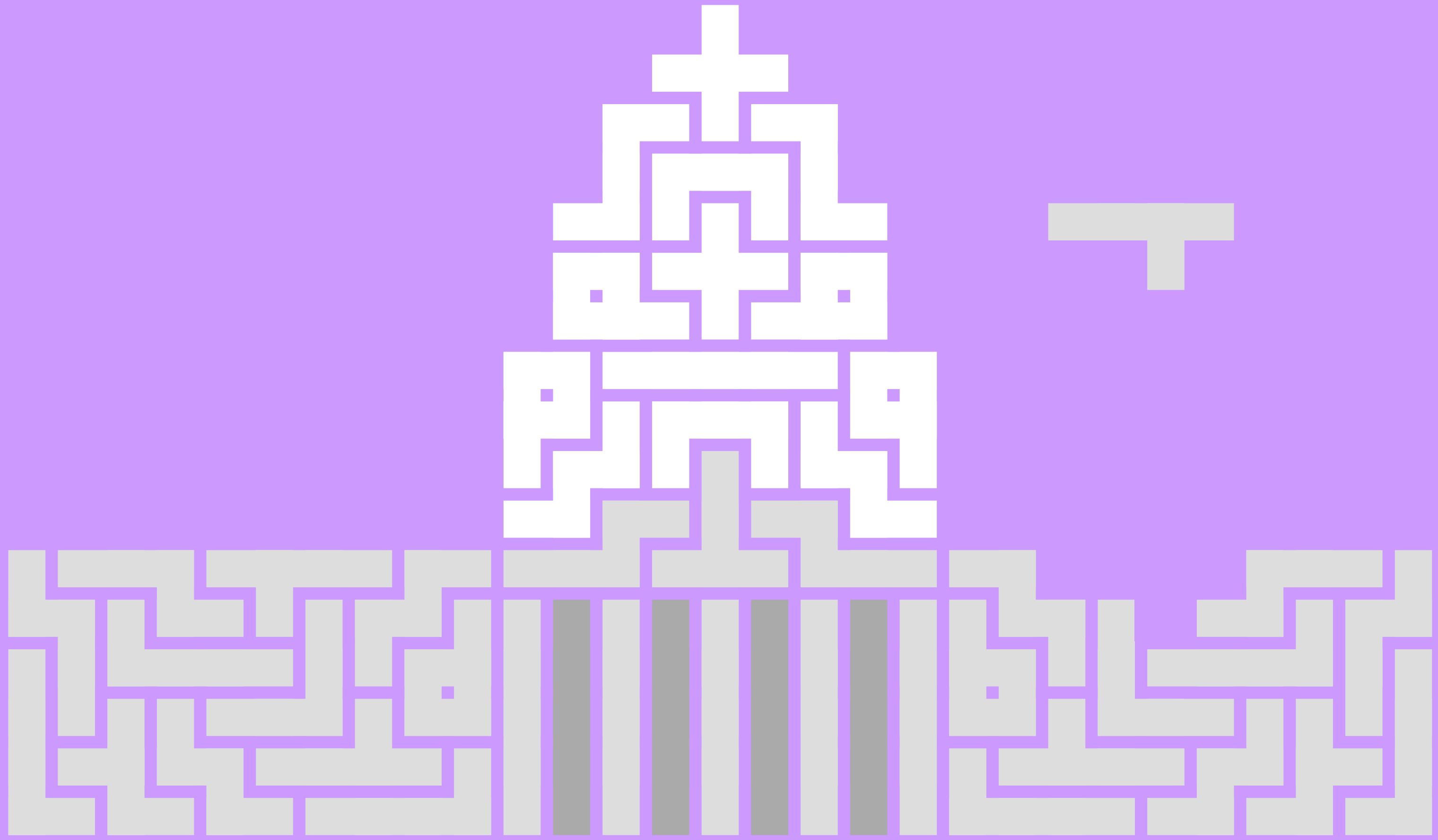

Algorithms For Democratic Decision-Making

Jamie Tucker-Foltz • Yale University • Spring 2026

Lecture 18: **Apportionment 1**

Announcements

Project updates due next Monday night. Like the proposal, this is counted for completion only, and is your chance to get feedback!

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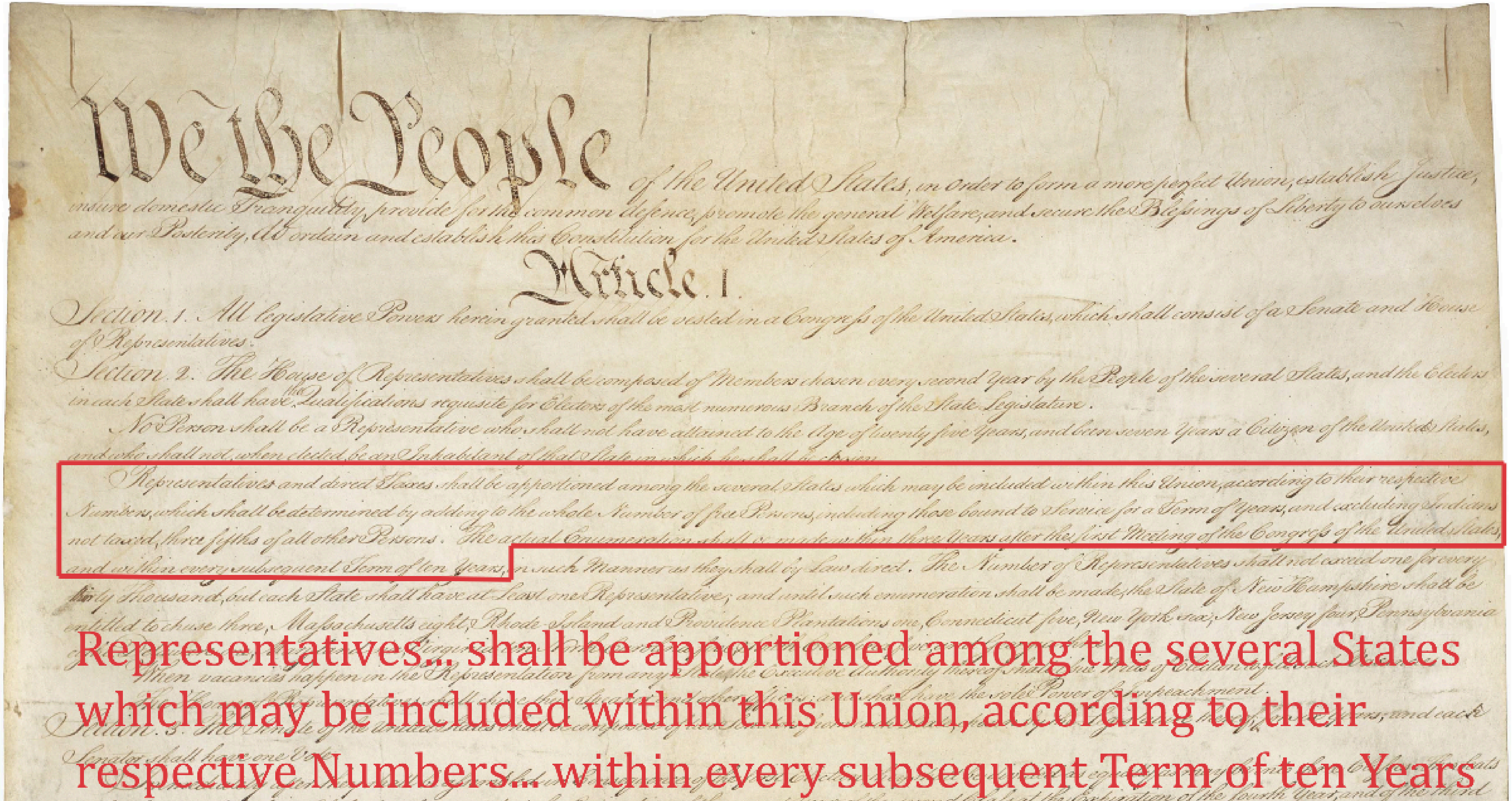
Project updates due next Monday night. Like the proposal, this is counted for completion only, and is your chance to get feedback!

This Wednesday: Lecture over Zoom again

Next Monday: Back to in-person

Next Wednesday: No class, individual meetings to discuss projects

Apportionment in the US



Representatives… shall be apportioned among the several States which may be included within this Union, according to their respective Numbers… within every subsequent Term of ten Years

Apportionment in parliamentary democracies

Party	Winning vote share (-20% / 30% / 40% / 50% / 60%+)	Statewide vote	Seats: Districts	Seats: Overall
Social Democratic Party *Sozialdemokratische Partei Deutschlands (SPD)*		28.29%	33	47
Christian Democratic Union *Christlich Demokratische Union (CDU)*		23.35%	25	39
Alliance 90/The Greens *Bündnis 90/Die Grünen (GRÜNE)*		17.59%	11	29
The Left *DIE LINKE*		11.71%	9	19
Pirate Party *Piratenpartei (PIRATEN)*		8.90%	0	15

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2023 Dutch general election

← 2021 **22 November 2023** *Next* →

All 150 seats in the House of Representatives
76 seats needed for a majority

Turnout 77.75% (▼ 0.96pp)

Party	Leader	%	Seats	+/–
PVV	Geert Wilders	23.49	37	+20
GL/PvdA	Frans Timmermans	15.75	25	+8
VVD	Dilan Yeşilgöz	15.24	24	−10
NSC	Pieter Omtzigt	12.88	20	New
D66	Rob Jetten	6.29	9	−15
BBB	Caroline van der Plas	4.65	7	+6
CDA	Henri Bontenbal	3.31	5	−10
SP	Lilian Marijnissen	3.15	5	−4
Denk	Stephan van Baarle	2.37	3	0
PvdD	Esther Ouwehand	2.25	3	−3
FvD	Thierry Baudet	2.23	3	−5
SGP	Chris Stoffer	2.08	3	0
CU	Mirjam Bikker	2.04	3	−2
Volt	Laurens Dassen	1.71	2	−1
JA21	Joost Eerdmans	0.68	1	−2

Running example

Example

Suppose there are 1100 voters and 11 seats in parliament.

Party 1 votes: 110

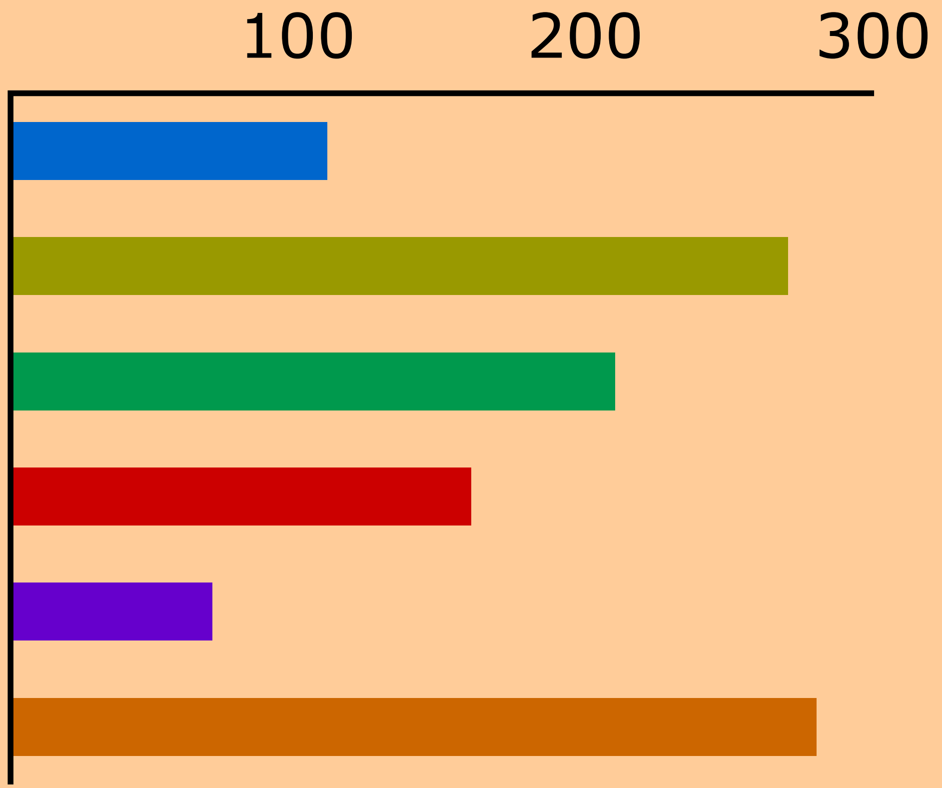
Party 2 votes: 270

Party 3 votes: 210

Party 4 votes: 160

Party 5 votes: 70

Party 6 votes: 280



Running example

Example

Suppose there are 1100 voters and 11 seats in parliament.

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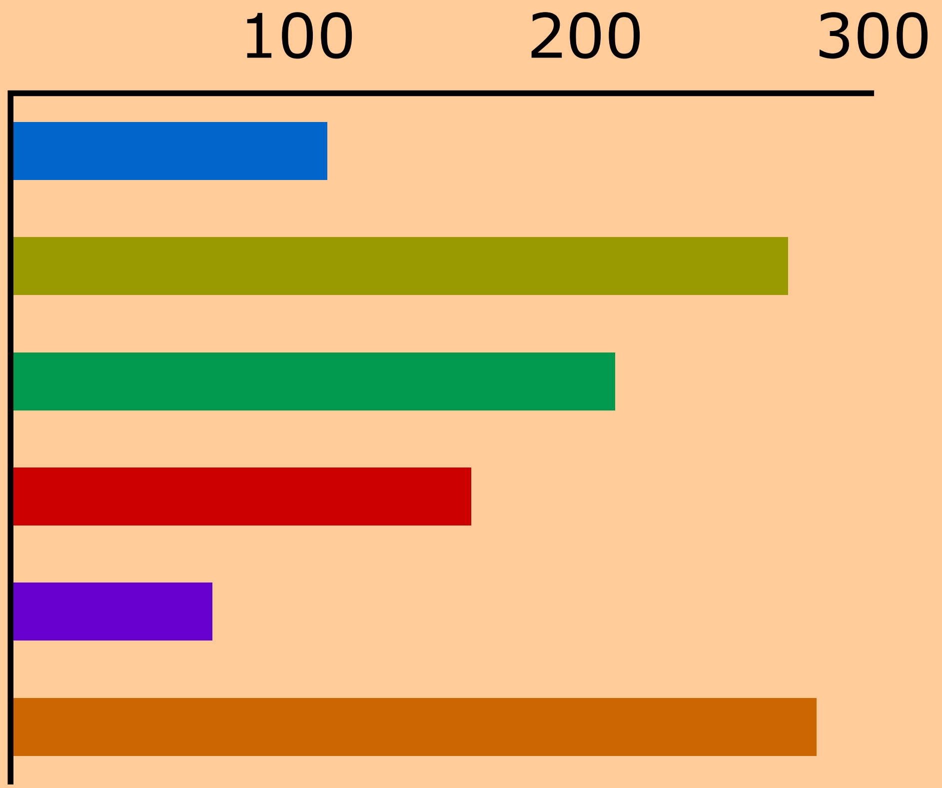

- **How many seats should each party get?**

(No right answer!)

