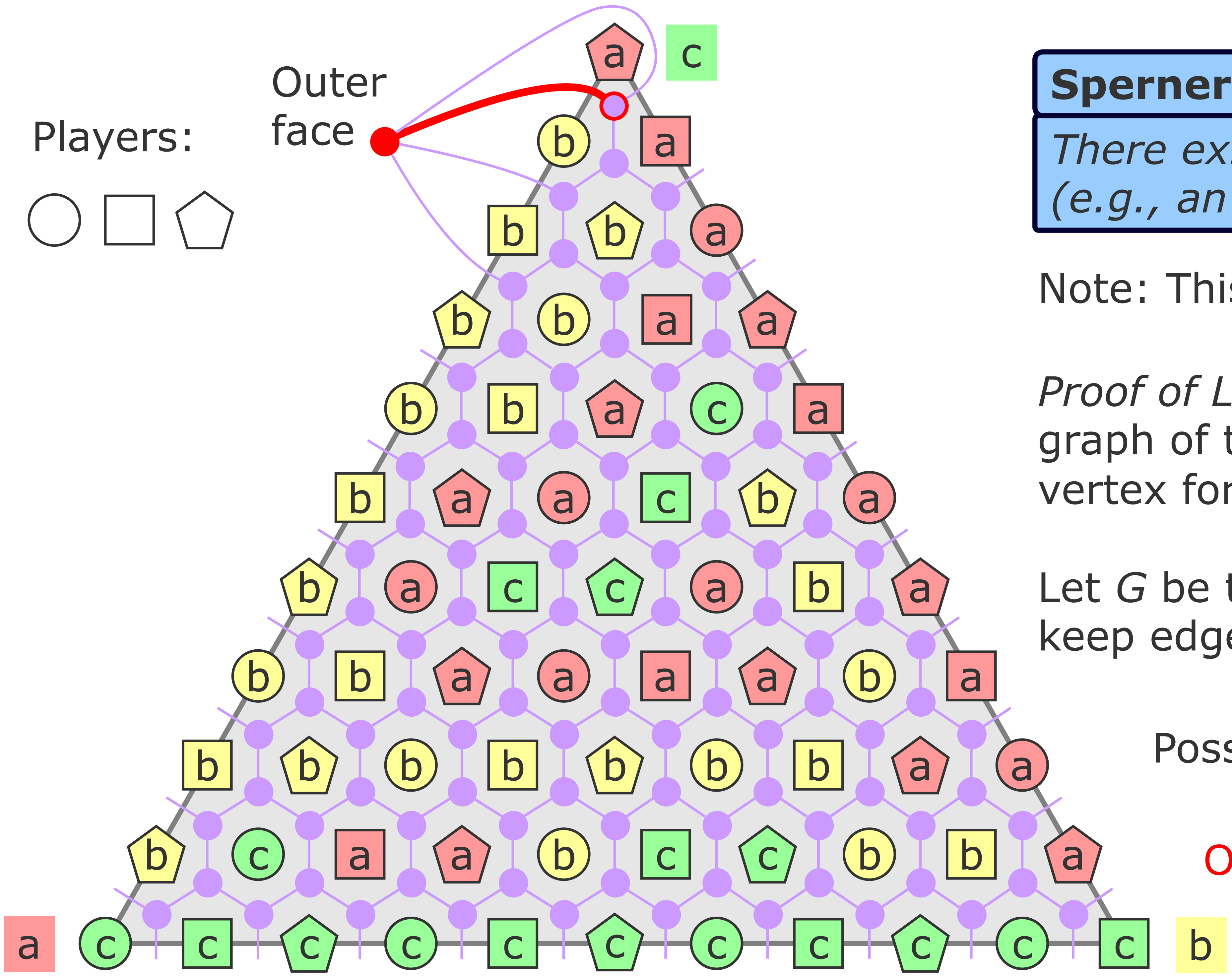
Proof of Lemma. Consider the dual graph of the triangulation, with one vertex for each triangular room.

Let G be the subgraph where we only keep edges crossing an a b door.

Possible degrees: 0, 1, 2, ~~3~~

Only possible in an abc triangle

Proof via Sperner's Lemma (n = 3)



Sperner's Lemma

There exists a simplex with all n labels (e.g., an abc triangle).

Note: This proves the previous claim!

Proof of Lemma. Consider the dual graph of the triangulation, with one vertex for each triangular room.

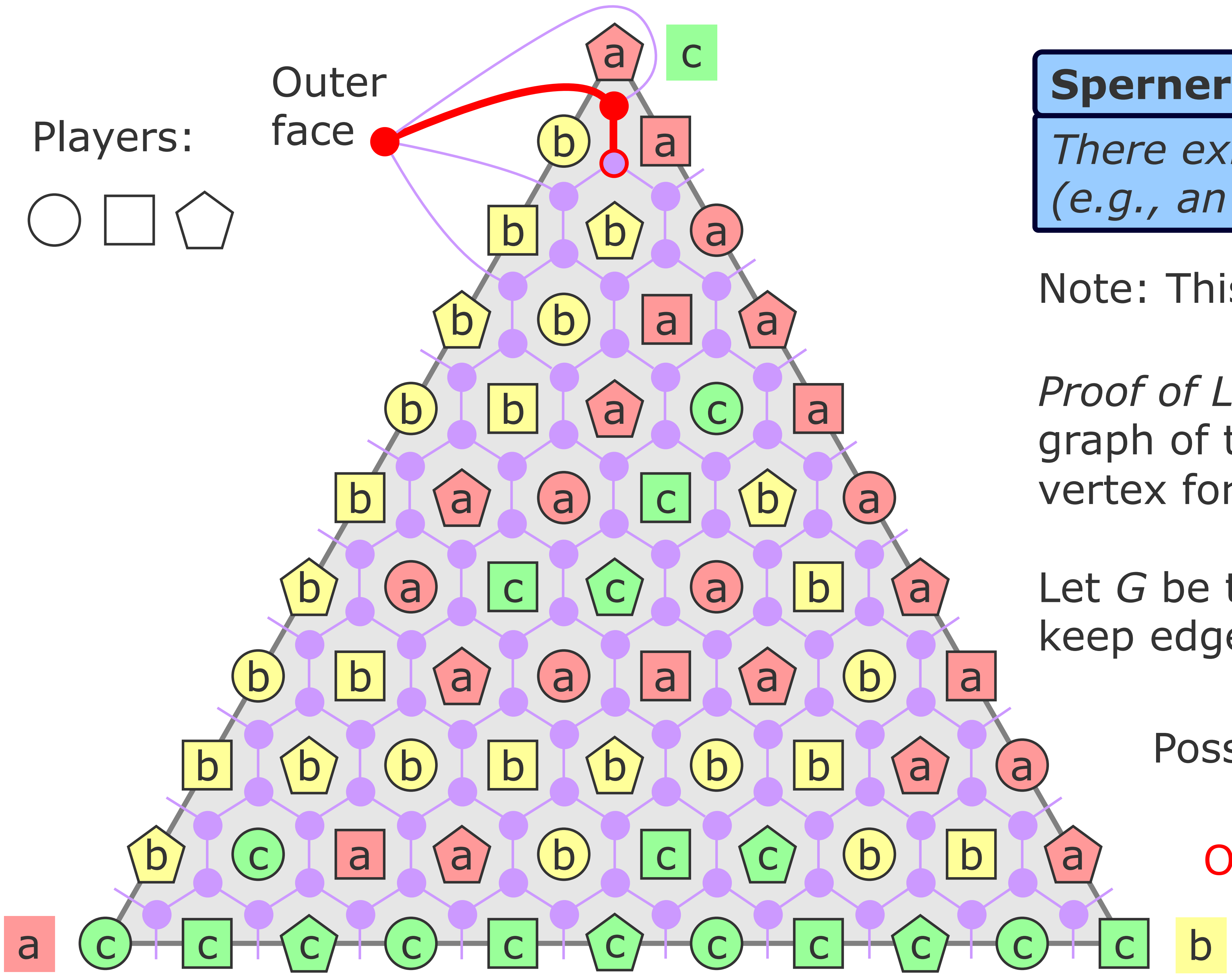
Let *G* be the subgraph where we only keep edges crossing an a b door.

Possible degrees: 0, 1, 2, ~~3~~

Only possible in an abc triangle

...and the outer face!

Proof via Sperner's Lemma (n = 3)



Sperner's Lemma

There exists a simplex with all n labels (e.g., an abc triangle).

Note: This proves the previous claim!

Proof of Lemma. Consider the dual graph of the triangulation, with one vertex for each triangular room.