Respond at:

pollev.com/jtuckerfoltz255 or

bit.ly/jtfpoll or

text jtuckerfoltz255 to 37607

Hamilton's method

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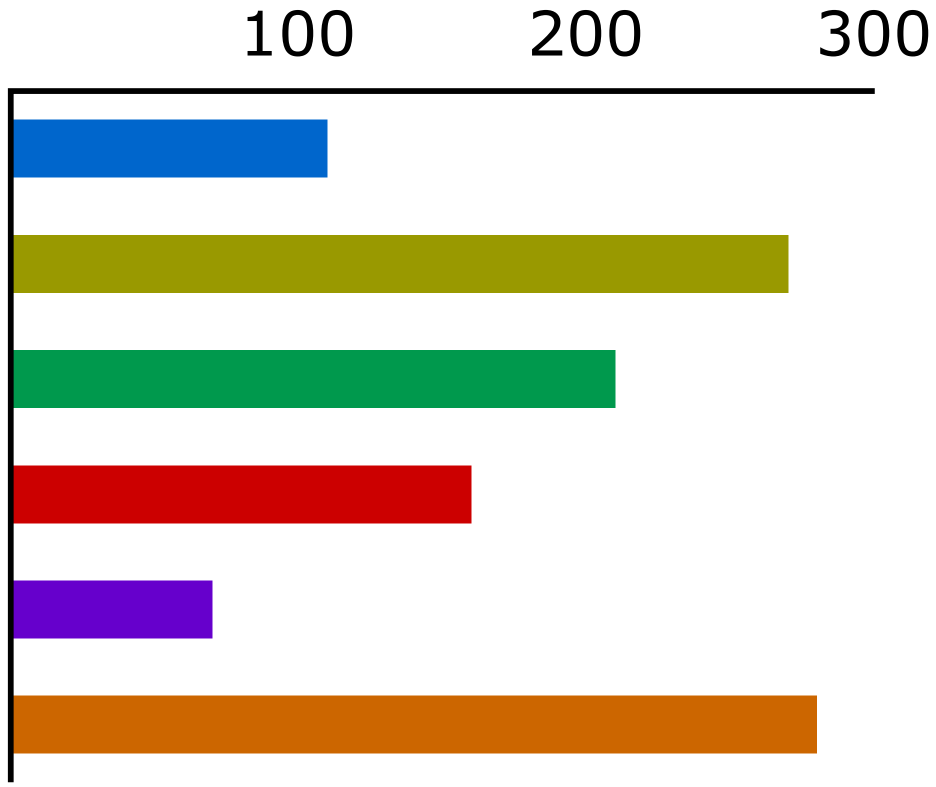
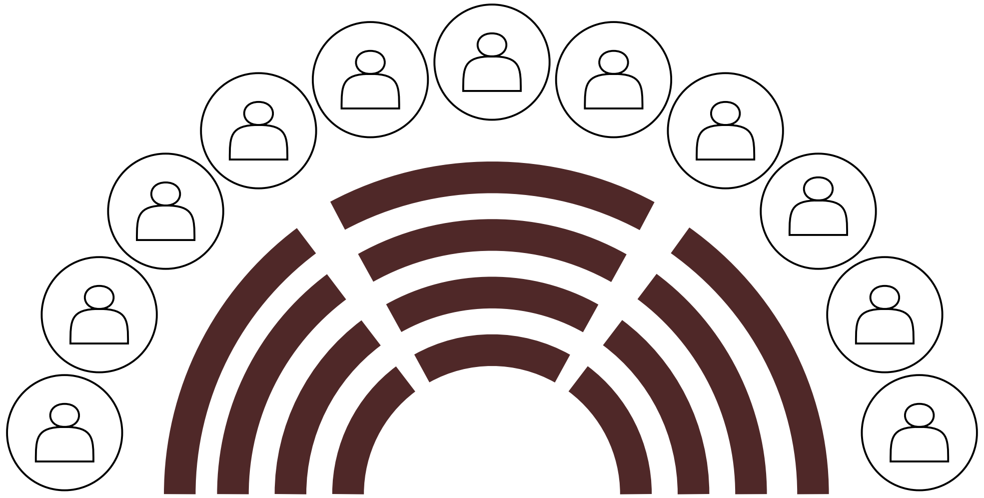




Hamilton's method

Party 1 quota: 1.1

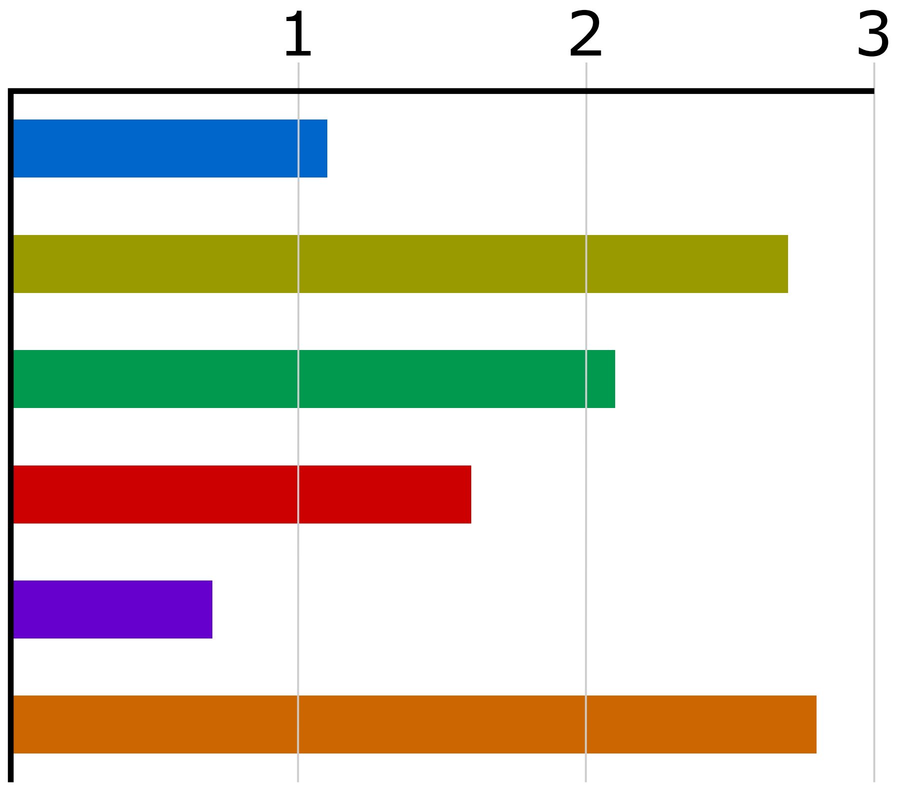
Party 2 quota: 2.7

Party 3 quota: 2.1

Party 4 quota: 1.6

Party 5 quota: 0.7

Party 6 quota: 2.8



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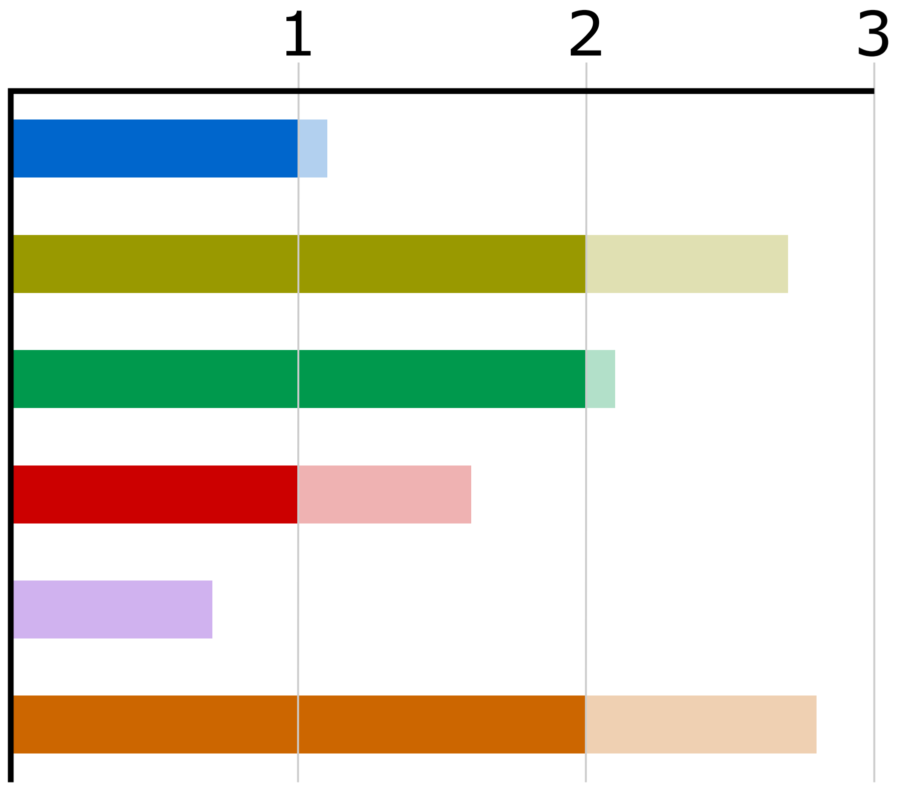

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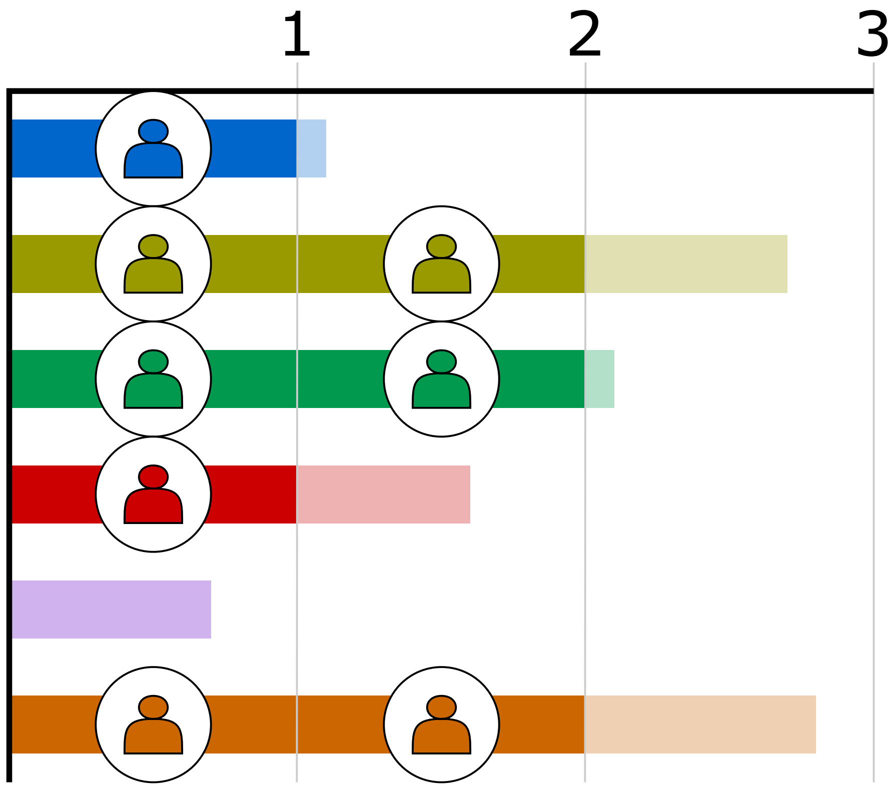

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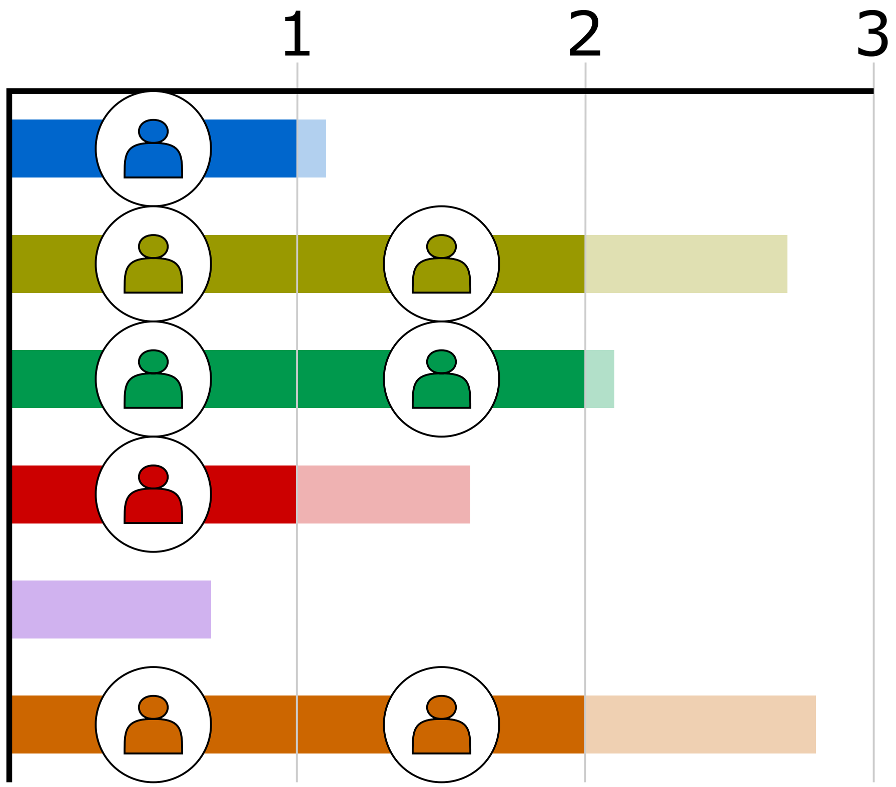
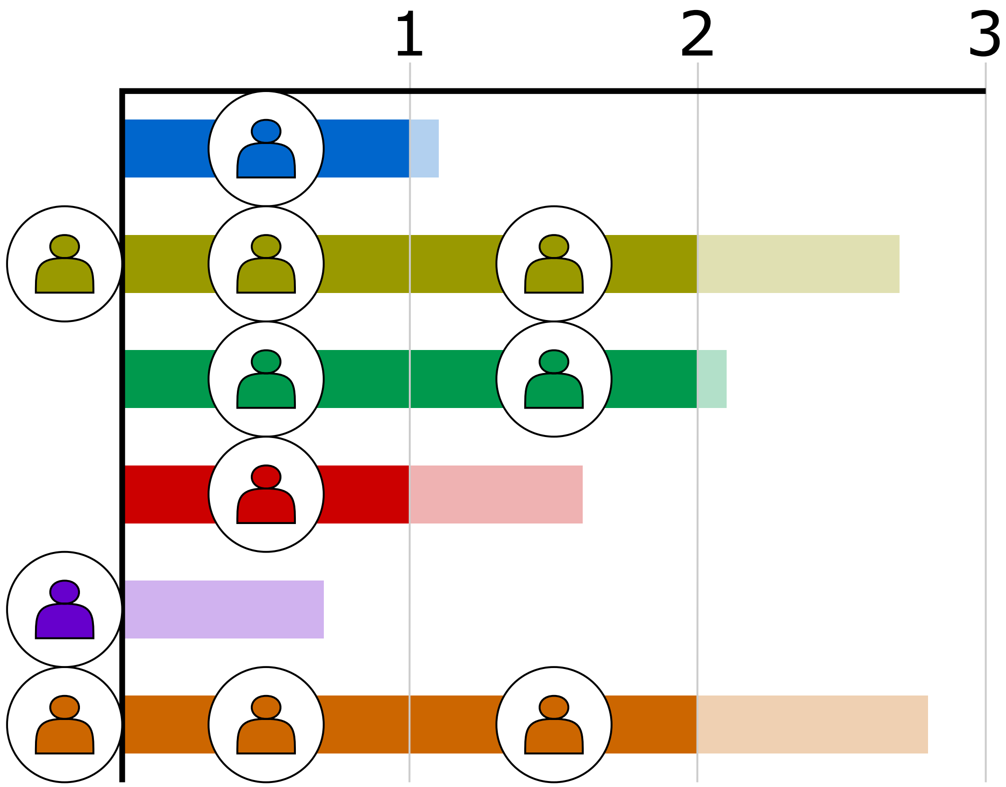


residue := $q_i - \lfloor q_i \rfloor$

Hamilton's method



#Seats	Ham
Party 1	1
Party 2	3
Party 3	2
Party 4	1
Party 5	1
Party 6	3

Jefferson's method

Divisor methods: Fix a rounding rule, then scale populations to get the right num seats.