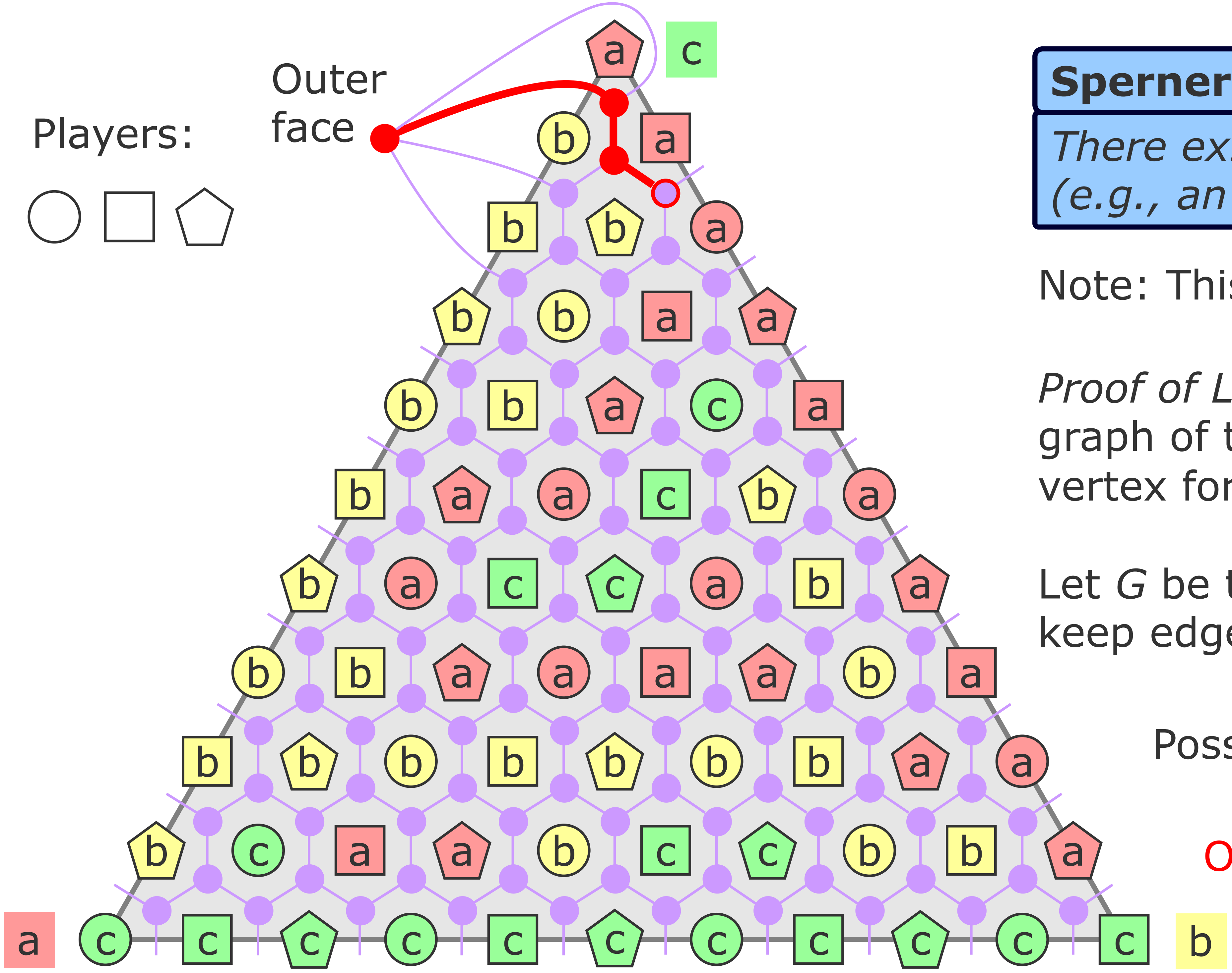
Let *G* be the subgraph where we only keep edges crossing an a b door.

Possible degrees: 0, 1, 2, ~~3~~

Only possible in an abc triangle

...and the outer face!

Proof via Sperner's Lemma (n = 3)



Sperner's Lemma

There exists a simplex with all n labels (e.g., an abc triangle).

Note: This proves the previous claim!

Proof of Lemma. Consider the dual graph of the triangulation, with one vertex for each triangular room.

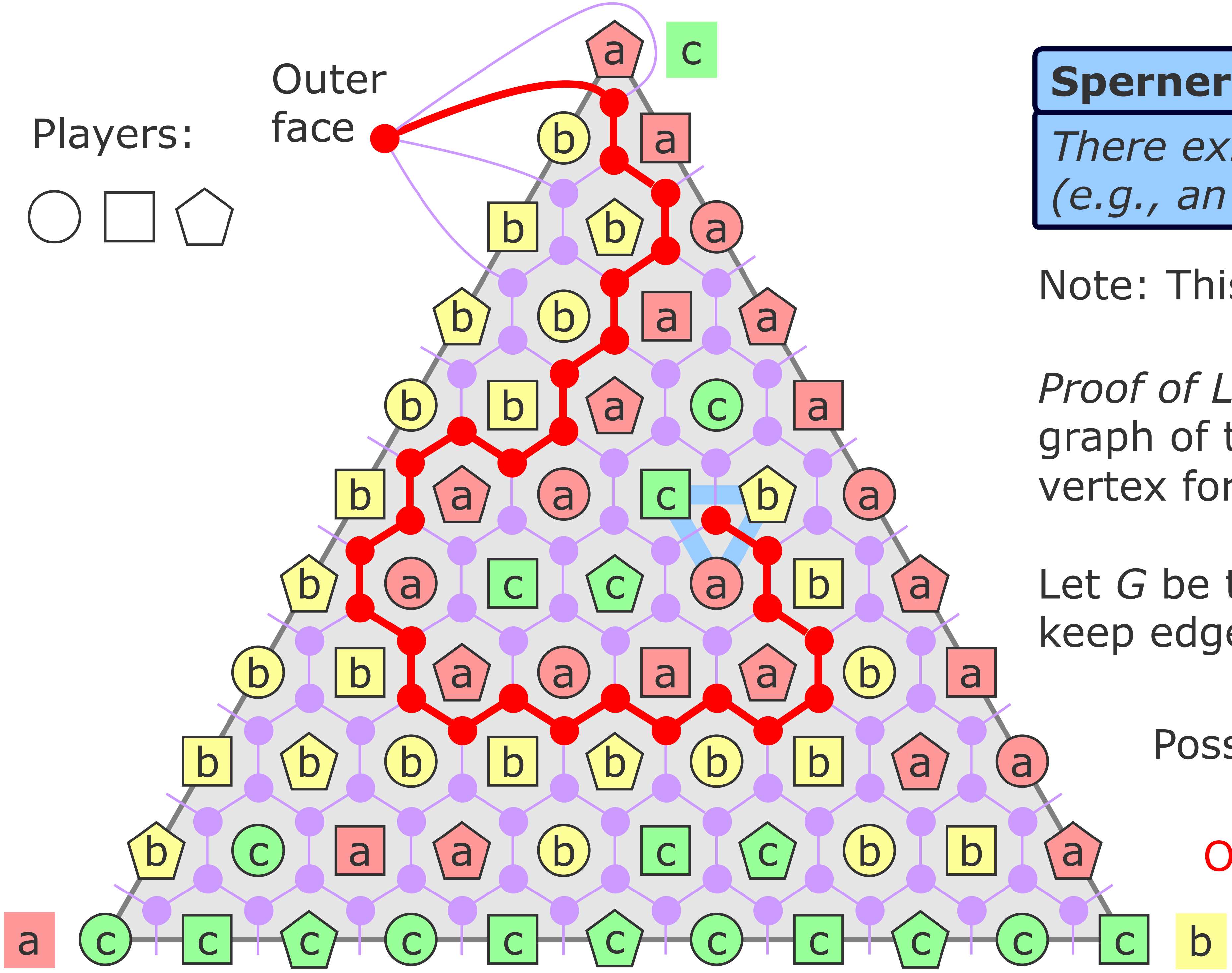
Let G be the subgraph where we only keep edges crossing an a b door.

Possible degrees: 0, 1, 2, ~~3~~

Only possible in an abc triangle

...and the outer face!

Proof via Sperner's Lemma (n = 3)



Sperner's Lemma

There exists a simplex with all n labels (e.g., an abc triangle).

Note: This proves the previous claim!

Proof of Lemma. Consider the dual graph of the triangulation, with one vertex for each triangular room.

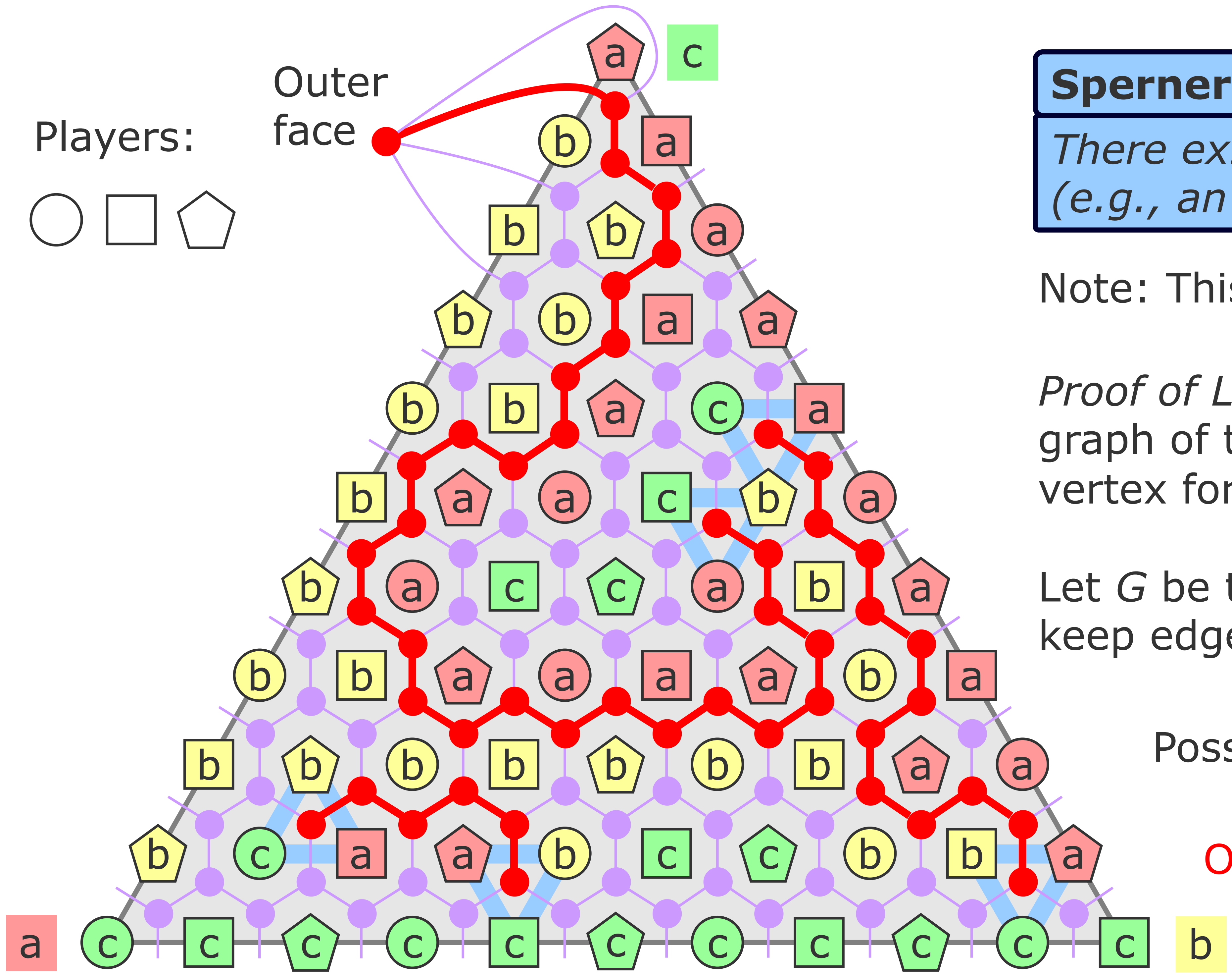
Let G be the subgraph where we only keep edges crossing an a b door.