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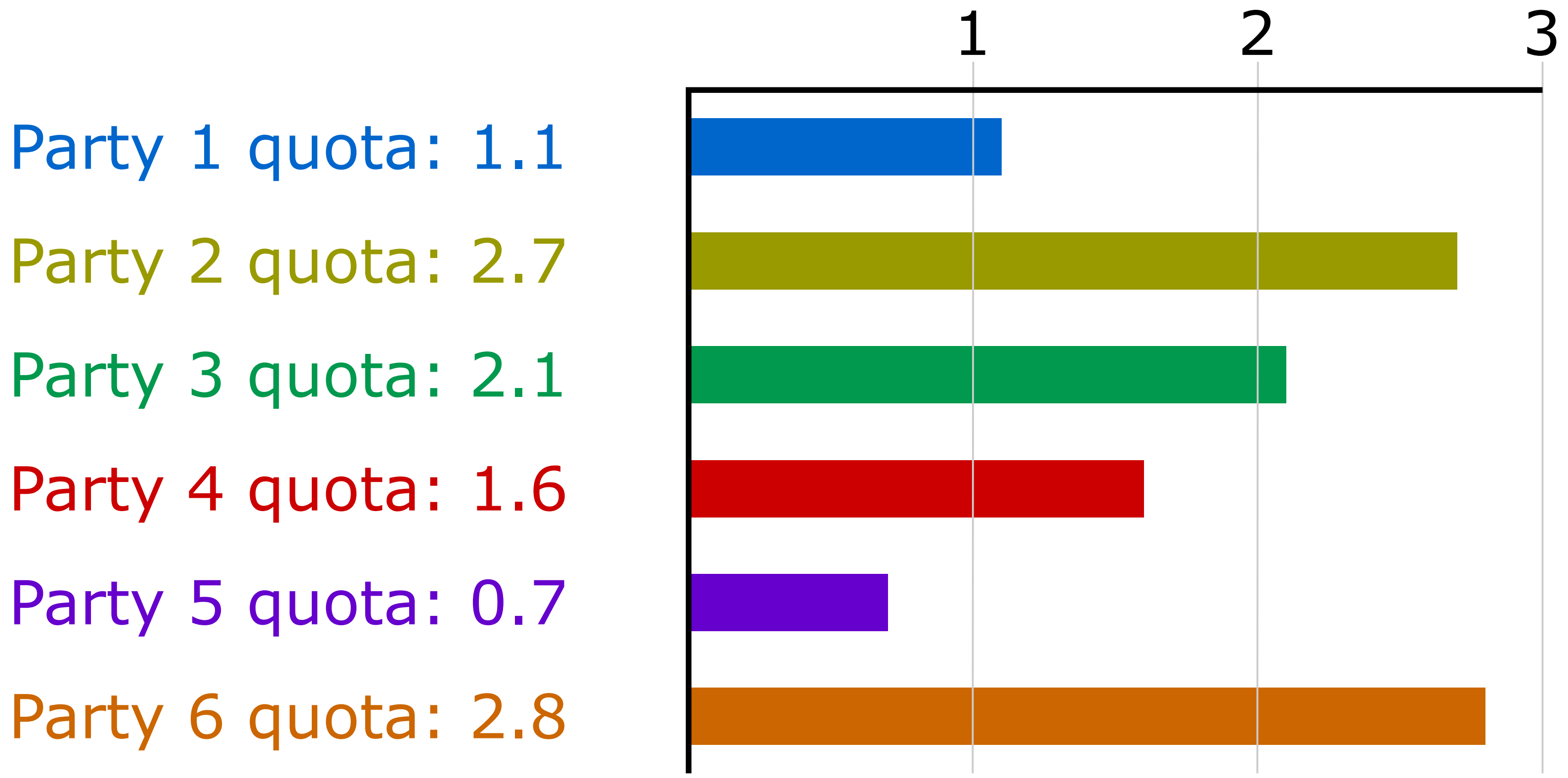

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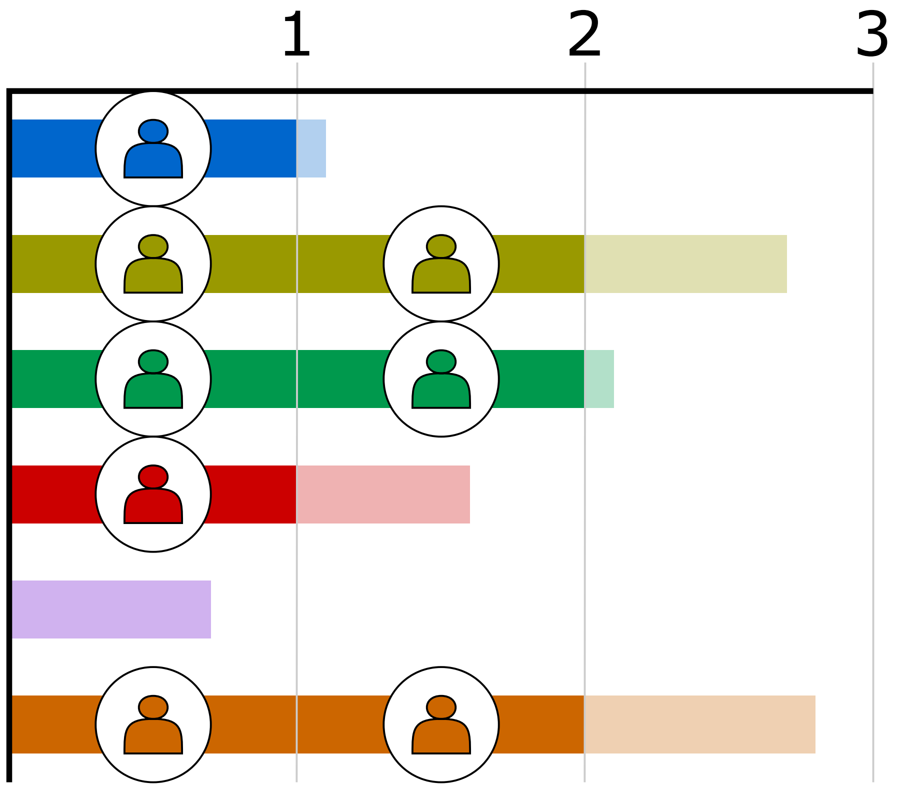

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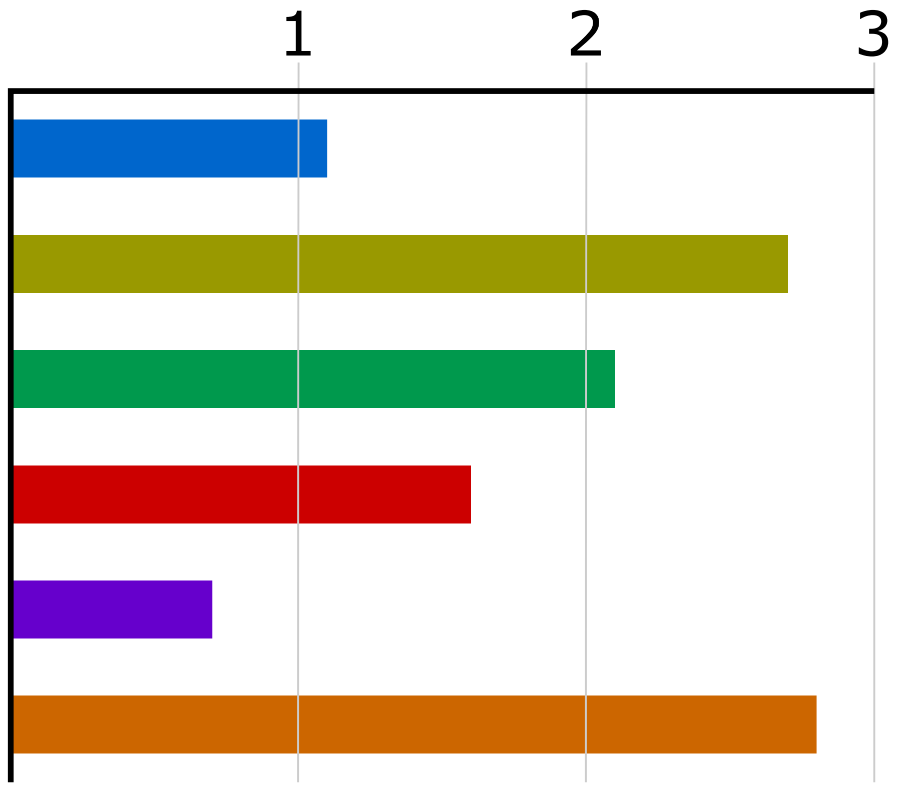

Jefferson's method: Round down - need to scale up first

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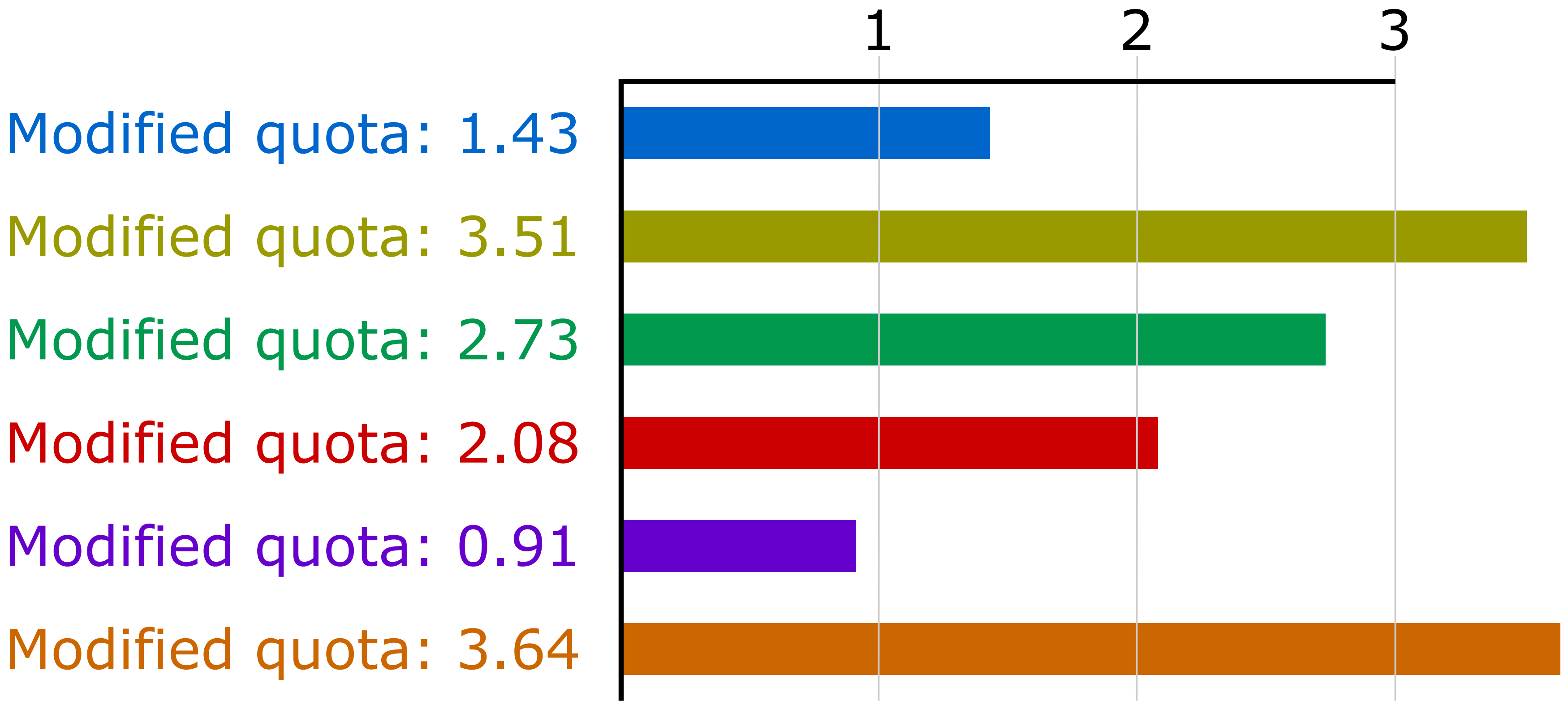

Jefferson's method: Round down - need to scale up first

The correct scale factor turns out to be 1.3

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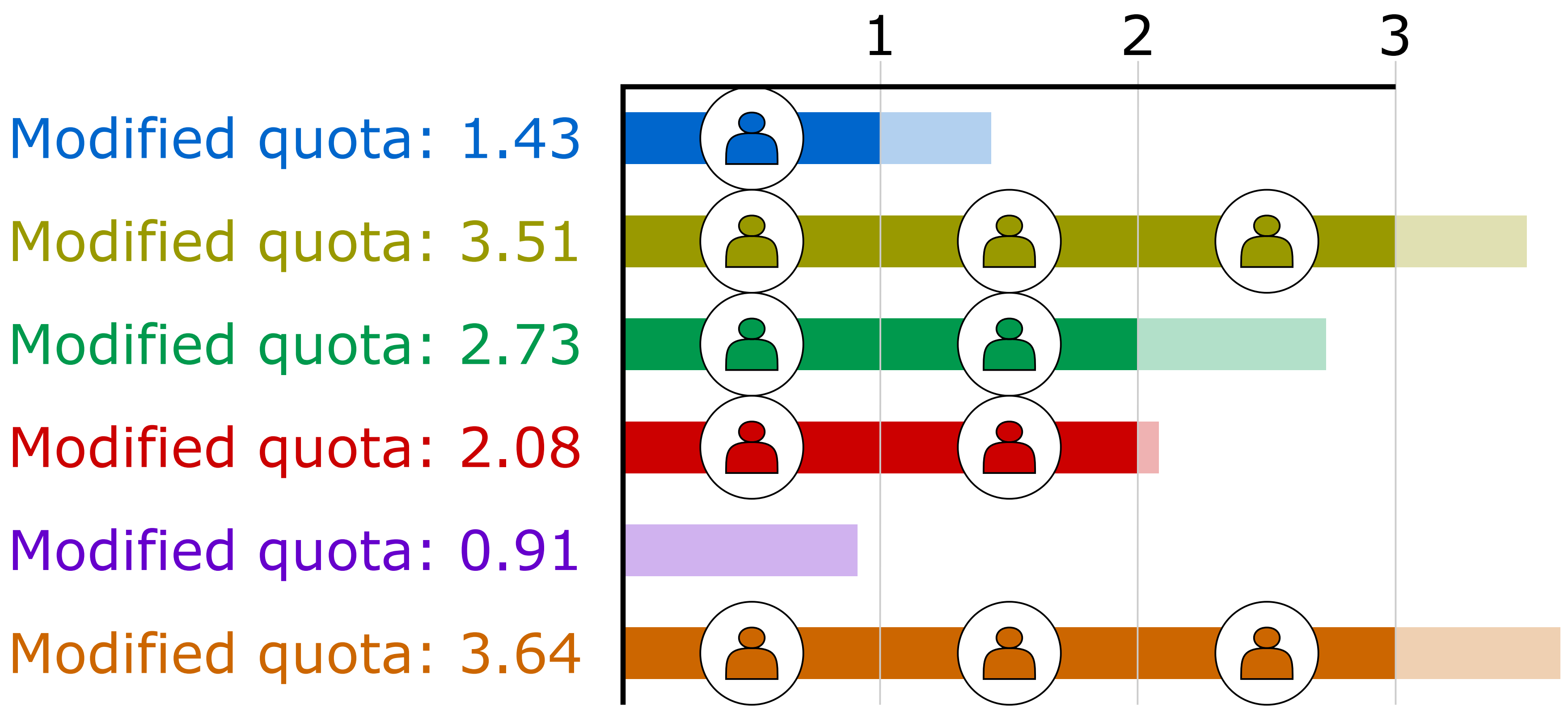

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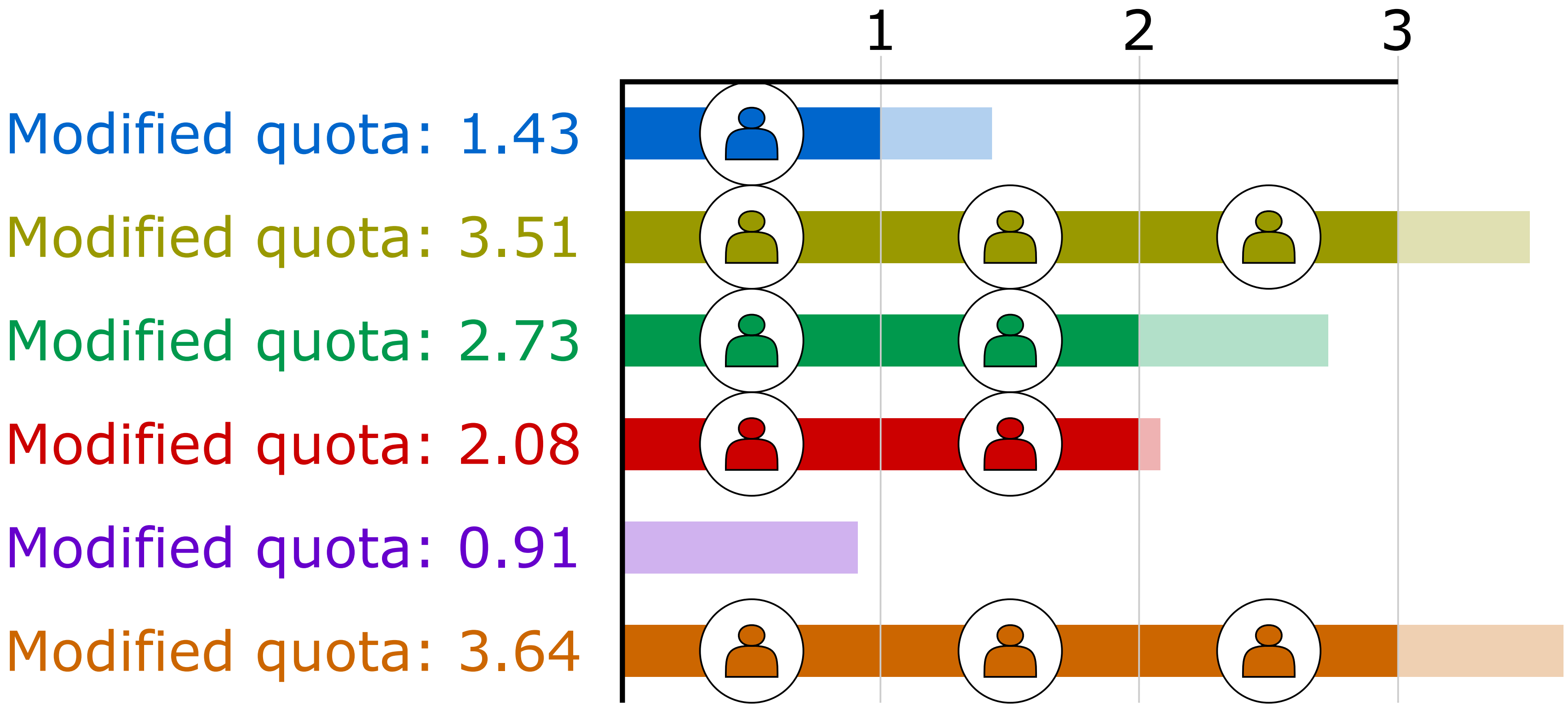

#Seats	Ham	Jeff
Party 1	1	1
Party 2	3	3
Party 3	2	2
Party 4	1	2
Party 5	1	0
Party 6	3	3