Possible degrees: 0, 1, 2, ~~3~~

Only possible in an abc triangle
...and the outer face!

Proof via Sperner's Lemma (n = 3)

Players:



Sperner's Lemma

There exists a simplex with all n labels (e.g., an abc triangle).

Note: This proves the previous claim!

Proof of Lemma. Consider the dual graph of the triangulation, with one vertex for each triangular room.

Let G be the subgraph where we only keep edges crossing an a b door.

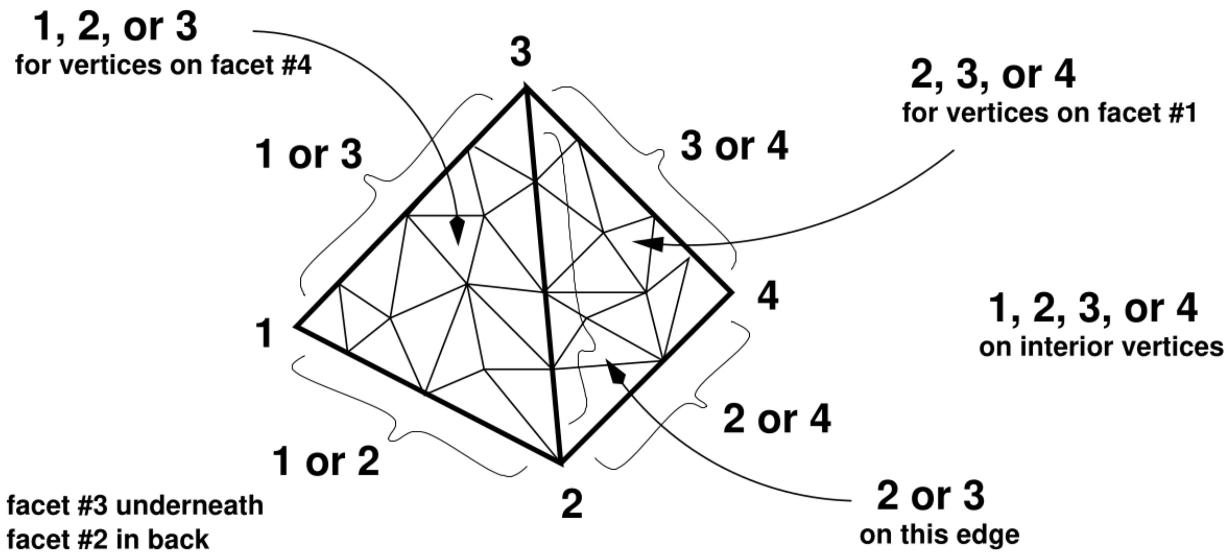
Possible degrees: 0, 1, 2, ~~3~~

Only possible in an abc triangle

...and the outer face!

Sperner's Lemma in higher dimensions

A *Sperner labeling* of a triangulated simplex on an n-dimensional simplex S on $n+1$ points is an assignment of labels in $[n+1]$ to each vertex such that, for each j, no vertices on the j^{th} facet of S have label j.

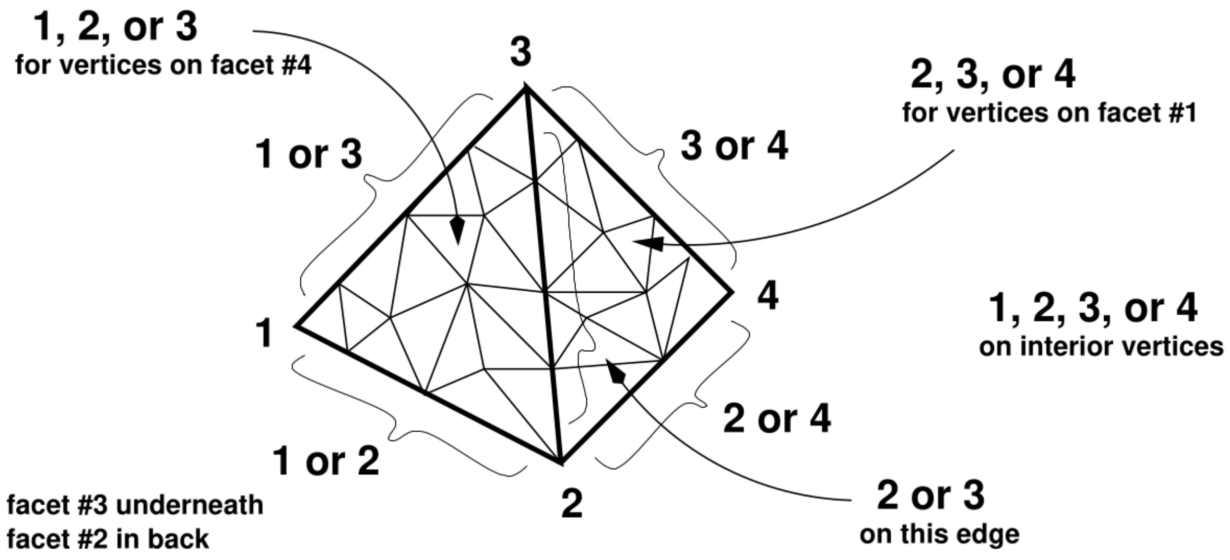


Sperner's Lemma in higher dimensions

A *Sperner labeling* of a triangulated simplex on an n-dimensional simplex S on $n+1$ points is an assignment of labels in $[n+1]$ to each vertex such that, for each j, no vertices on the j^{th} facet of S have label j.

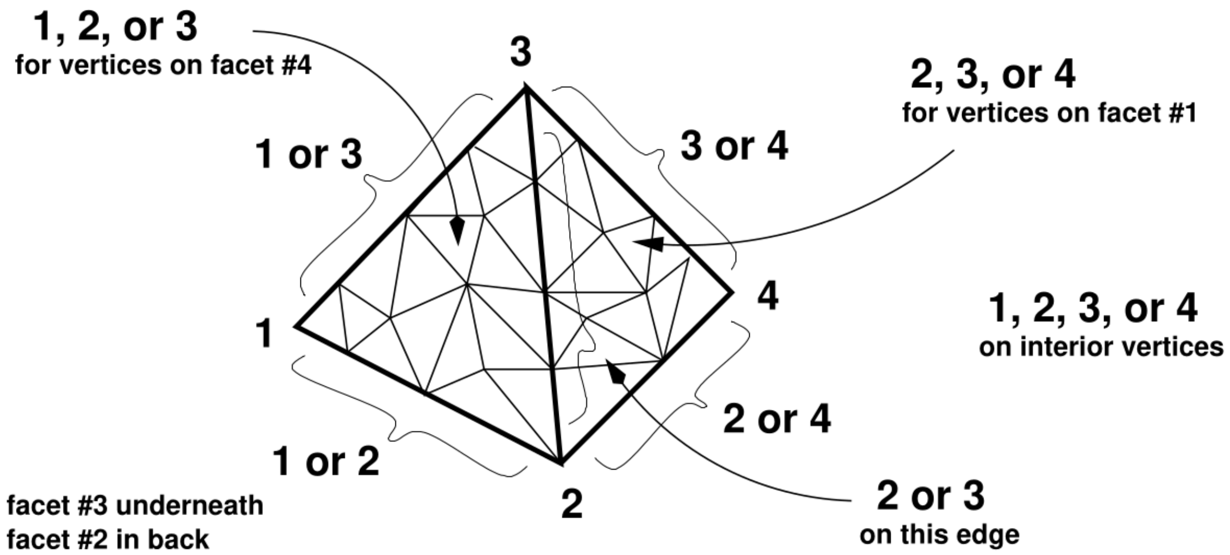
Sperner's Lemma

In any Sperner labeling, an odd number of simplices have all $n+1$ labels.



Sperner's Lemma in higher dimensions

A *Sperner labeling* of a triangulated simplex on an n-dimensional simplex S on $n+1$ points is an assignment of labels in $[n+1]$ to each vertex such that, for each j, no vertices on the j^{th} facet of S have label j.



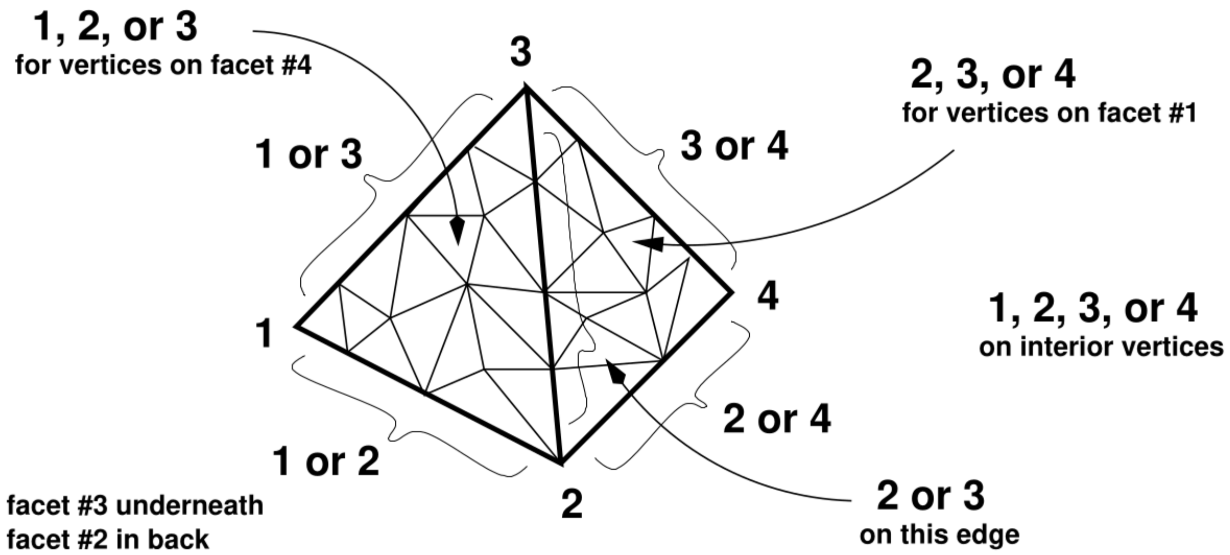
Sperner's Lemma

In any Sperner labeling, an odd number of simplices have all $n+1$ labels.

Proof. By induction on n. Base case ($n=1$):

Sperner's Lemma in higher dimensions

A *Sperner labeling* of a triangulated simplex on an n-dimensional simplex S on $n+1$ points is an assignment of labels in $[n+1]$ to each vertex such that, for each j, no vertices on the j^{th} facet of S have label j.



Sperner's Lemma

In any Sperner labeling, an odd number of simplices have all $n+1$ labels.

Proof. By induction on n. Base case ($n=1$):

Sperner's Lemma in higher dimensions

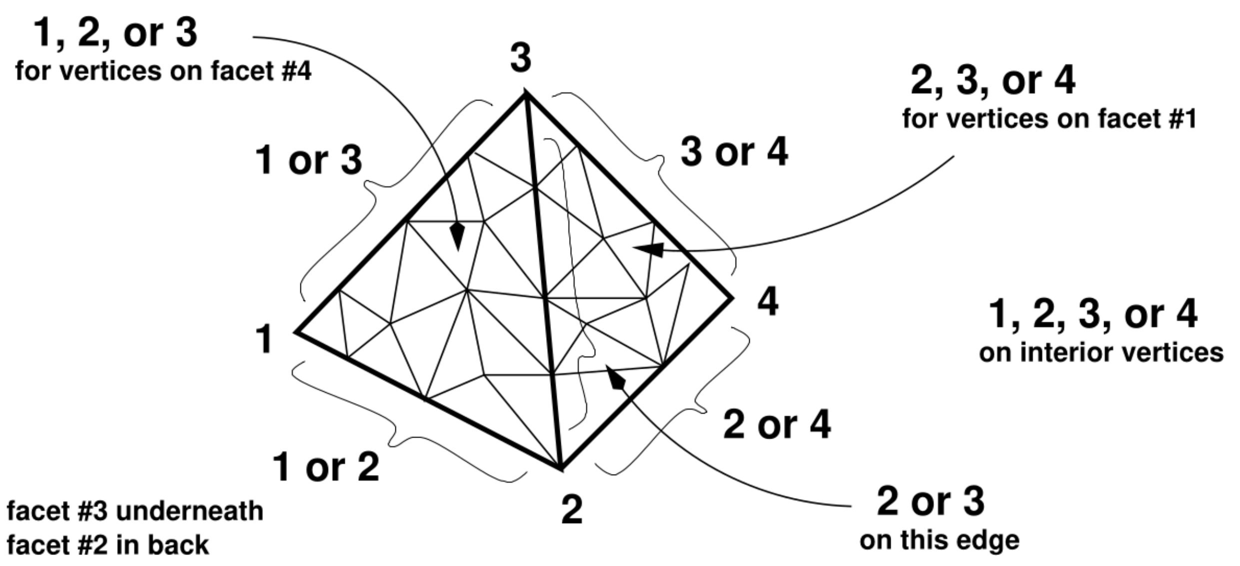
A *Sperner labeling* of a triangulated simplex on an n-dimensional simplex S on $n+1$ points is an assignment of labels in $[n+1]$ to each vertex such that, for each j, no vertices on the j^{th} facet of S have label j.

Sperner's Lemma

In any Sperner labeling, an odd number of simplices have all $n+1$ labels.

Proof. By induction on n. Base case ($n = 1$):

There are an odd number of transitions between 1 and 2.



Now suppose the lemma holds for $n-1$. Define the dual subgraph G as before, with edges across $(n-1)$-dimensional faces labeled 1 through n.

Sperner's Lemma in higher dimensions

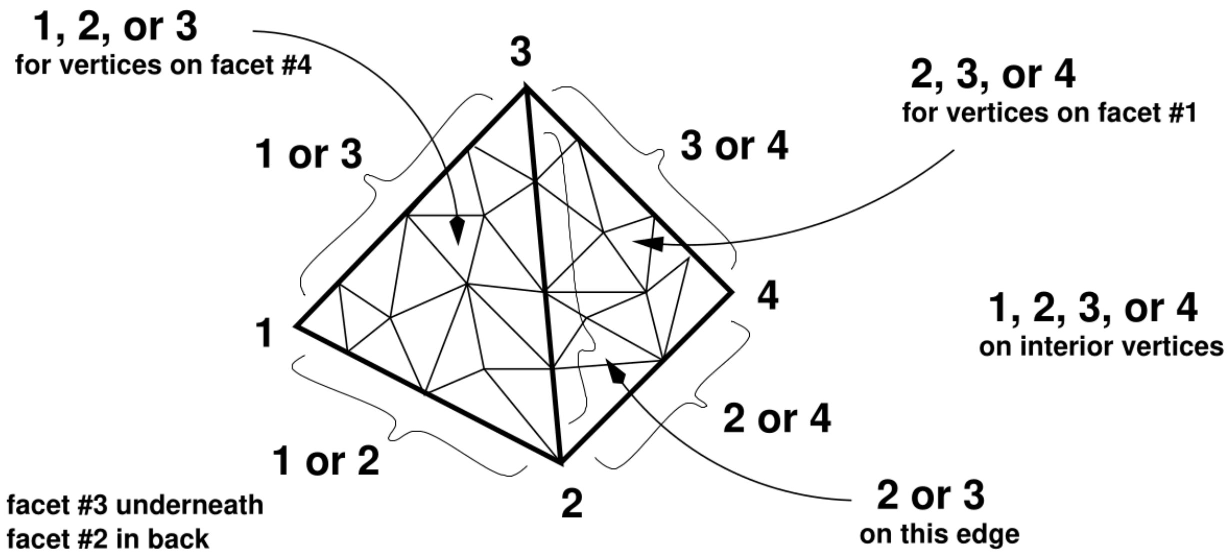
A *Sperner labeling* of a triangulated simplex on an n-dimensional simplex S on $n+1$ points is an assignment of labels in $[n+1]$ to each vertex such that, for each j, no vertices on the j^{th} facet of S have label j.

Sperner's Lemma

In any Sperner labeling, an odd number of simplices have all $n+1$ labels.

Proof. By induction on n. Base case ($n = 1$):

There are an odd number of transitions between 1 and 2.



Now suppose the lemma holds for $n-1$. Define the dual subgraph G as before, with edges across $(n-1)$-dimensional faces labeled 1 through n.

- Internal rooms have degree 0, 1, or 2, with degree 1 iff all labels present.

Sperner's Lemma in higher dimensions

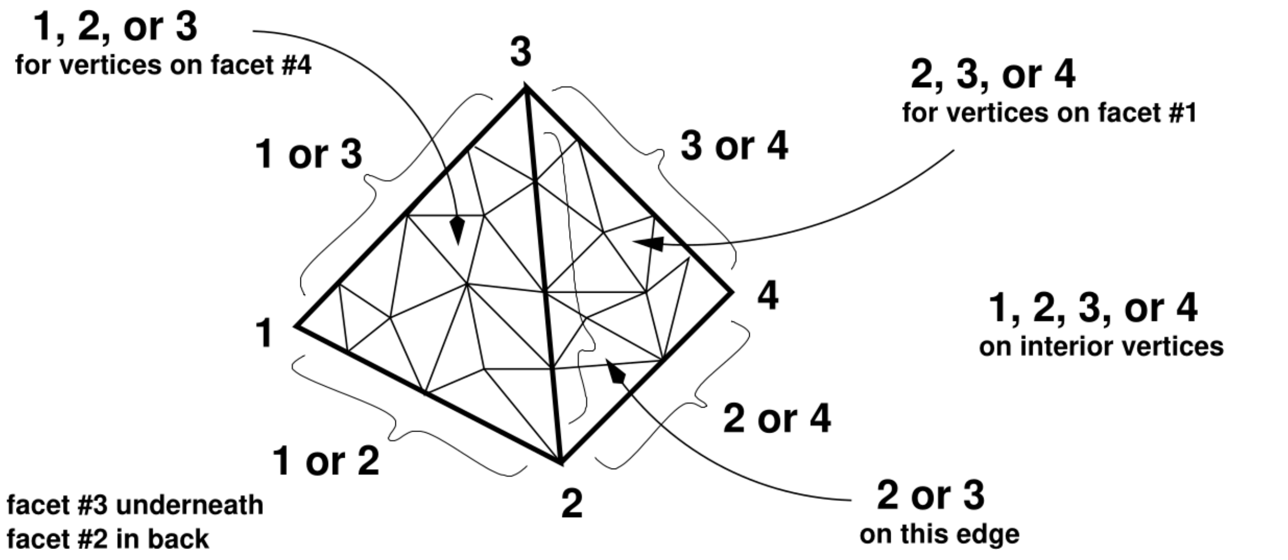
A *Sperner labeling* of a triangulated simplex on an n-dimensional simplex S on $n+1$ points is an assignment of labels in $[n+1]$ to each vertex such that, for each j, no vertices on the j^{th} facet of S have label j.

Sperner's Lemma

In any Sperner labeling, an odd number of simplices have all $n+1$ labels.

Proof. By induction on n. Base case ($n = 1$):

There are an odd number of transitions between 1 and 2.



Now suppose the lemma holds for $n-1$. Define the dual subgraph G as before, with edges across $(n-1)$-dimensional faces labeled 1 through n.

- Internal rooms have degree 0, 1, or 2, with degree 1 iff all labels present.
- Outer edges are only on the j^{th} facet.

Sperner's Lemma in higher dimensions

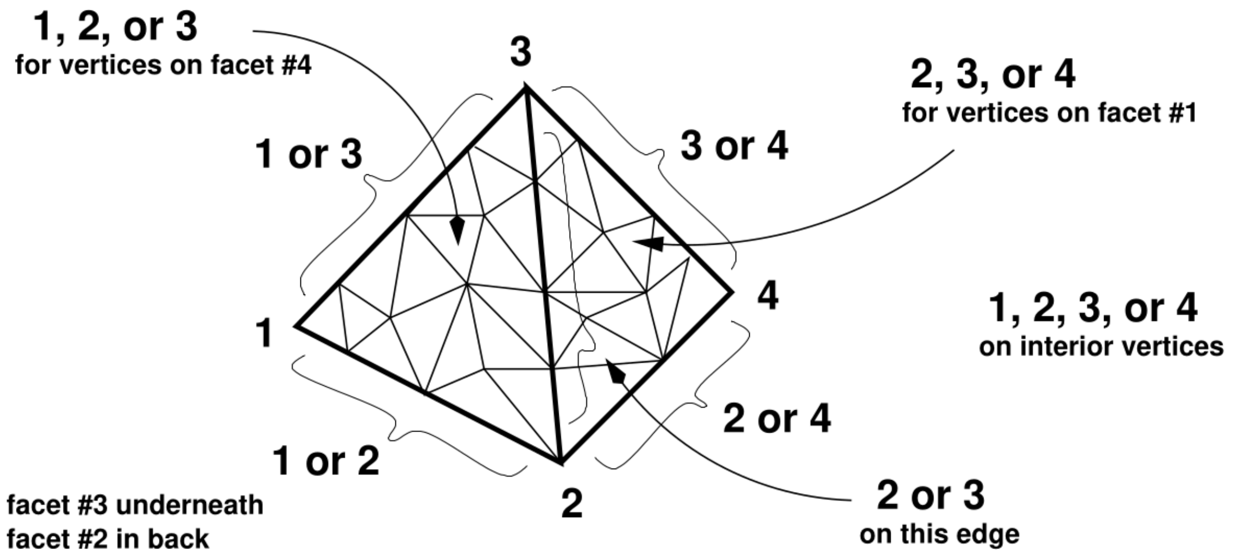
A *Sperner labeling* of a triangulated simplex on an n-dimensional simplex S on $n+1$ points is an assignment of labels in $[n+1]$ to each vertex such that, for each j, no vertices on the j^{th} facet of S have label j.