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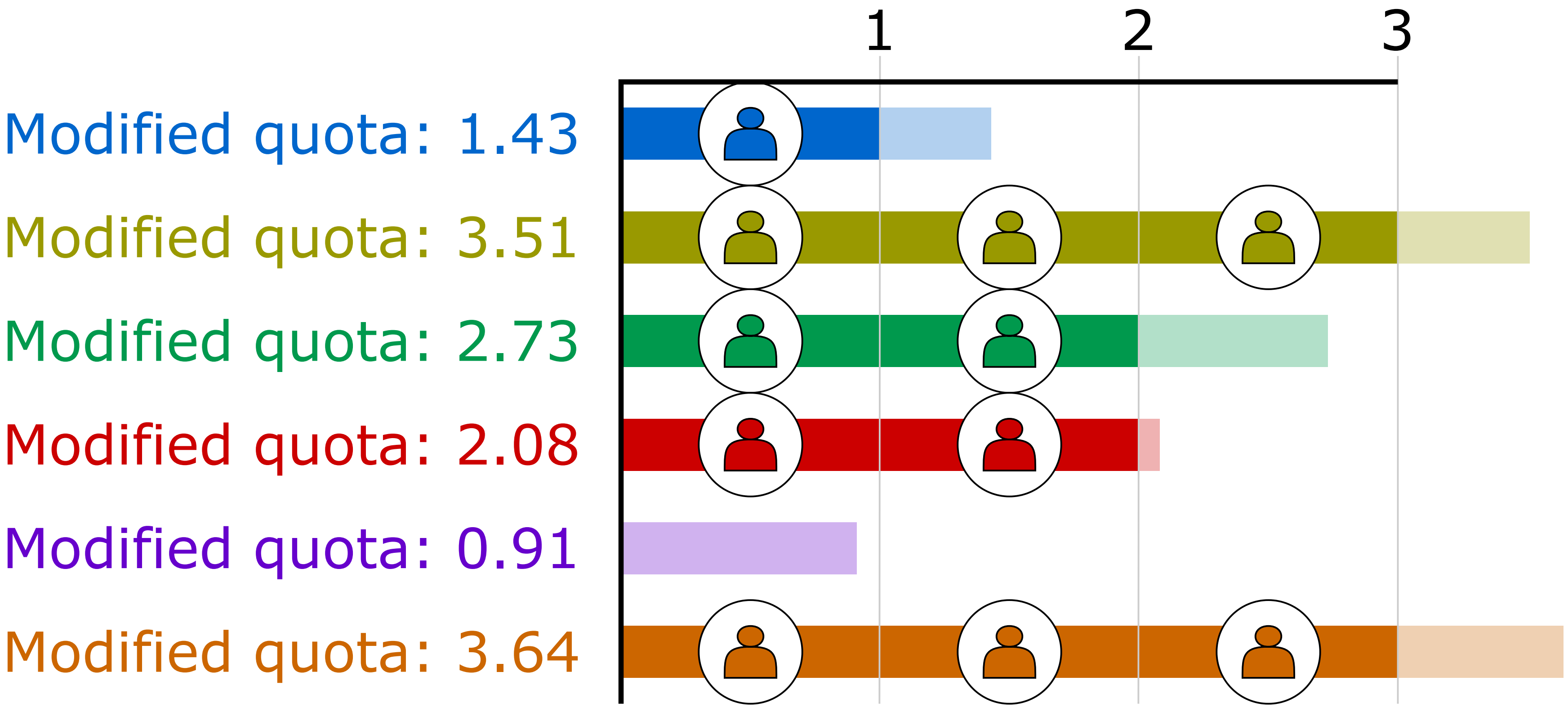

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Jefferson's has large state bias!

Adams' method

Round up instead of down

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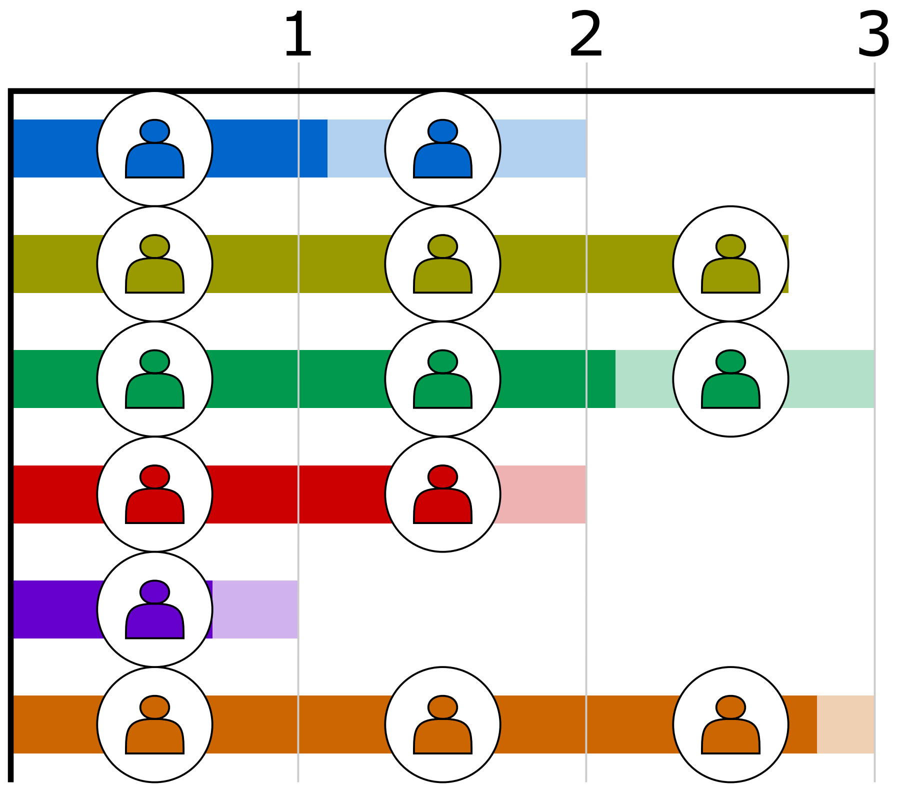

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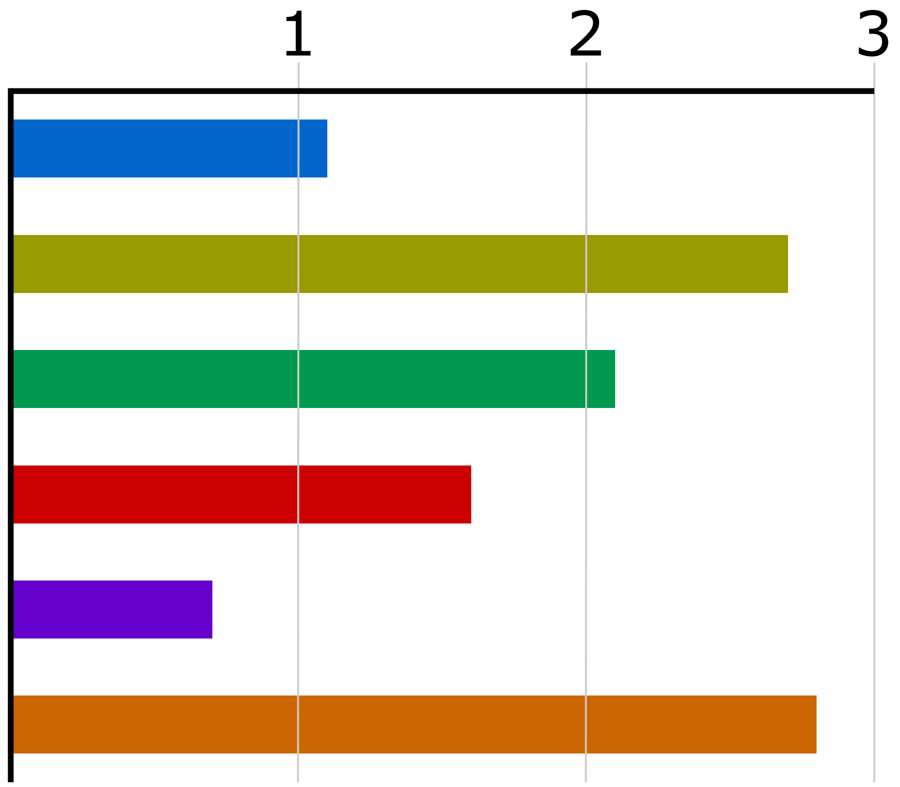

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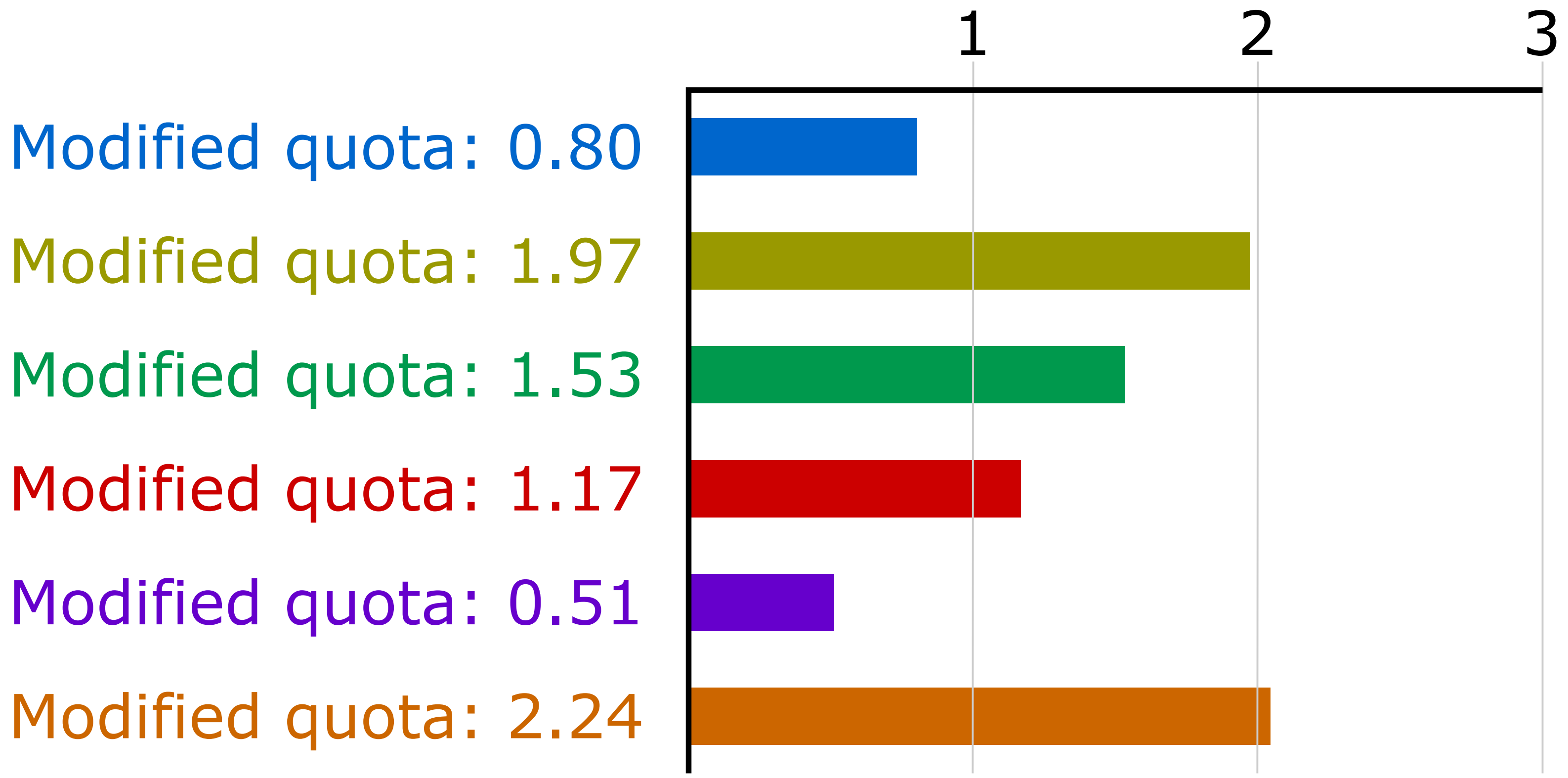

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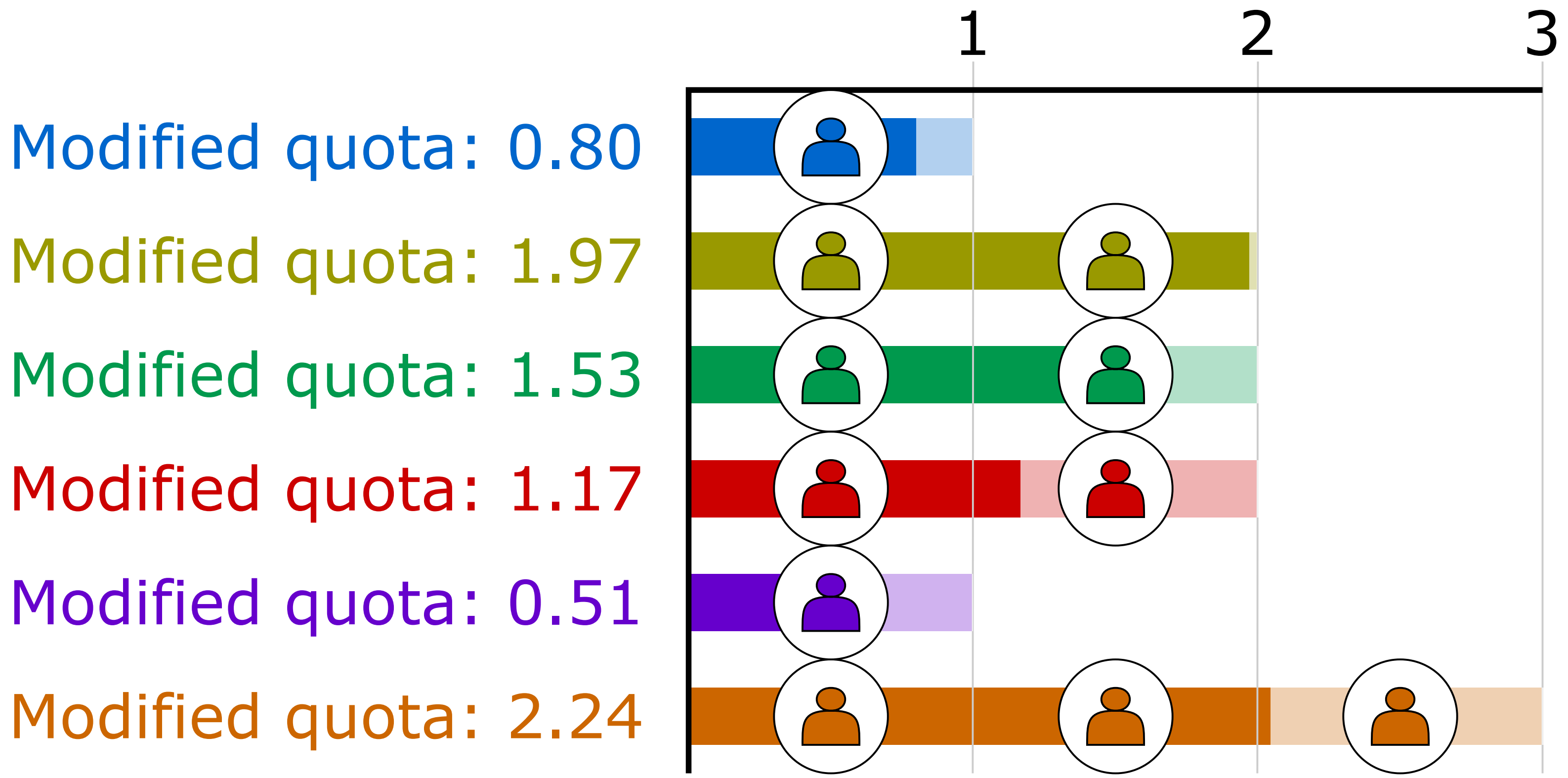

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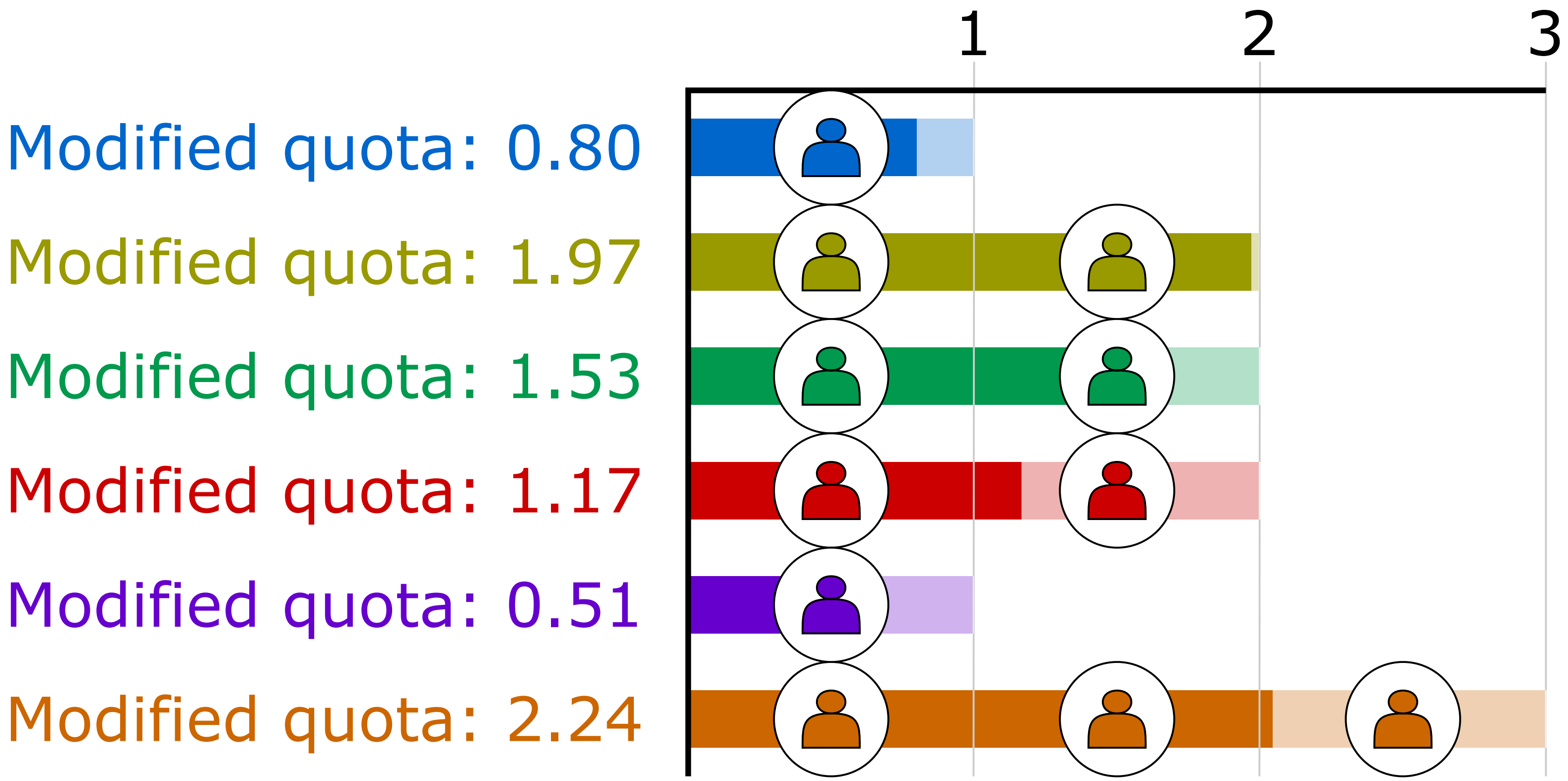

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Party 1	1	1	1
Party 2	3	3	2
Party 3	2	2	2
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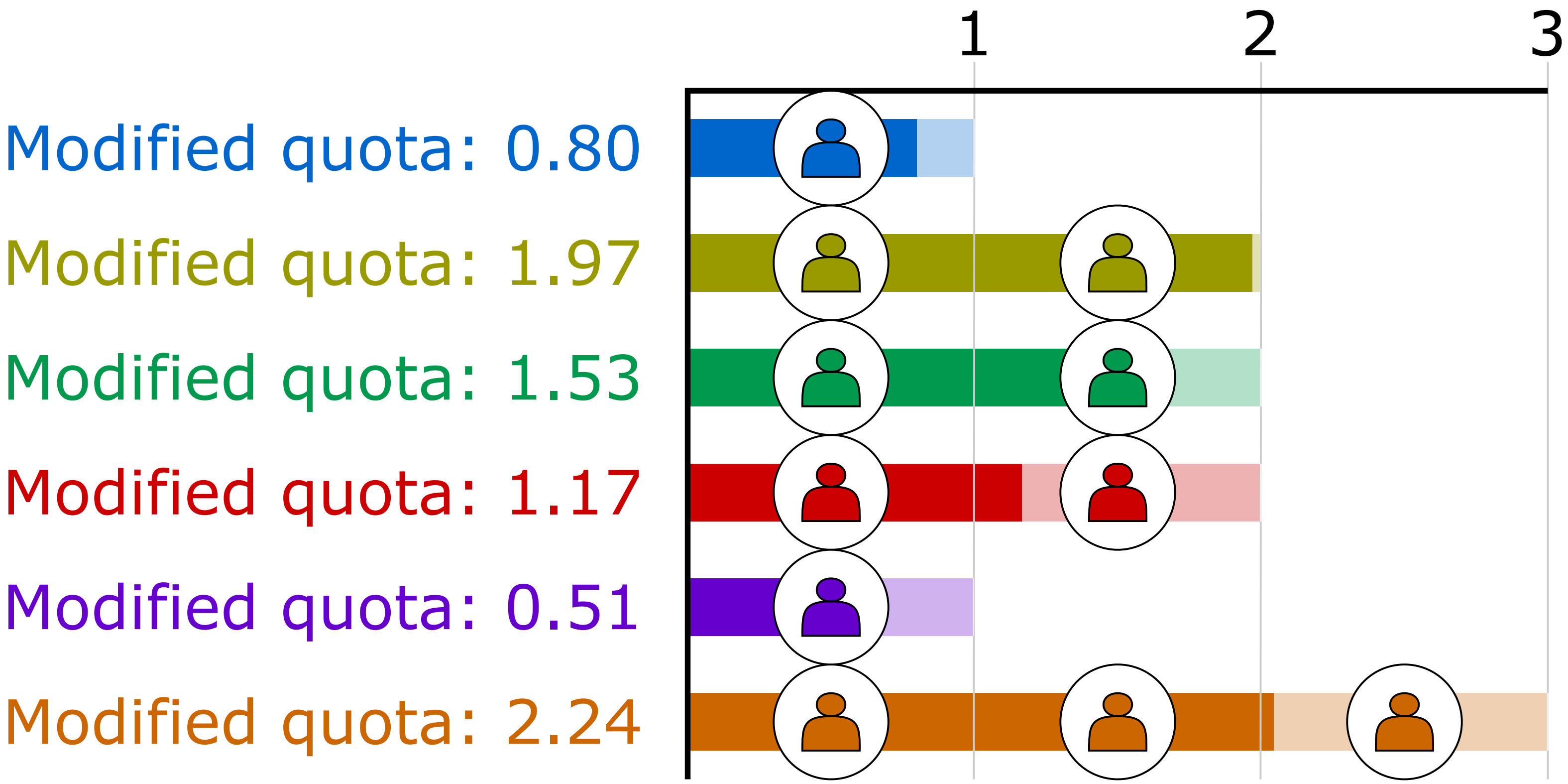

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Adams' has small state bias!

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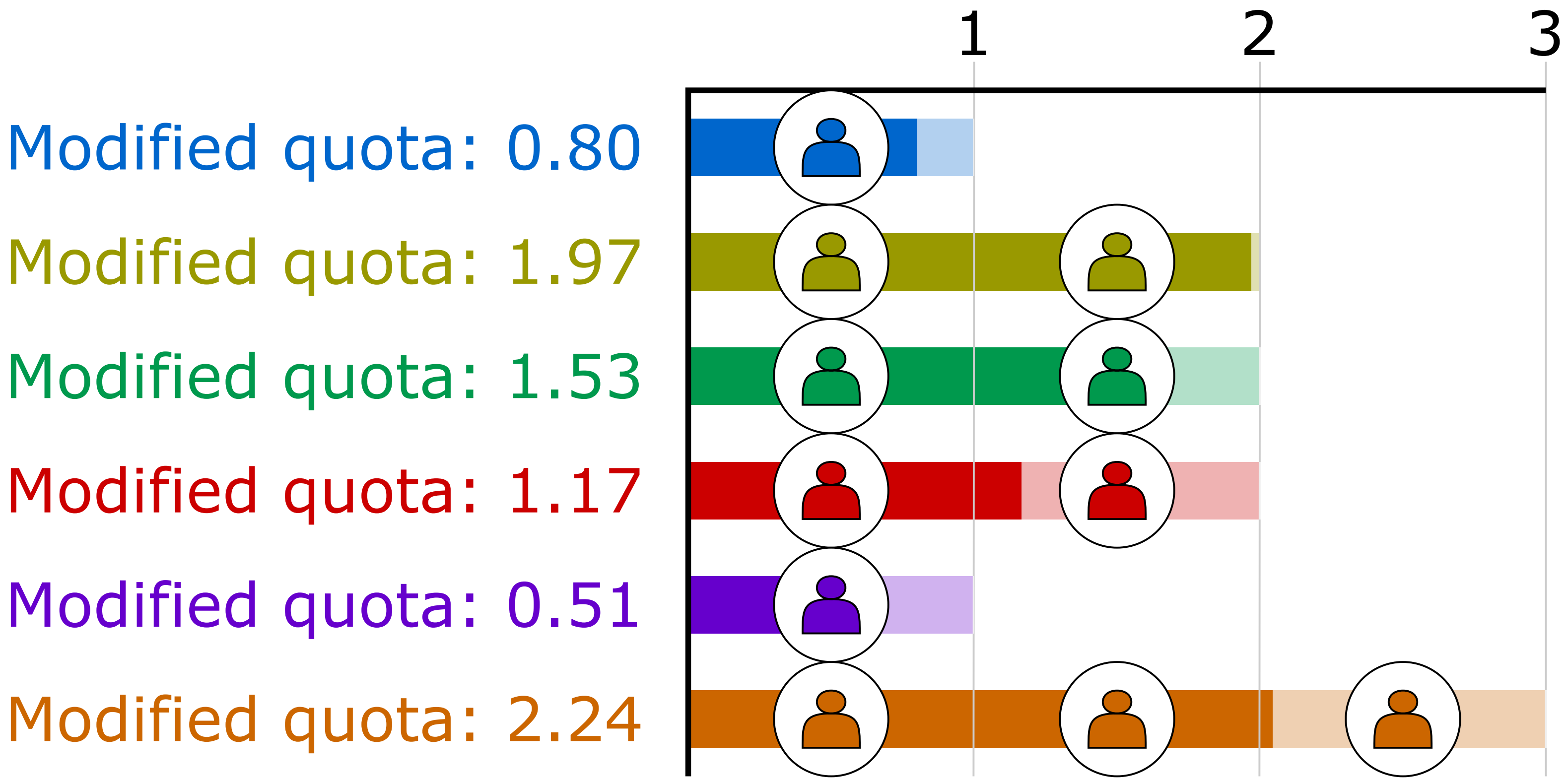

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Also coincides with *Webster's method*: round to nearest.

Stationary divisor methods

Let $[r]_\delta$ denote the rounding of r, where we round r down to $\lfloor r \rfloor$ if $r - \lfloor r \rfloor < \delta$ and up to $\lceil r \rceil$ if $r - \lfloor r \rfloor > \delta$. Examples: $[2.7]_{0.5} = 3$, $[2.7]_{0.9} = 2$.

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Given input populations $p \in \mathbb{Z}^n_{>0}$ and house size $H \in \mathbb{Z}_{>0}$, the δ-*divisor method* returns any vector $x \in \mathbb{Z}_{\geq 0}$ such that

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Questions:

1. Is this even well-defined? Does x always exist and is it unique?

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Questions:

1. Is this even well-defined? Does x always exist and is it unique?
2. Where do the large-state and small-state biases come from? Does larger δ necessarily mean bias toward larger states?
3. How many different apportionments x can we get by choosing different δ?

Associated line arrangement

The following ideas are all from:
(Cembrano, Correa, Tsigonias-Dimitriadis, Schmidt-Kraepelin, Verdugo, 2025)

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For a given (p, H), consider the collection of lines

$$\mathcal{L}(p, H) - \left\{ \ell_{i,t}(\delta) = \frac{t}{p_i} + \frac{\delta}{p_i} \;\middle|\; i \in [n],\ t \in \{0, 1, 2, \ldots, H-1\} \right\}.$$

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λ gives $x \iff \forall i,\ x_i - 1 + \delta \leq \lambda p_i \leq x_i + \delta$

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$$\begin{aligned} \lambda \text{ gives } x &\iff \forall i,\ x_i - 1 + \delta \leq \lambda p_i \leq x_i + \delta \\ &\iff \forall i,\ (x_i - 1)/p_i + \delta/p_i \leq \lambda \leq x_i/p_i + \delta/p_i \end{aligned}$$

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$$\begin{aligned}
\lambda \text{ gives } x &\iff \forall i,\ x_i - 1 + \delta \le \lambda p_i \le x_i + \delta \\
&\iff \forall i,\ (x_i - 1)/p_i + \delta/p_i \le \lambda \le x_i/p_i + \delta/p_i \\
&\iff \forall i,\ \ell_{i,x_i-1}(\delta) \le \lambda \le \ell_{i,x_i}(\delta) \\
&\iff \forall i,\ \ell_{i,0}(\delta) < \ell_{i,1}(\delta) < \cdots < \ell_{i,x_i-1}(\delta) \le \lambda \le \ell_{i,x_i}(\delta)
\end{aligned}$$

$\iff$ The point (δ, λ) is above exactly $\sum_{i=1}^{n} x_i = H$ lines from $\mathcal{L}$, namely above x_i of each type i.

The $(H-1)$-level defines all δ-divisor rules

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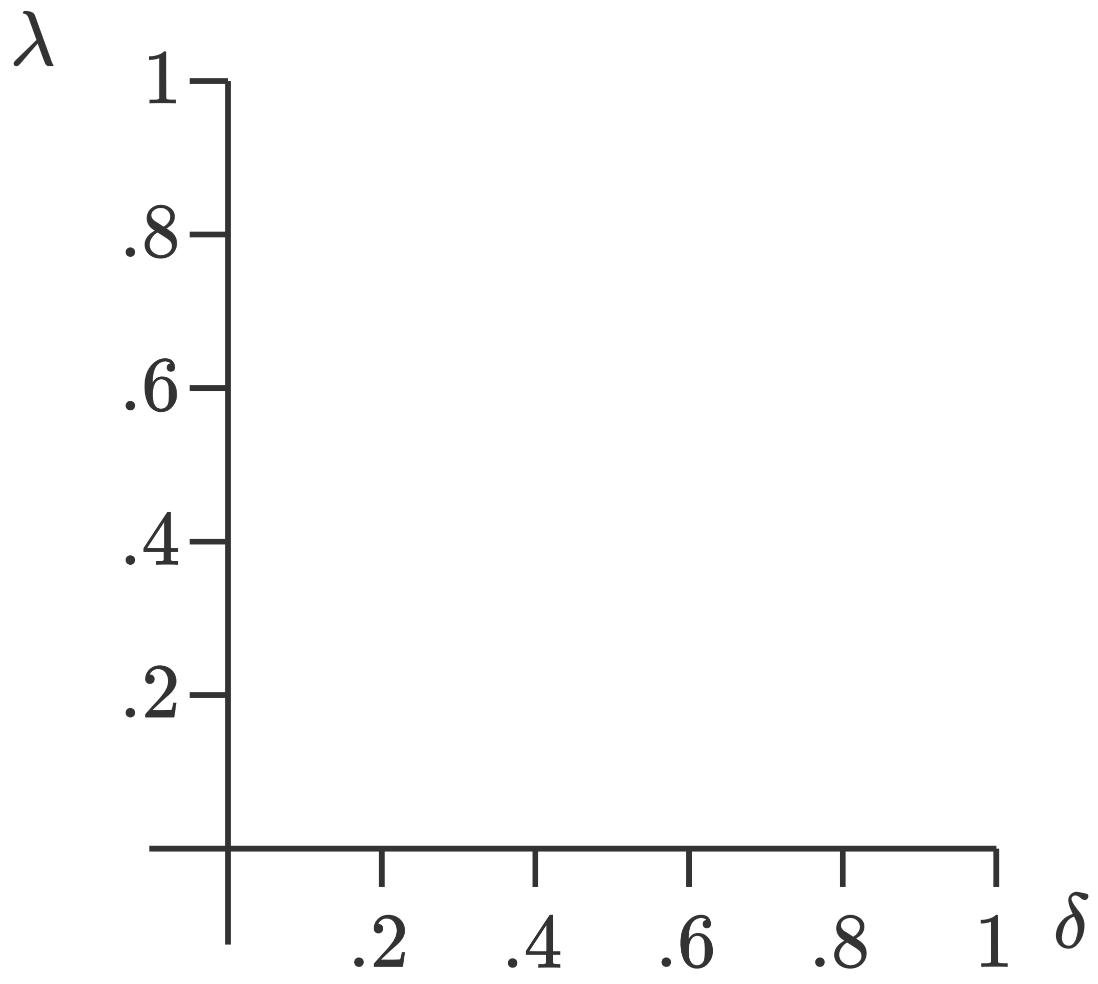