Sperner's Lemma

In any Sperner labeling, an odd number of simplices have all $n+1$ labels.

Proof. By induction on n. Base case ($n = 1$):

There are an odd number of transitions between 1 and 2.



Now suppose the lemma holds for $n-1$. Define the dual subgraph G as before, with edges across $(n-1)$-dimensional faces labeled 1 through n.

- Internal rooms have degree 0, 1, or 2, with degree 1 iff all labels present.
- Outer edges are only on the j^{th} facet.
- There are an odd number by induction!

Sperner's Lemma in higher dimensions

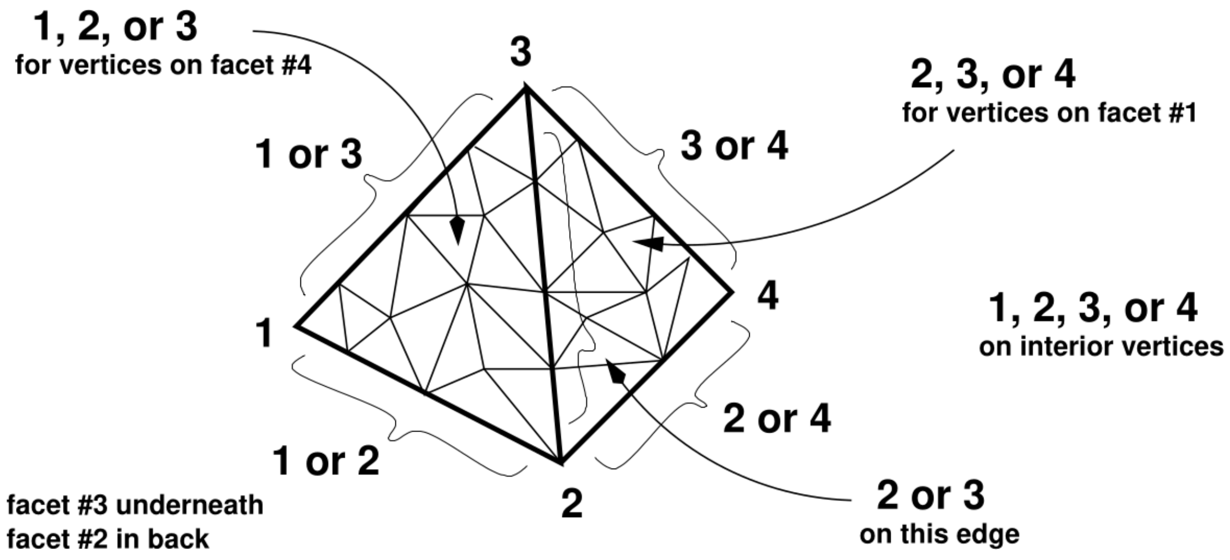
A *Sperner labeling* of a triangulated simplex on an n-dimensional simplex S on $n+1$ points is an assignment of labels in $[n+1]$ to each vertex such that, for each j, no vertices on the j^{th} facet of S have label j.

Sperner's Lemma

In any Sperner labeling, an odd number of simplices have all $n+1$ labels.

Proof. By induction on n. Base case ($n=1$):

There are an odd number of transitions between 1 and 2.



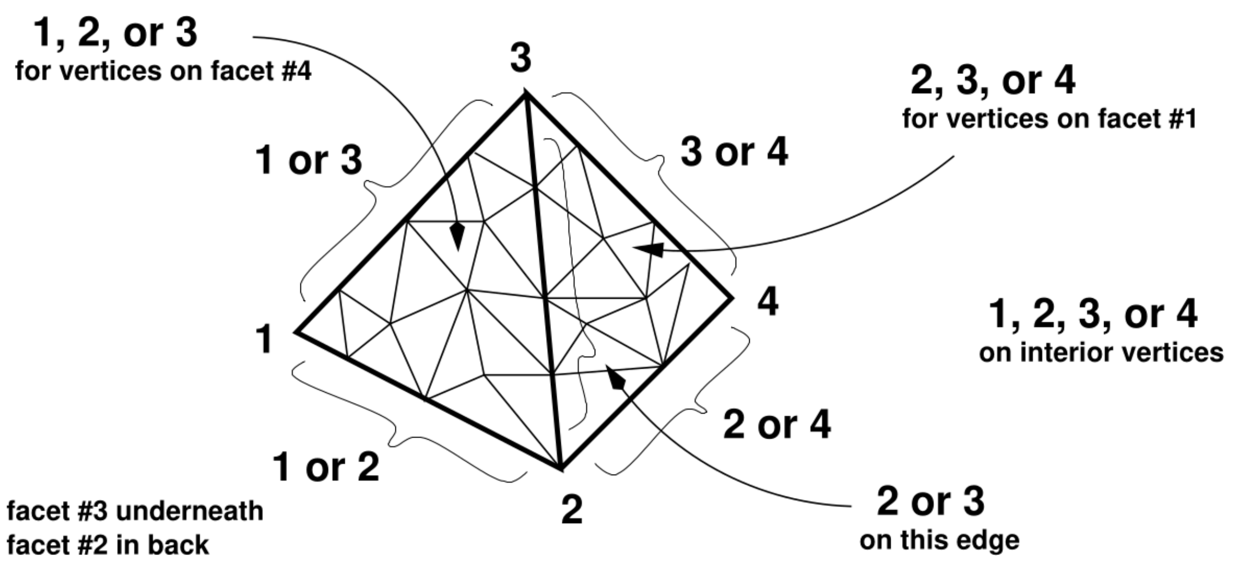
Now suppose the lemma holds for $n-1$. Define the dual subgraph G as before, with edges across $(n-1)$-dimensional faces labeled 1 through n.

- Internal rooms have degree 0, 1, or 2, with degree 1 iff all labels present.
- Outer edges are only on the j^{th} facet.
- There are an odd number by induction!

Thus, an odd number have all labels. ■

Sperner's Lemma in higher dimensions

A *Sperner labeling* of a triangulated simplex on an n-dimensional simplex S on $n+1$ points is an assignment of labels in $[n+1]$ to each vertex such that, for each j, no vertices on the j^{th} facet of S have label j.



Sperner's Lemma

In any Sperner labeling, an odd number of simplices have all $n+1$ labels.

Corollary (from Simmons, 1980):

Theorem (Stromquist, 1980)

There is always an envy-free allocation in which each piece is connected.

Sperner's Lemma in higher dimensions

A *Sperner labeling* of a triangulated simplex on an n-dimensional simplex S on $n+1$ points is an assignment of labels in $[n+1]$ to each vertex such that, for each j, no vertices on the j^{th} facet of S have label j.

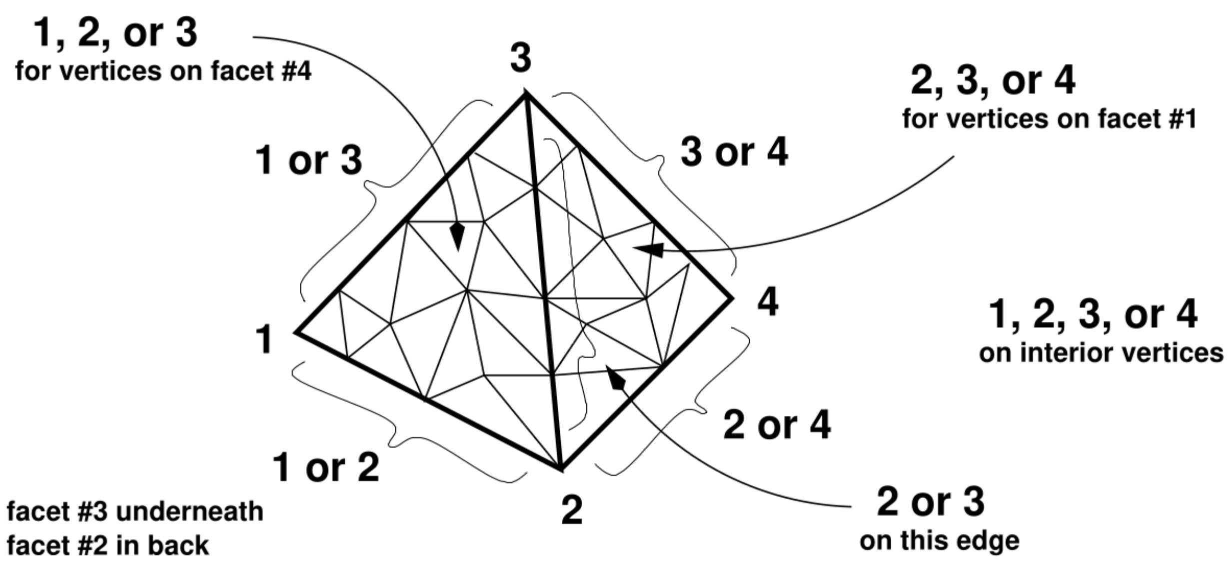
Sperner's Lemma

In any Sperner labeling, an odd number of simplices have all $n+1$ labels.