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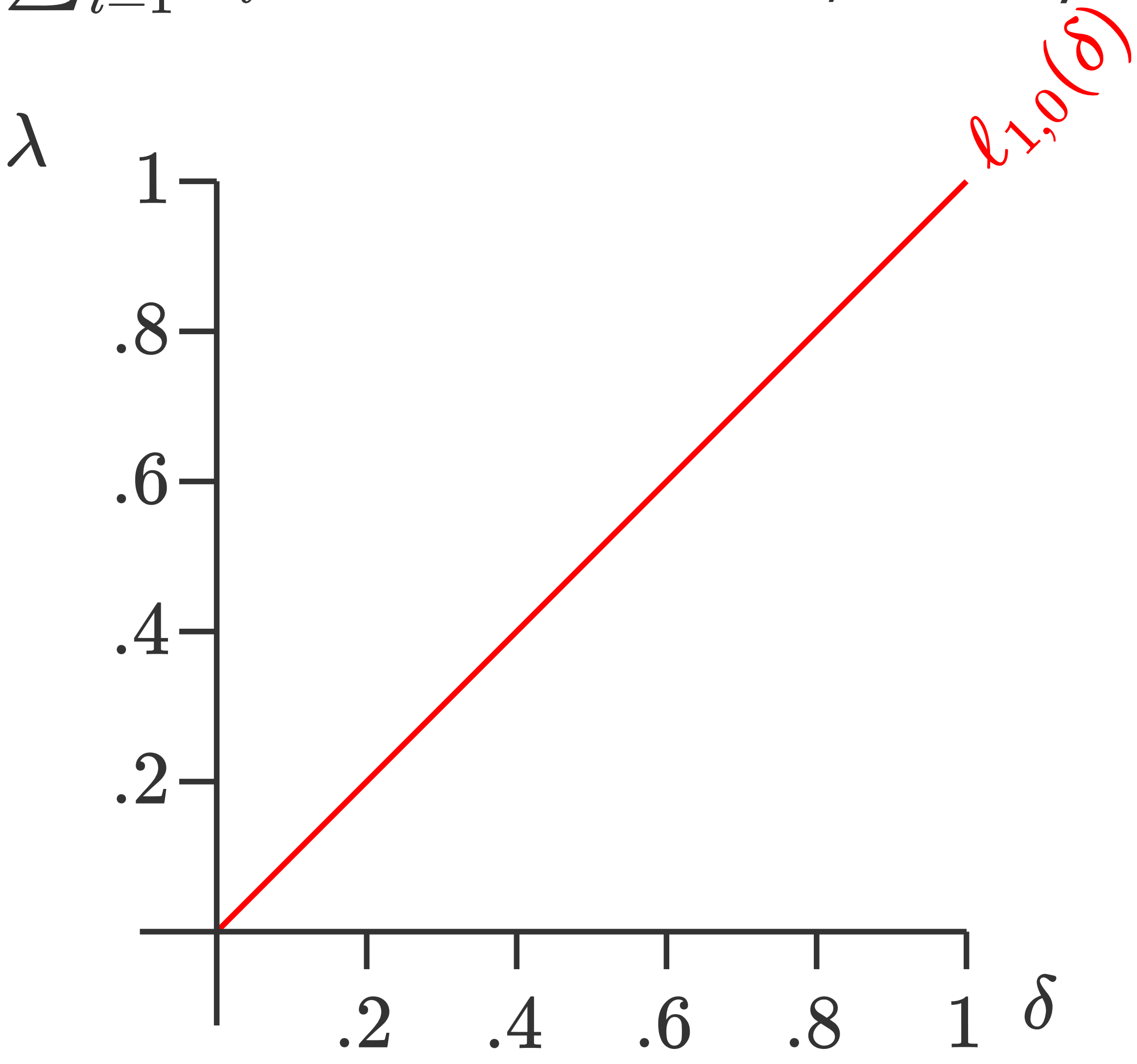

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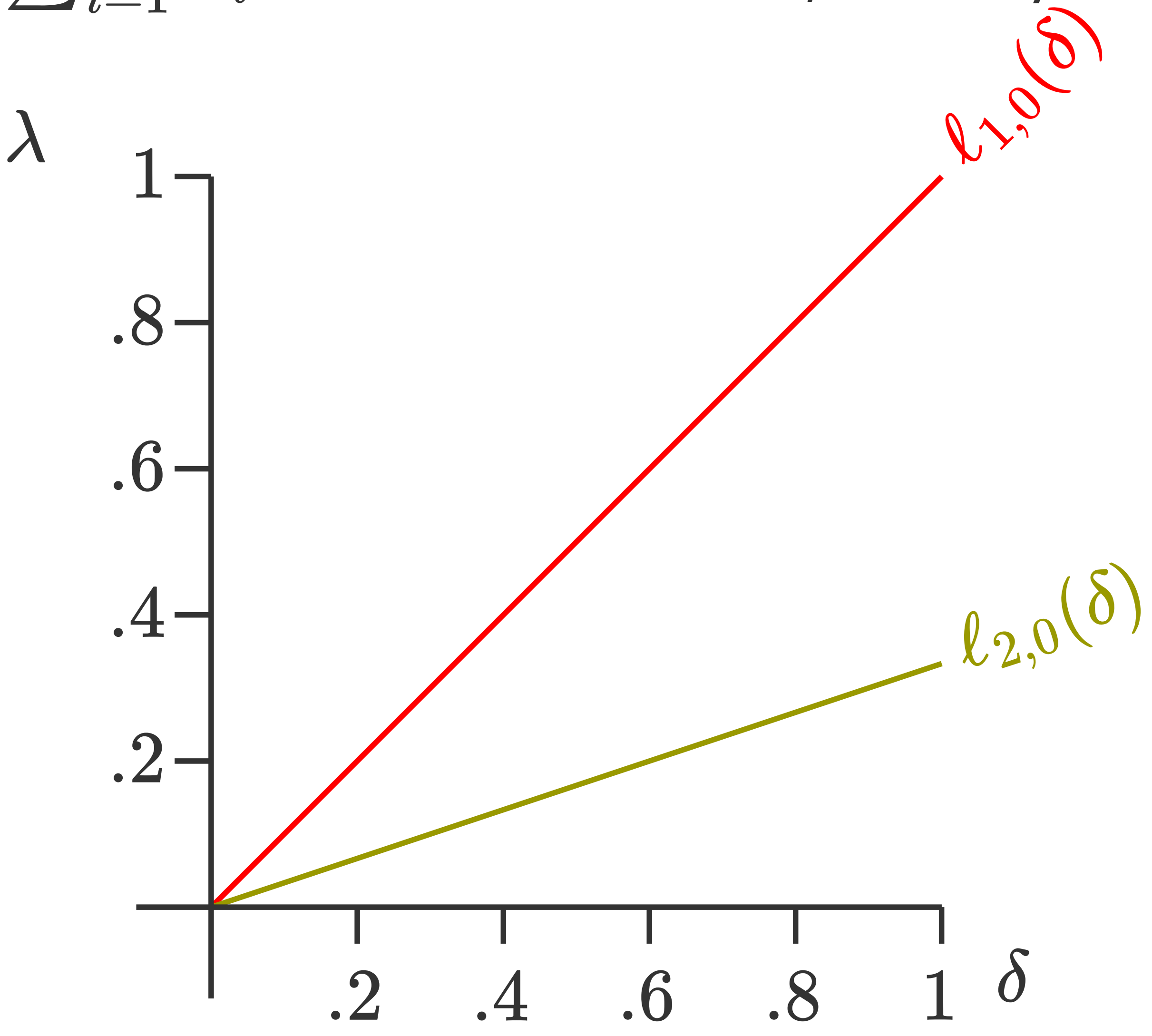

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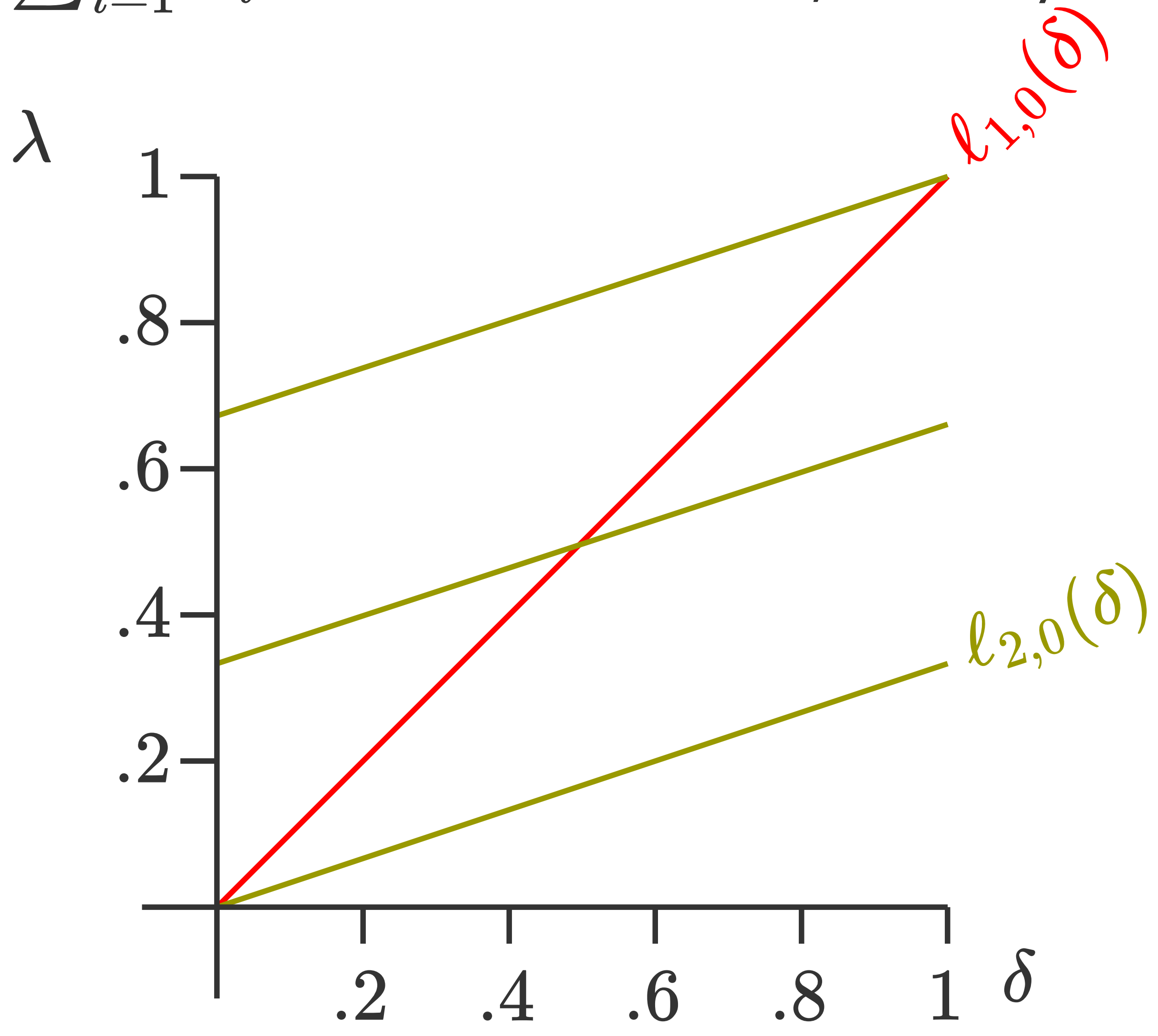

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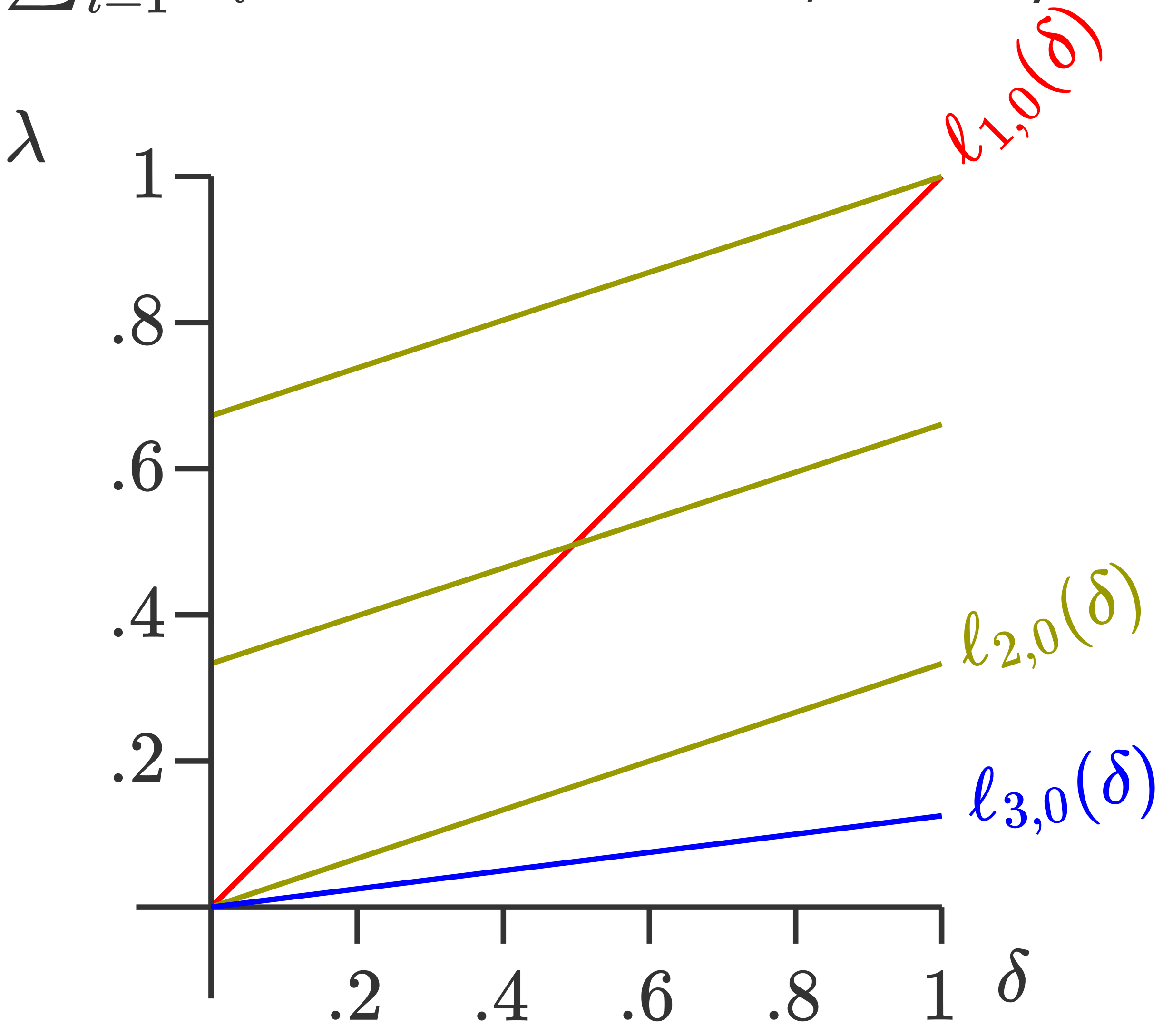

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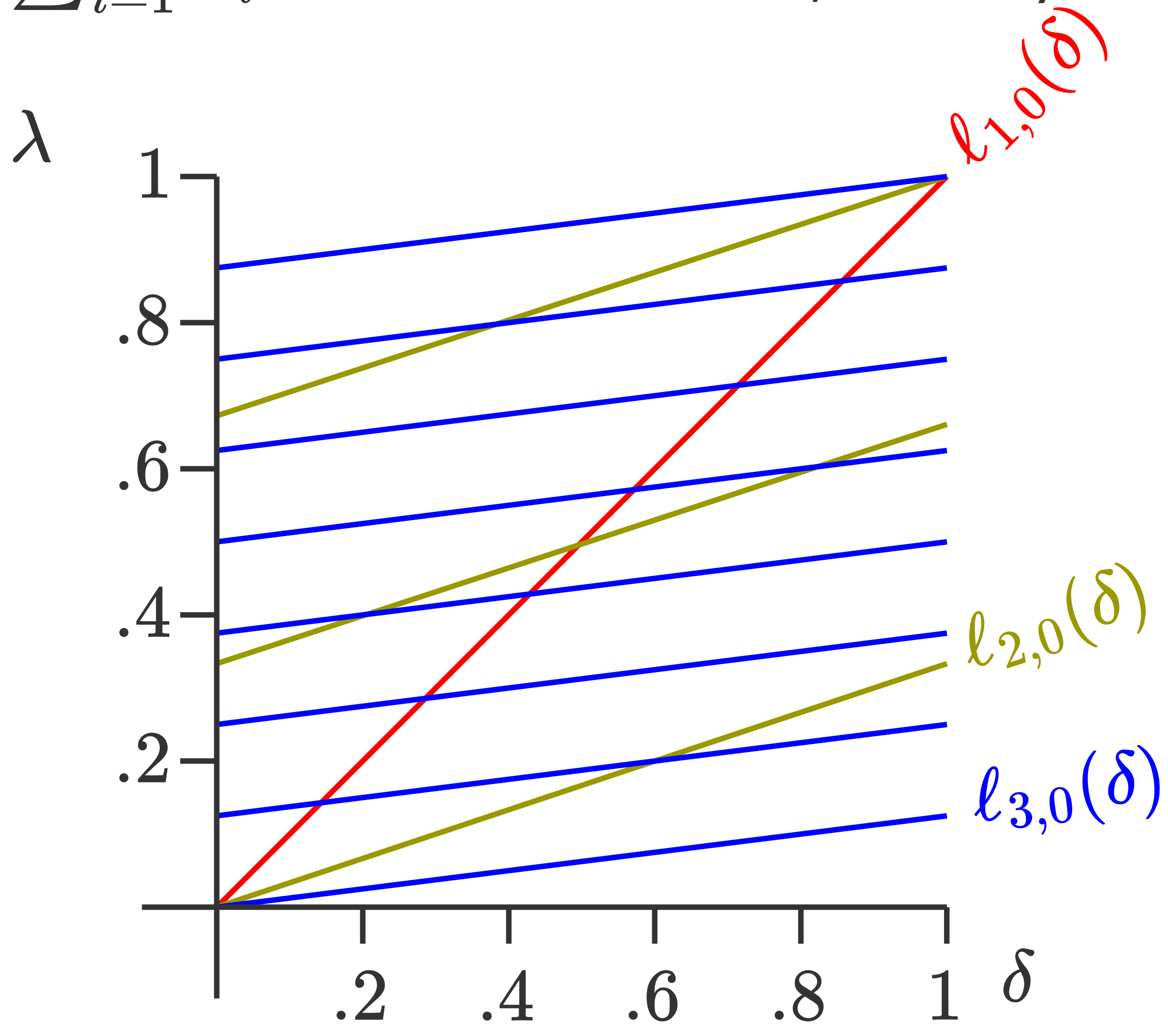

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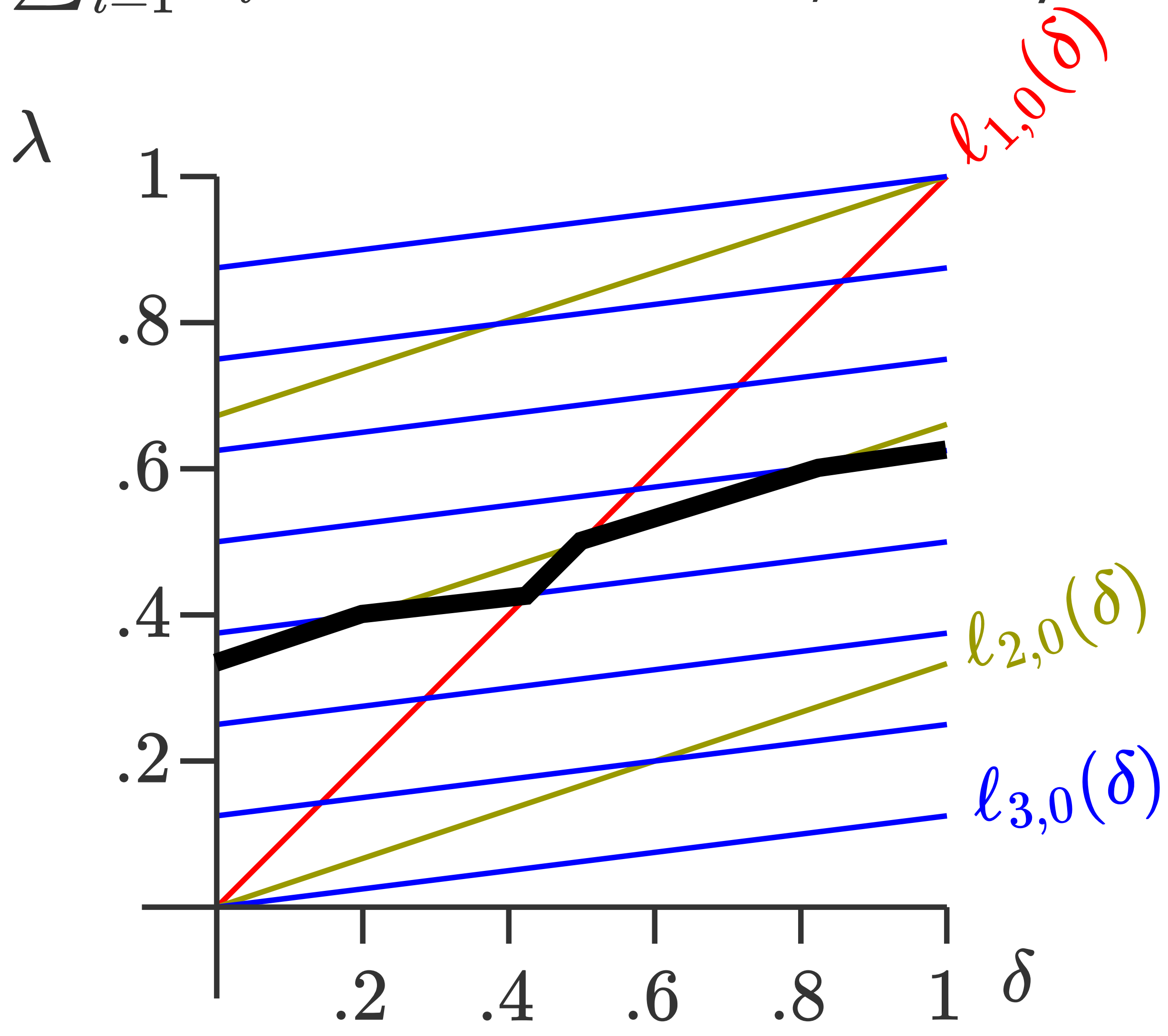

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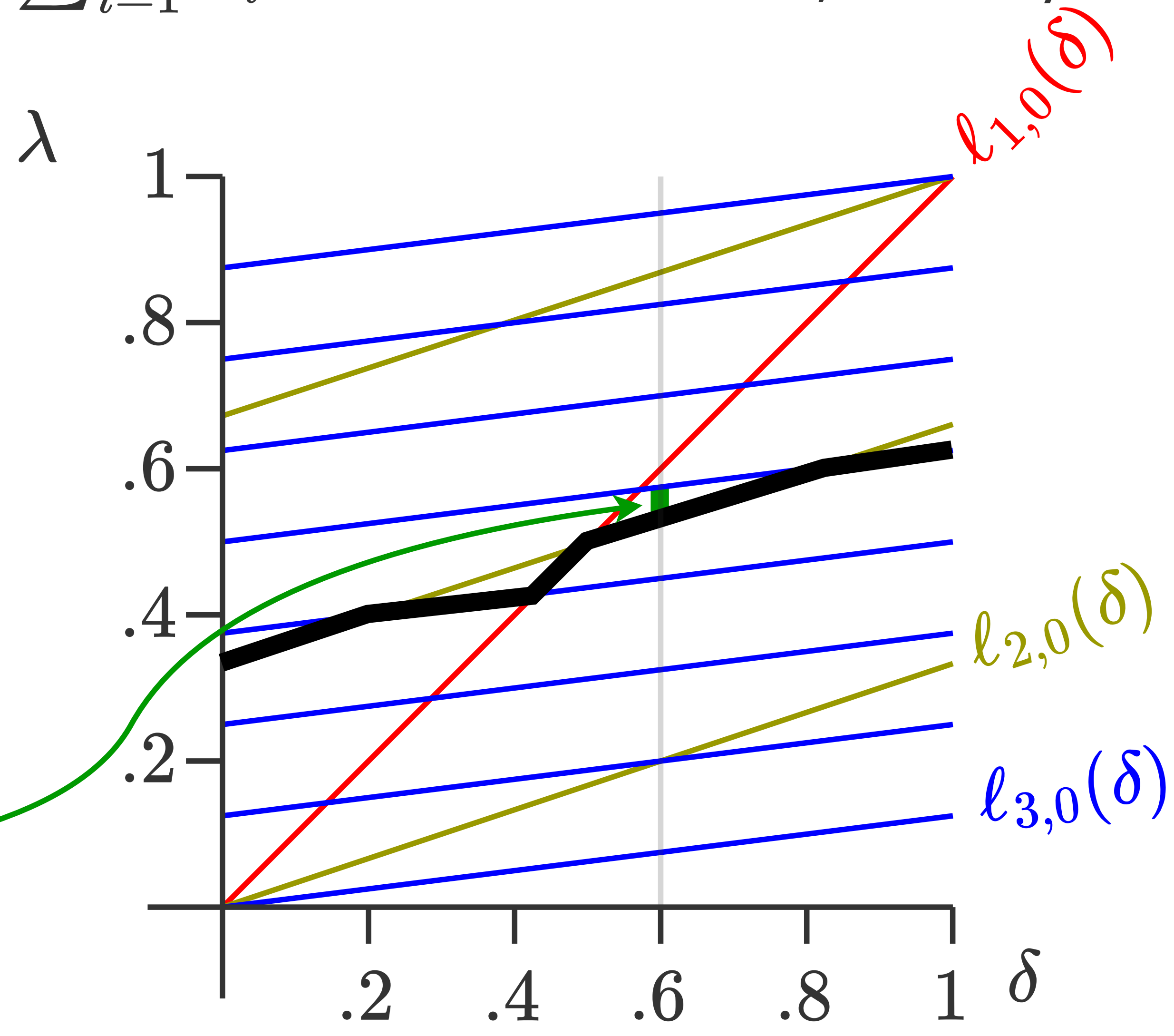
Generically, the i index of lines at or below the $H-1$ uniquely determines the allocation of seats, even if λ is not unique.



Small vs large states

Theorem (Marshall, 2002)

Increasing δ leads to a transfer of seats from smaller states to larger ones.

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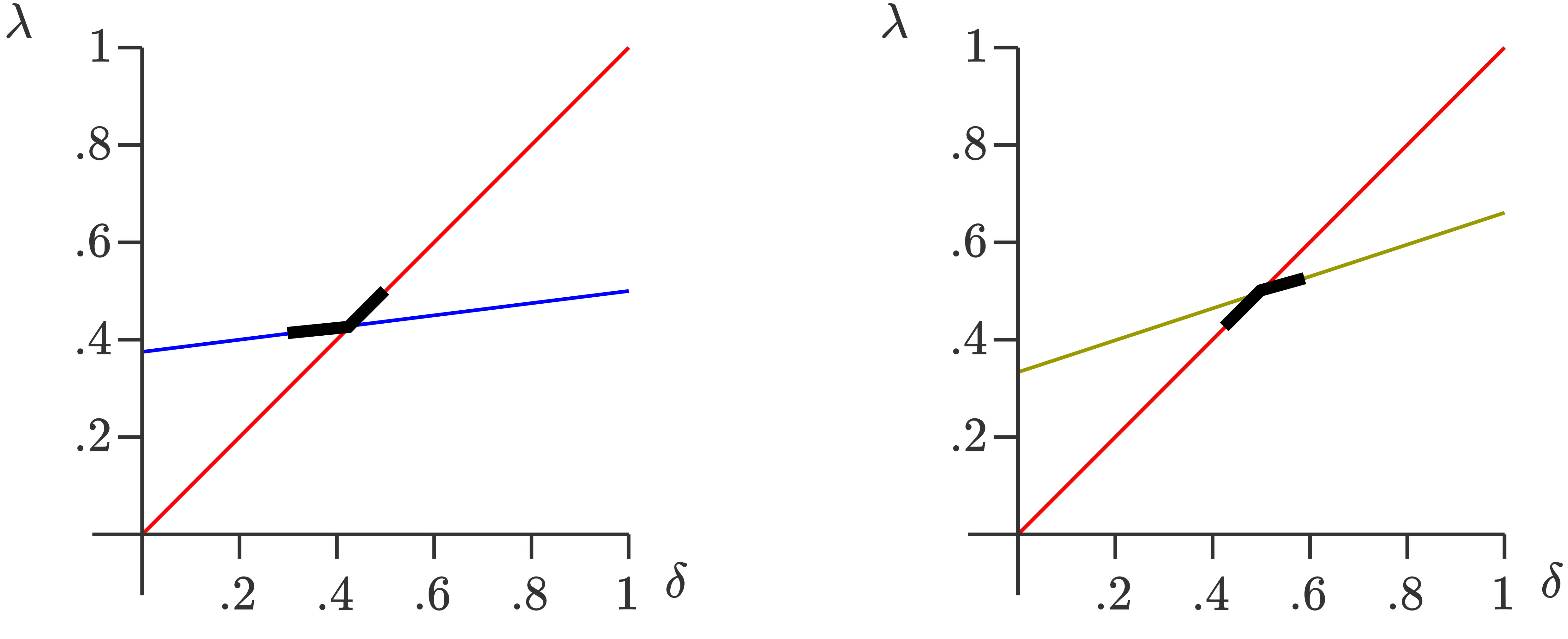
Proof. As we traverse the $(H-1)$-level, two possible things can happen:



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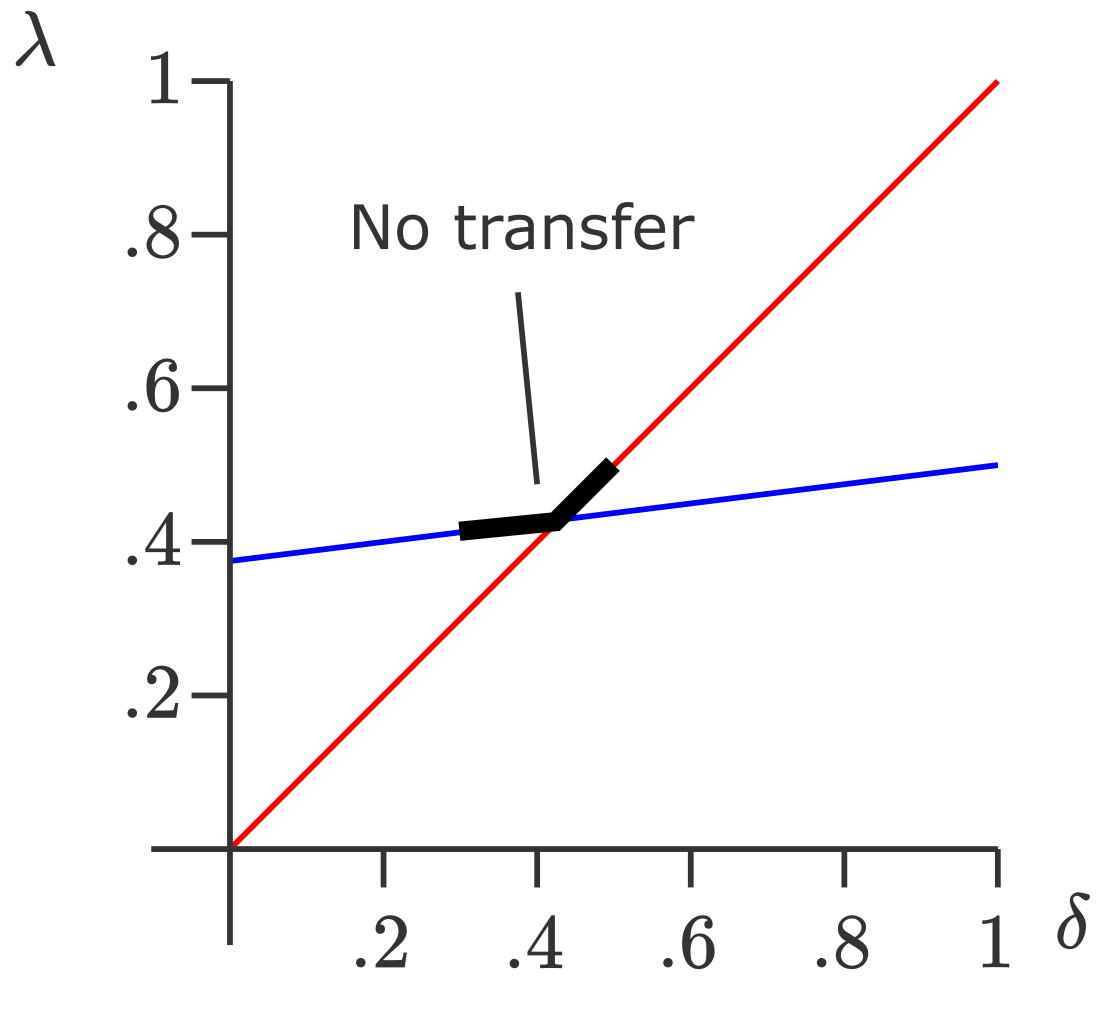