Corollary (from Simmons, 1980):

Theorem (Stromquist, 1980)

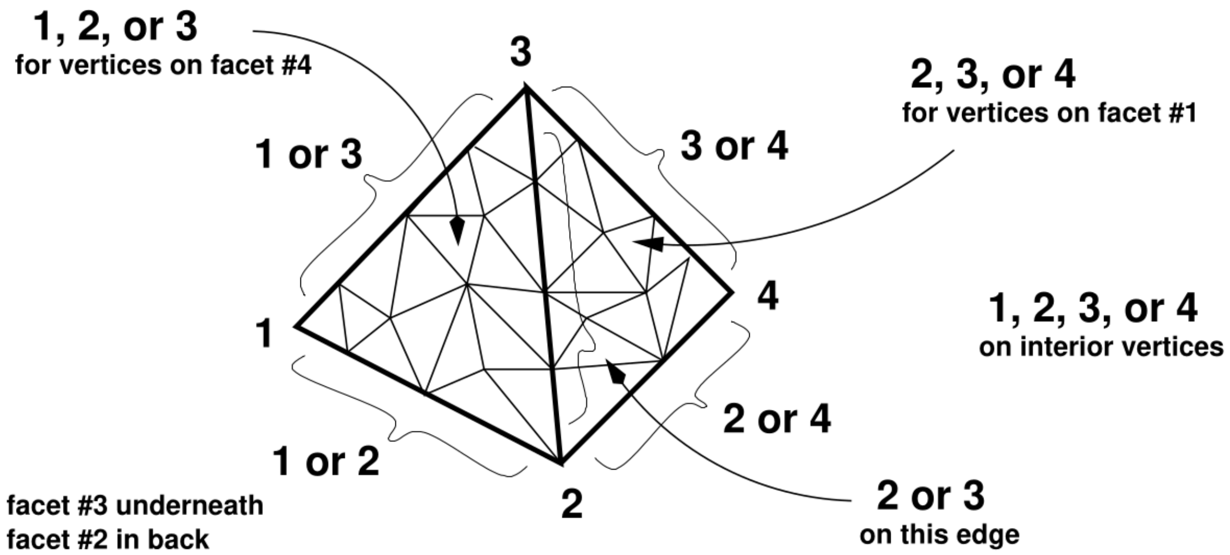
There is always an envy-free allocation in which each piece is connected.



Proof. Represent a cake cut into $n+1$ pieces as a point in the n-dimensional simplex.

Sperner's Lemma in higher dimensions

A *Sperner labeling* of a triangulated simplex on an n-dimensional simplex S on $n+1$ points is an assignment of labels in $[n+1]$ to each vertex such that, for each j, no vertices on the j^{th} facet of S have label j.



Sperner's Lemma

In any Sperner labeling, an odd number of simplices have all $n+1$ labels.

Corollary (from Simmons, 1980):

Theorem (Stromquist, 1980)

There is always an envy-free allocation in which each piece is connected.

Proof. Represent a cake cut into $n+1$ pieces as a point in the n-dimensional simplex.

Apply Sperner's lemma to the labeling where each player identifies their favorite piece, then take the limit. ■

Random assignment

What if we have to assign indivisible items and there's no money involved?
Use randomness!

Random assignment

What if we have to assign indivisible items and there's no money involved?
Use randomness!

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

Random assignment

What if we have to assign indivisible items and there's no money involved?
Use randomness!

Each player picks their favorite option in random order:

- With probability 1/6, J-I-P → bca

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

Random assignment

What if we have to assign indivisible items and there's no money involved?
Use randomness!

Each player picks their favorite option in random order:

- With probability 1/6, J-I-P → bca
- With probability 1/6, J-P-I → bac
- With probability 1/6, P-J-I → bac
- With probability 1/6, P-I-J → abc
- With probability 1/6, I-P-J → abc
- With probability 1/6, I-J-P → abc

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

Random assignment

What if we have to assign indivisible items and there's no money involved?
Use randomness!

Each player picks their favorite option in random order:

- With probability 1/6, J-I-P → bca
- With probability 1/6, J-P-I → bac
- With probability 1/6, P-J-I → bac
- With probability 1/6, P-I-J → abc
- With probability 1/6, I-P-J → abc
- With probability 1/6, I-J-P → abc

So the allocation is abc w.p. 1/2, bac w.p. 1/3, bca w.p. 1/6.

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

Random assignment

What if we have to assign indivisible items and there's no money involved?
Use randomness!

Each player picks their favorite option in random order:

- With probability 1/6, J-I-P → bca
- With probability 1/6, J-P-I → bac
- With probability 1/6, P-J-I → bac
- With probability 1/6, P-I-J → abc
- With probability 1/6, I-P-J → abc
- With probability 1/6, I-J-P → abc

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

So the allocation is abc w.p. 1/2, bac w.p. 1/3, bca w.p. 1/6.

This method is known as *Random Serial Dictatorship (RSD)*.

RSD properties

Image credit: Gemini 3 - "Generate a picture embodying 'random serial dictatorship'"

RSD properties

Image credit: Gemini 3 - "Generate a picture embodying 'random serial dictatorship'"

Assume preferences are strict. Which properties does RSD satisfy?

- **Ex ante strategyproofness**
- **Ex post Pareto efficiency**
- **Both**
- **Neither**

Respond at:

pollev.com/jtuckerfoltz255 or

bit.ly/jtfpoll or

text jtuckerfoltz255 to 37607

RSD properties

Image credit: Gemini 3 - "Generate a picture embodying 'random serial dictatorship'"

Assume preferences are strict. Which properties does RSD satisfy?

- Ex ante strategyproofness
- Ex post Pareto efficiency
- **Both ✓**
- Neither

RSD properties

Image credit: Gemini 3 - "Generate a picture embodying 'random serial dictatorship'"

Assume preferences are strict. Which properties does RSD satisfy?

- Ex ante strategyproofness
- Ex post Pareto efficiency
- **Both** ✓
- Neither

Proof of PE. WLOG, assume we have an allocation A where players picked in order 1, 2, ..., n.

RSD properties

Image credit: Gemini 3 - "Generate a picture embodying 'random serial dictatorship'"

Assume preferences are strict. Which properties does RSD satisfy?

- Ex ante strategyproofness
- Ex post Pareto efficiency
- **Both ✔**
- Neither

Proof of PE. WLOG, assume we have an allocation A where players picked in order 1, 2, ..., n.

Consider an alternative assignment A'. Let i be the minimum index where A' differs from A.

RSD properties

Image credit: Gemini 3 - "Generate a picture embodying 'random serial dictatorship'"

▸ **Assume preferences are strict. Which properties does RSD satisfy?**

- **Ex ante strategyproofness**
- **Ex post Pareto efficiency**
- **Both ✔**
- **Neither**

Proof of PE. WLOG, assume we have an allocation A where players picked in order 1, 2, ..., n.

Consider an alternative assignment A'. Let i be the minimum index where A' differs from A.

Since i can only be assigned something worse than what they would have picked as the dictator, they prefer A to A'. Hence, A' is not a Pareto improvement. ■

Is ex post Pareto efficiency enough?

P1	P2	P3	P4
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

Is ex post Pareto efficiency enough?

P1	P2	P3	P4
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

Under RSD, probability each player gets their

- 4^{th} choice:

Is ex post Pareto efficiency enough?

P1	P2	P3	P4
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

Under RSD, probability each player gets their

- 4^{th} choice: 1/4

Is ex post Pareto efficiency enough?

P1	P2	P3	P4
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

Under RSD, probability each player gets their

- 4^{th} choice: 1/4
- 3^{rd} choice:

Is ex post Pareto efficiency enough?

P1	P2	P3	P4
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

Under RSD, probability each player gets their

- 4^{th} choice: 1/4
- 3^{rd} choice: 1/4

Is ex post Pareto efficiency enough?

P1	P2	P3	P4
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

Under RSD, probability each player gets their

- 4^{th} choice: 1/4
- 3^{rd} choice: 1/4
- 2^{nd} choice:

Is ex post Pareto efficiency enough?

P1	P2	P3	P4
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

Under RSD, probability each player gets their

- 4^{th} choice: 1/4
- 3^{rd} choice: 1/4
- 2^{nd} choice: > 0

Is ex post Pareto efficiency enough?

P1	P2	P3	P4
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

Under RSD, probability each player gets their

- 4th choice: 1/4
- 3rd choice: 1/4
- 2nd choice: > 0
- 1st choice: < 1/2

Is ex post Pareto efficiency enough?

P1	P2	P3	P4
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

Under RSD, probability each player gets their

- 4^{th} choice: 1/4
- 3^{rd} choice: 1/4
- 2^{nd} choice: > 0
- 1^{st} choice: < 1/2

Consider the following alternative lottery:

- With probability 1/4, acbd

Is ex post Pareto efficiency enough?

P1	P2	P3	P4
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

Under RSD, probability each player gets their

- 4^{th} choice: 1/4
- 3^{rd} choice: 1/4
- 2^{nd} choice: > 0
- 1^{st} choice: < 1/2

Consider the following alternative lottery:

- With probability 1/4, acbd
- With probability 1/4, adbc

Is ex post Pareto efficiency enough?

P1	P2	P3	P4
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

Under RSD, probability each player gets their

- 4^{th} choice: 1/4
- 3^{rd} choice: 1/4
- 2^{nd} choice: > 0
- 1^{st} choice: < 1/2

Consider the following alternative lottery:

- With probability 1/4, acbd
- With probability 1/4, adbc
- With probability 1/4, cadb

Is ex post Pareto efficiency enough?

P1	P2	P3	P4
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

Under RSD, probability each player gets their

- 4^{th} choice: 1/4
- 3^{rd} choice: 1/4
- 2^{nd} choice: > 0
- 1^{st} choice: < 1/2

Consider the following alternative lottery:

- With probability 1/4, acbd
- With probability 1/4, adbc
- With probability 1/4, cadb
- With probability 1/4, dacb

Is ex post Pareto efficiency enough?

P1	P2	P3	P4
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

Under RSD, probability each player gets their

- 4^{th} choice: 1/4
- 3^{rd} choice: 1/4
- 2^{nd} choice: > 0
- 1^{st} choice: $< 1/2$

Consider the following alternative lottery:

- With probability 1/4, acbd
- With probability 1/4, adbc
- With probability 1/4, cadb
- With probability 1/4, dacb

Under this new lottery, the probabilities are:

- 4^{th} choice: 1/4
- 3^{rd} choice: 1/4
- 2^{nd} choice: 0
- 1^{st} choice: 1/2

Is ex post Pareto efficiency enough?