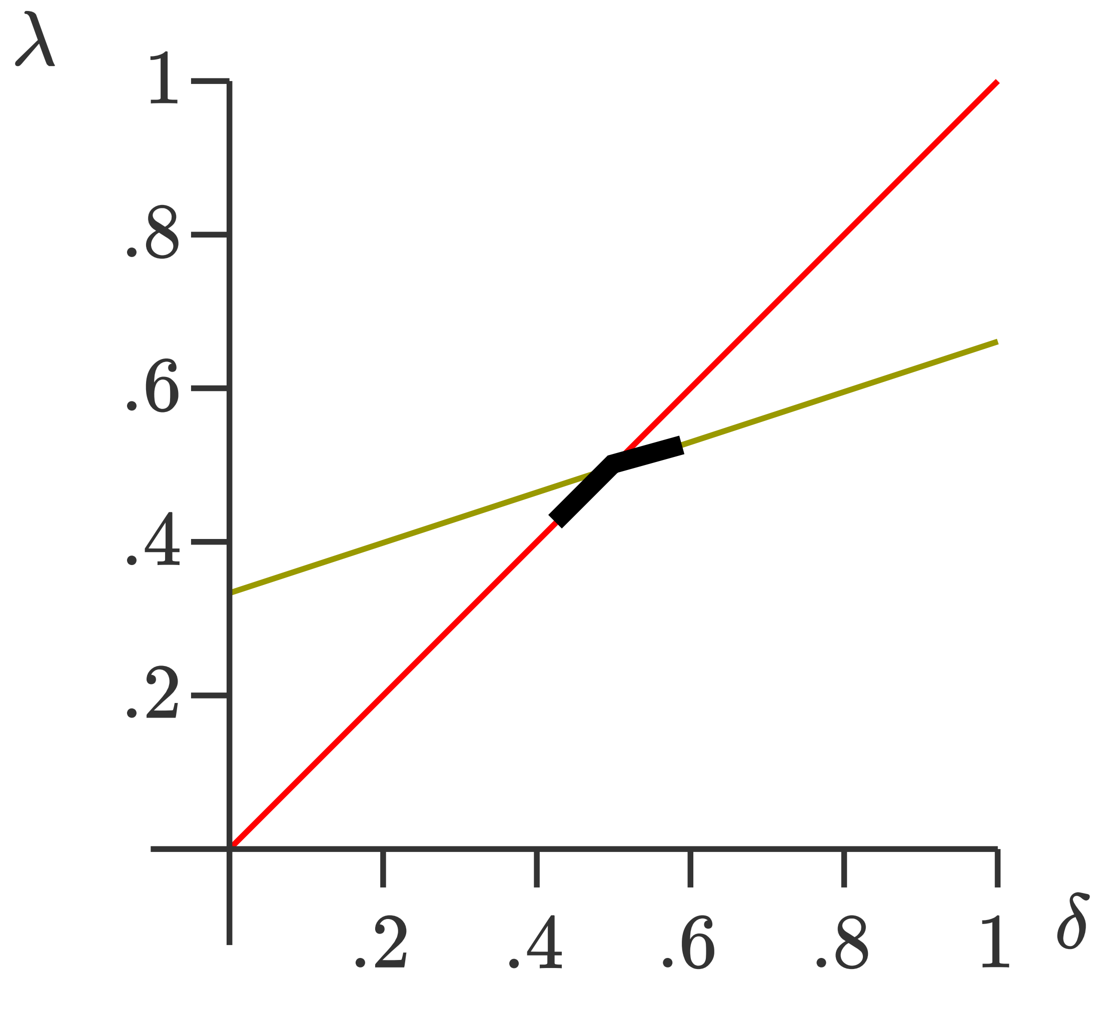

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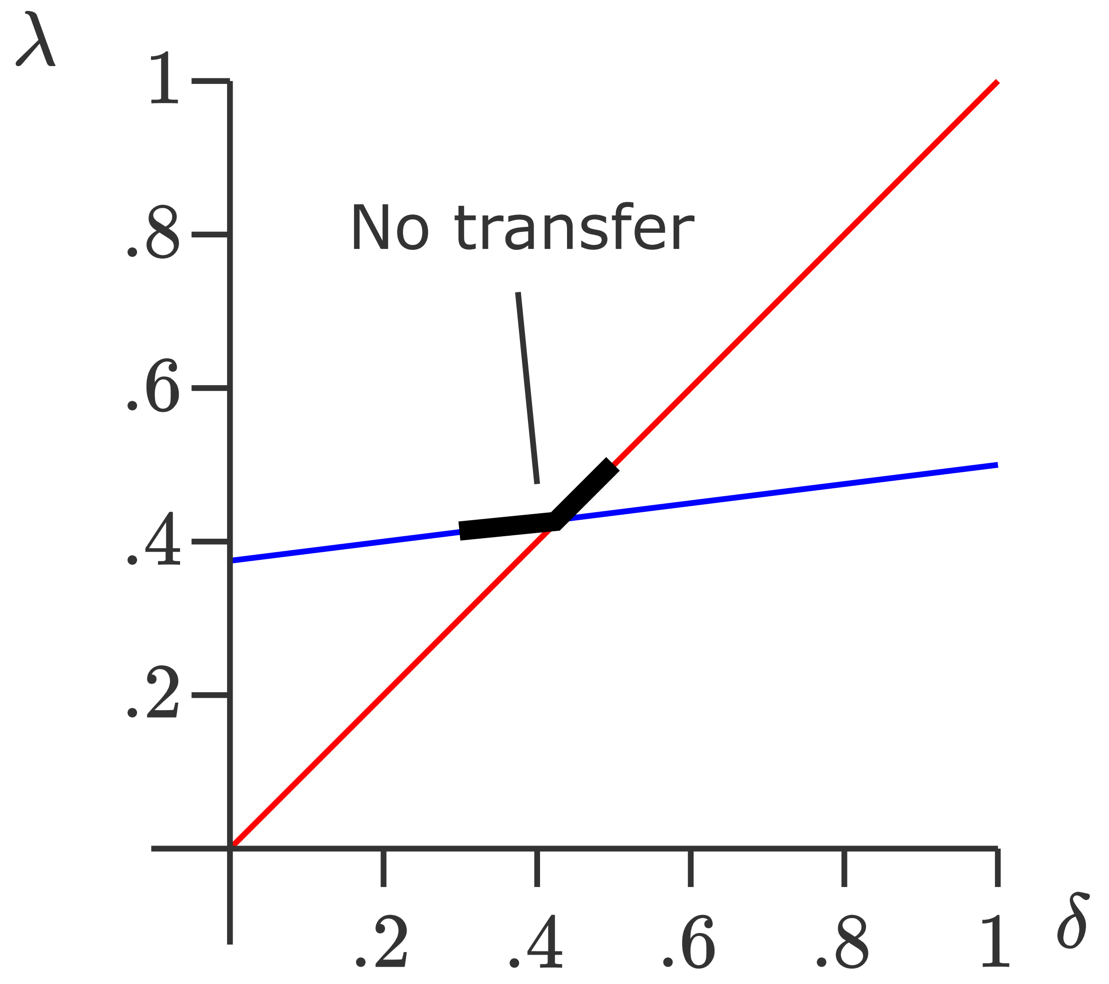

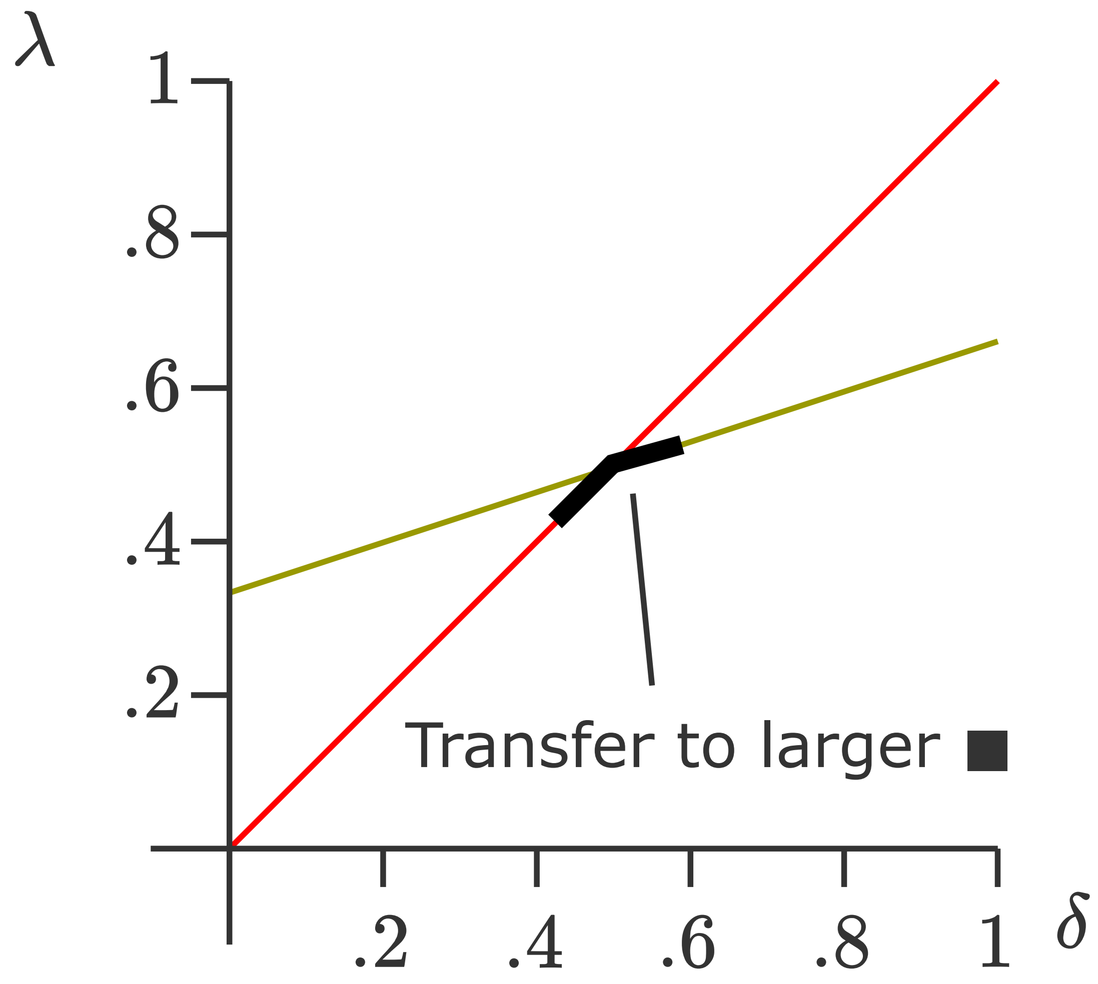

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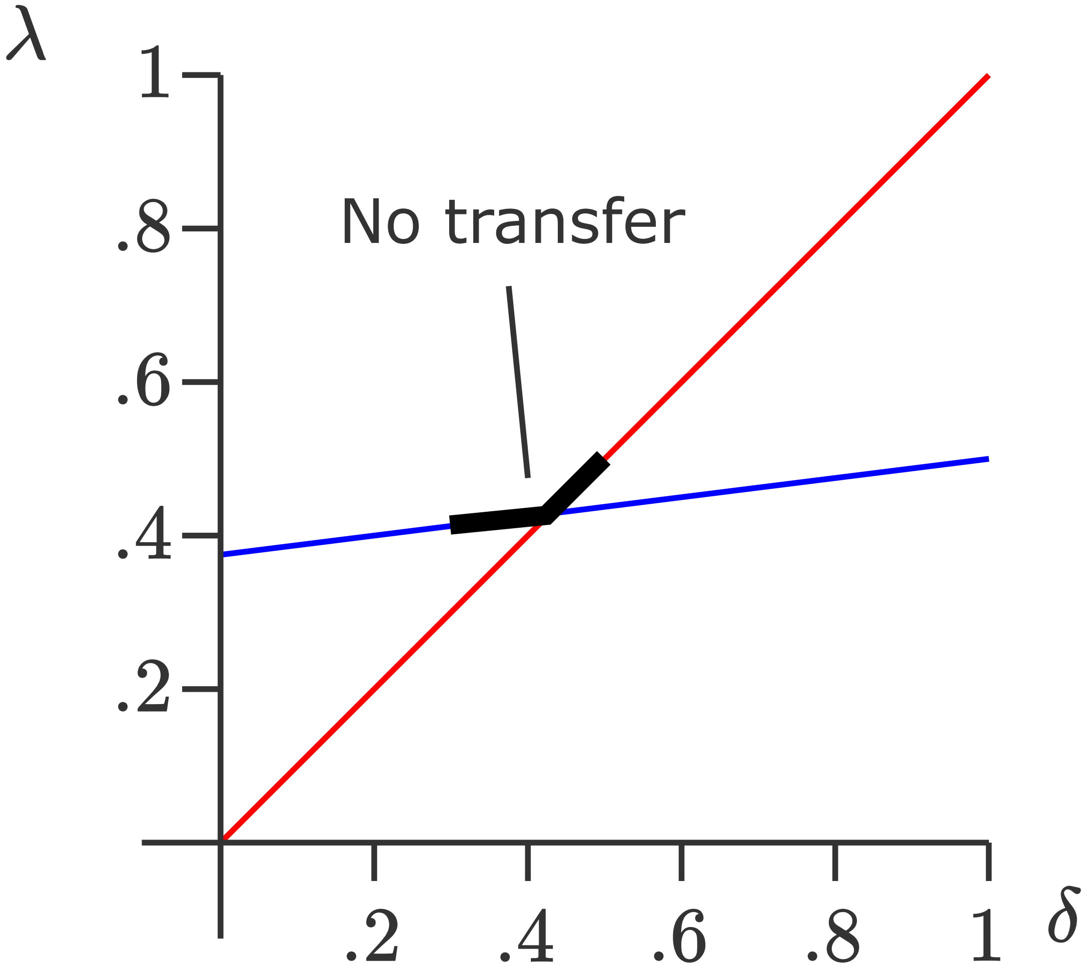

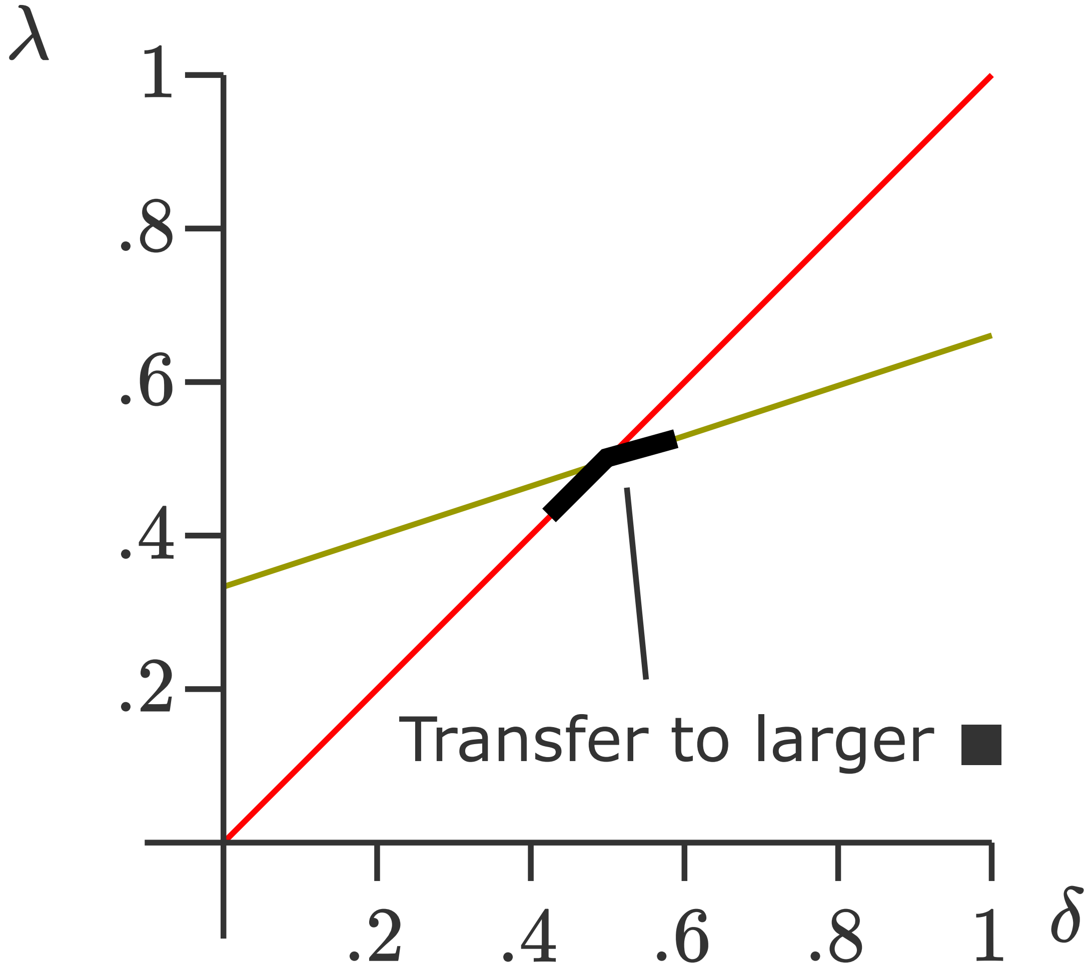

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Theorem (Cembrano et al., 2025)

Whatever the answer to the above question is, the maximum number of possible distinct outputs of stationary diviser methods is the same up to constant factors.