P1	P2	P3	P4
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

Consider the following alternative lottery:

- With probability 1/4, acbd
- With probability 1/4, adbc
- With probability 1/4, cadb
- With probability 1/4, dacb

Under RSD, probability each player gets their

- 4^{th} choice: 1/4
- 3^{rd} choice: 1/4
- 2^{nd} choice: > 0
- 1^{st} choice: < 1/2

Under this new lottery, the probabilities are:

- 4^{th} choice: 1/4
- 3^{rd} choice: 1/4
- 2^{nd} choice: 0
- 1^{st} choice: 1/2

Thus, RSD is Pareto-dominated ex ante for *any* utility function consistent with strict preferences. We thus say that it violates *ordinal efficiency*.

Probabilistic Serial (PS)

Each player "eats" their favorite remaining alternative at a constant rate. This defines a randomized allocation for each individual.

Probabilistic Serial (PS)

Each player "eats" their favorite remaining alternative at a constant rate. This defines a randomized allocation for each individual.

A fancy theorem then implies we can correlate these marginal allocations into a global lottery over assignments.

Probabilistic Serial (PS)

Each player "eats" their favorite remaining alternative at a constant rate. This defines a randomized allocation for each individual.

A fancy theorem then implies we can correlate these marginal allocations into a global lottery over assignments.

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

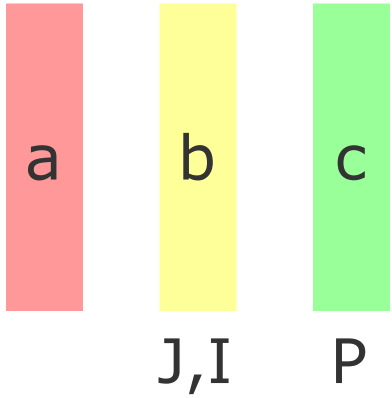
Probabilistic Serial (PS)

Each player "eats" their favorite remaining alternative at a constant rate. This defines a randomized allocation for each individual.

A fancy theorem then implies we can correlate these marginal allocations into a global lottery over assignments.

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

	a	b	c
Jamie	0	0	0
Irina	0	0	0
Paul	0	0	0



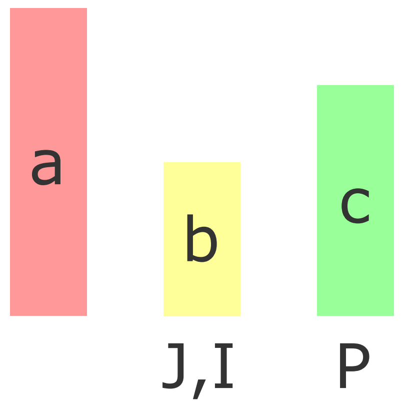
Probabilistic Serial (PS)

Each player "eats" their favorite remaining alternative at a constant rate. This defines a randomized allocation for each individual.

A fancy theorem then implies we can correlate these marginal allocations into a global lottery over assignments.

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

	a	b	c
Jamie	0	1/4	0
Irina	0	1/4	0
Paul	0	0	1/4



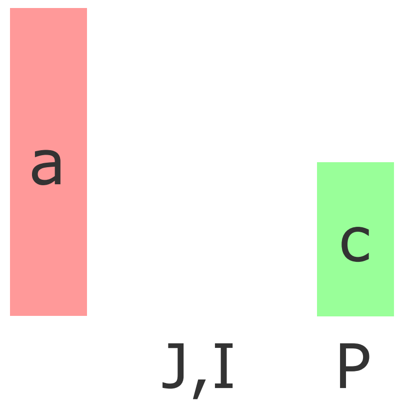
Probabilistic Serial (PS)

Each player "eats" their favorite remaining alternative at a constant rate. This defines a randomized allocation for each individual.

A fancy theorem then implies we can correlate these marginal allocations into a global lottery over assignments.

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

	a	b	c
Jamie	0	1/2	0
Irina	0	1/2	0
Paul	0	0	1/2



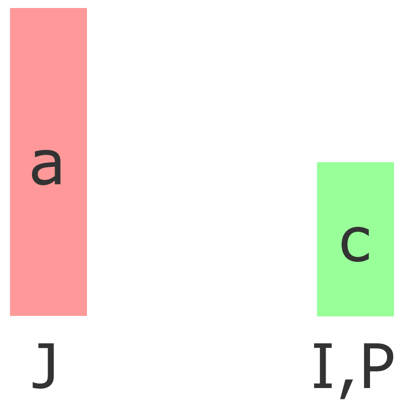
Probabilistic Serial (PS)

Each player "eats" their favorite remaining alternative at a constant rate. This defines a randomized allocation for each individual.

A fancy theorem then implies we can correlate these marginal allocations into a global lottery over assignments.

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

	a	b	c
Jamie	0	1/2	0
Irina	0	1/2	0
Paul	0	0	1/2



Probabilistic Serial (PS)

Each player "eats" their favorite remaining alternative at a constant rate. This defines a randomized allocation for each individual.

A fancy theorem then implies we can correlate these marginal allocations into a global lottery over assignments.

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

	a	b	c
Jamie	1/4	1/2	0
Irina	0	1/2	1/4
Paul	0	0	3/4

a

J I,P

Probabilistic Serial (PS)

Each player "eats" their favorite remaining alternative at a constant rate. This defines a randomized allocation for each individual.

A fancy theorem then implies we can correlate these marginal allocations into a global lottery over assignments.

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

	a	b	c
Jamie	1/4	1/2	0
Irina	0	1/2	1/4
Paul	0	0	3/4

a

J,I,P

Probabilistic Serial (PS)

Each player "eats" their favorite remaining alternative at a constant rate. This defines a randomized allocation for each individual.

A fancy theorem then implies we can correlate these marginal allocations into a global lottery over assignments.

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

	a	b	c
Jamie	1/2	1/2	0
Irina	1/4	1/2	1/4
Paul	1/4	0	3/4

Probabilistic Serial (PS)

Each player "eats" their favorite remaining alternative at a constant rate. This defines a randomized allocation for each individual.

A fancy theorem then implies we can correlate these marginal allocations into a global lottery over assignments.

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

	a	b	c
Jamie	1/2	1/2	0
Irina	1/4	1/2	1/4
Paul	1/4	0	3/4

Birkoff-von Neumann Theorem

Any bistochastic matrix is a convex combination of permutation matrices.

Probabilistic Serial (PS)

Each player "eats" their favorite remaining alternative at a constant rate. This defines a randomized allocation for each individual.

A fancy theorem then implies we can correlate these marginal allocations into a global lottery over assignments.

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

$$\begin{array}{c|ccc} & a & b & c \\ \hline \text{Jamie} & 1/2 & 1/2 & 0 \\ \text{Irina} & 1/4 & 1/2 & 1/4 \\ \text{Paul} & 1/4 & 0 & 3/4 \end{array} = \frac{1}{2}\left(\begin{array}{ccc} a & b & c \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) + \frac{1}{4}\left(\begin{array}{ccc} a & b & c \\ \hline 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right) + \frac{1}{4}\left(\begin{array}{ccc} a & b & c \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right)$$

Birkoff-von Neumann Theorem

Any bistochastic matrix is a convex combination of permutation matrices.

Probabilistic Serial (PS)

Each player "eats" their favorite remaining alternative at a constant rate. This defines a randomized allocation for each individual.

A fancy theorem then implies we can correlate these marginal allocations into a global lottery over assignments.

Jamie	Irina	Paul
b	b	c
a	c	a
c	a	b

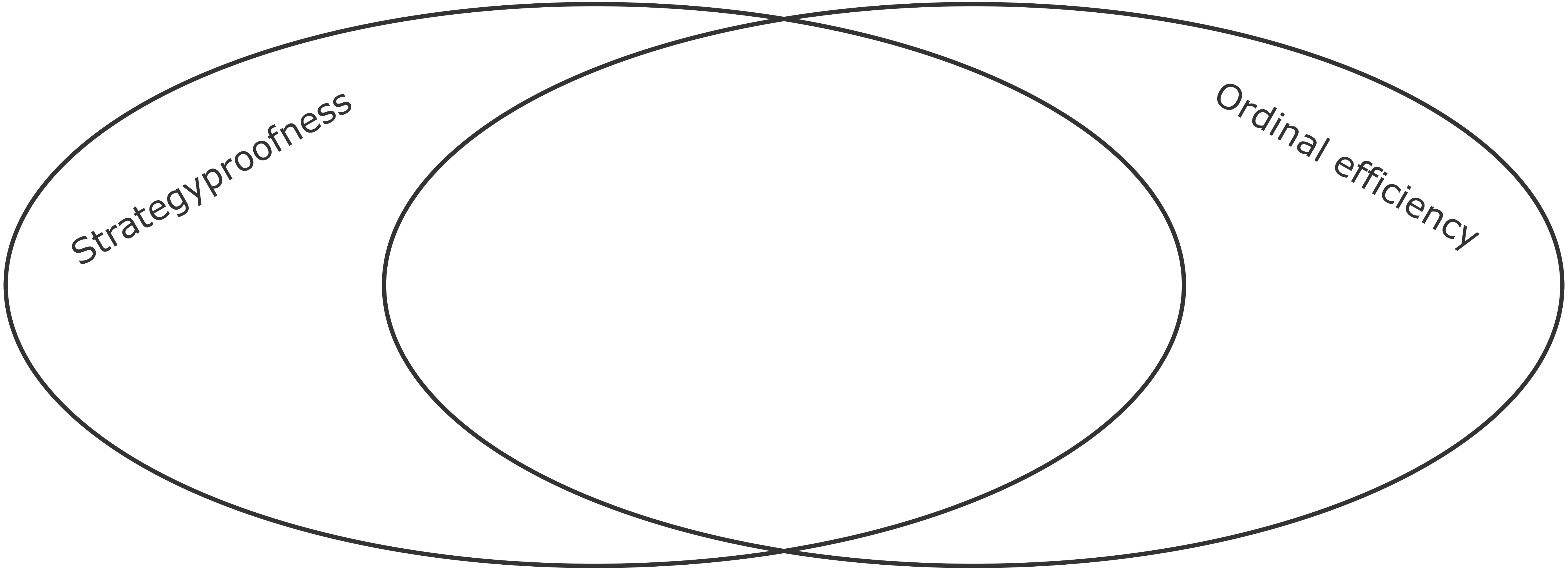
$$
\begin{array}{c|ccc} & a & b & c \\ \hline \text{Jamie} & 1/2 & 1/2 & 0 \\ \text{Irina} & 1/4 & 1/2 & 1/4 \\ \text{Paul} & 1/4 & 0 & 3/4 \end{array}
= \frac{1}{2}\left(\begin{array}{ccc} a & b & c \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)
+ \frac{1}{4}\left(\begin{array}{ccc} a & b & c \\ \hline 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right)
+ \frac{1}{4}\left(\begin{array}{ccc} a & b & c \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right)
$$

Birkoff-von Neumann Theorem

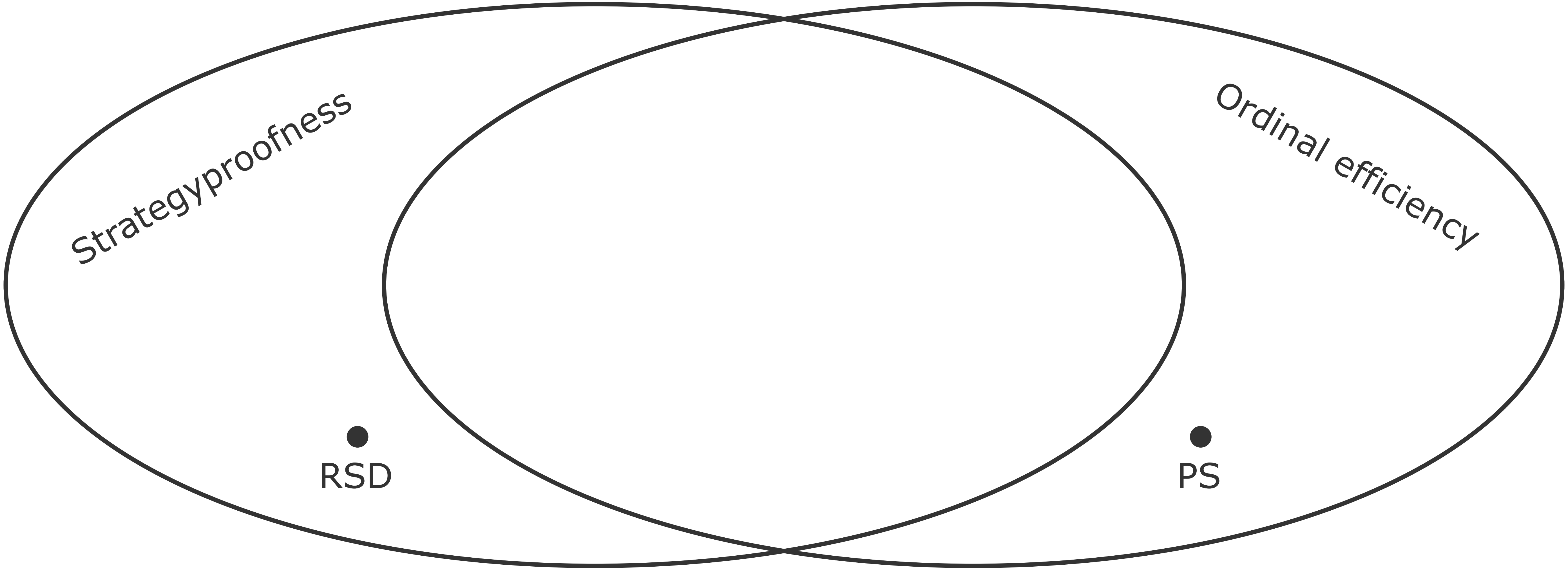
Any bistochastic matrix is a convex combination of permutation matrices.

So the final assignment is
abc w.p. 1/2,
bac w.p. 1/4,
bca w.p. 1/4.

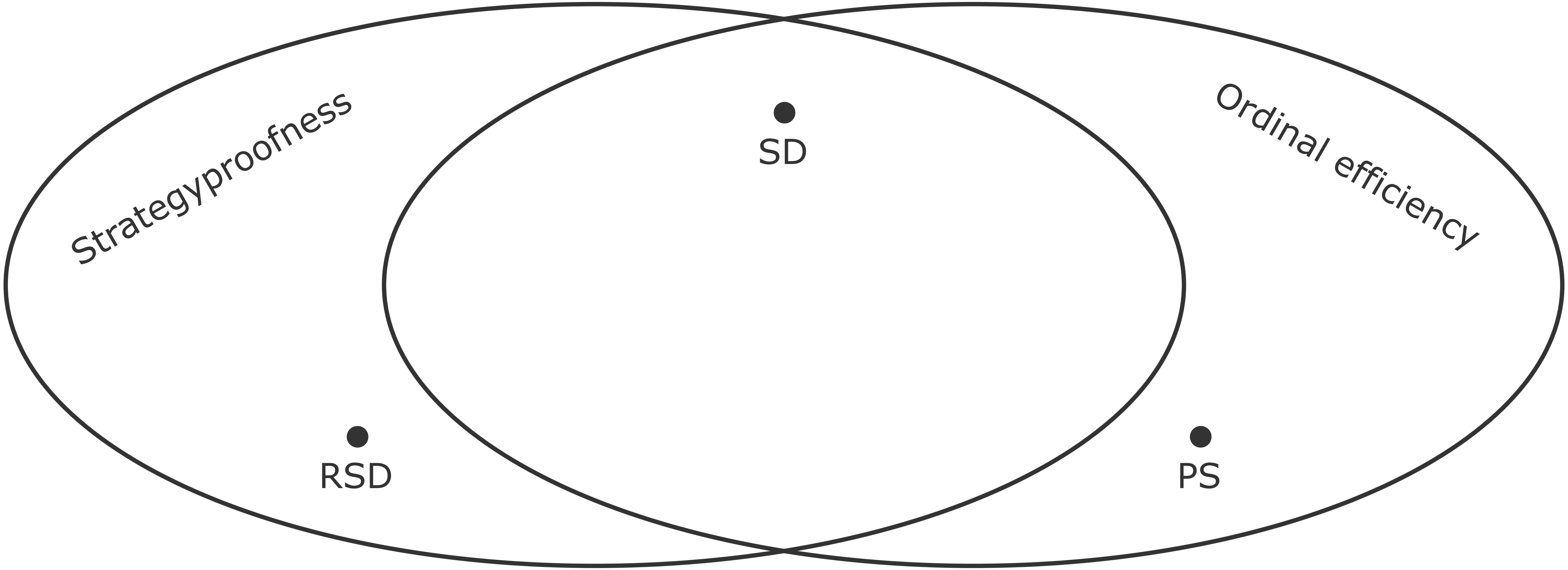
Comparison of random assignment rules (Bogomolnaia, Moulin, 2001)



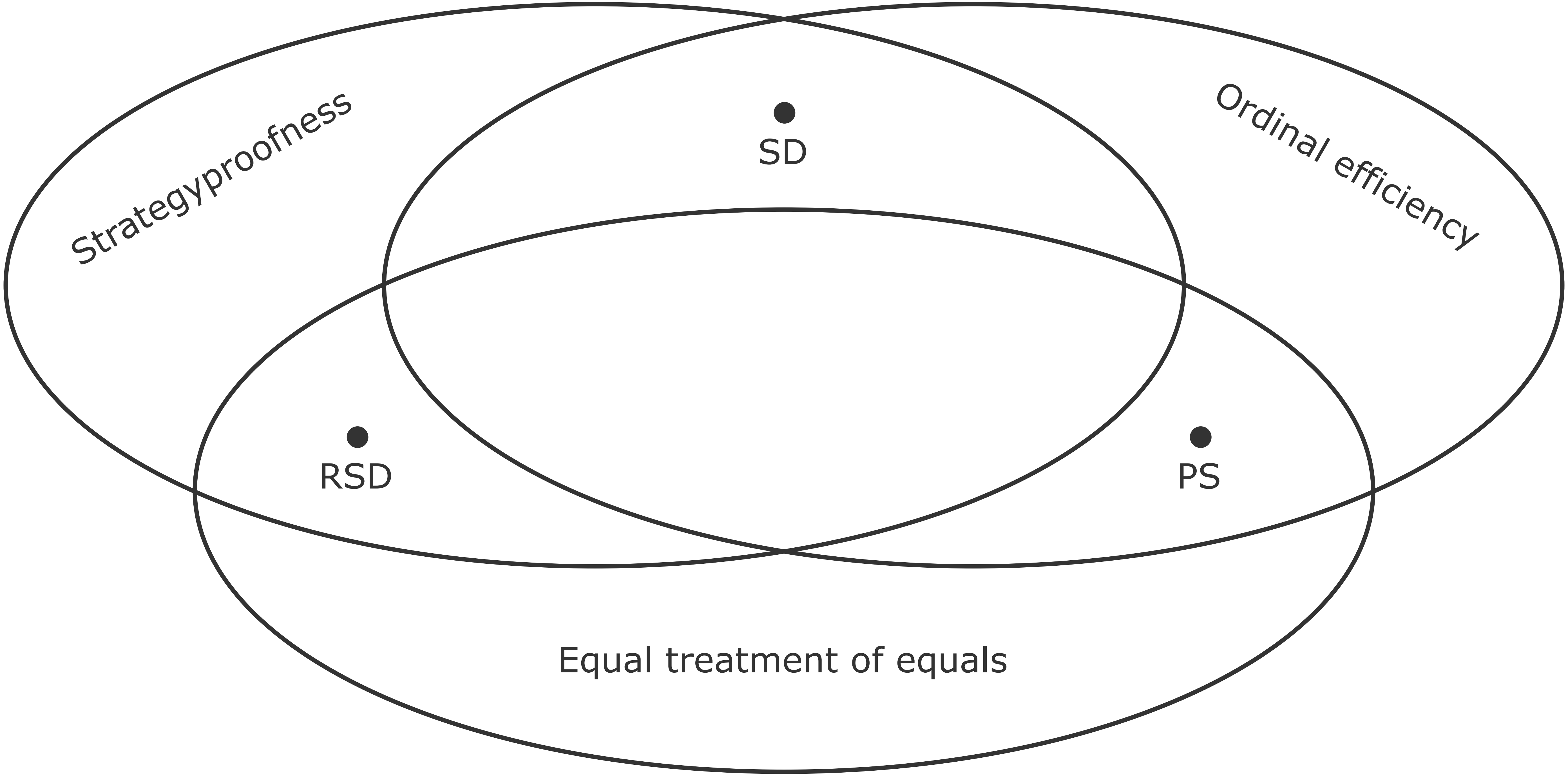
Comparison of random assignment rules (Bogomolnaia, Moulin, 2001)



Comparison of random assignment rules (Bogomolnaia, Moulin, 2001)



Comparison of random assignment rules (Bogomolnaia, Moulin, 2001)



Comparison of random assignment rules (Bogomolnaia, Moulin, 2001)