

Algorithms For Democratic Decision-Making

Jamie Tucker-Foltz • Yale University • Spring 2026

Lecture 18: **Apportionment 1**

Announcements

Project updates due next Monday night. Like the proposal, this is counted for completion only, and is your chance to get feedback!

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This Wednesday: Lecture over Zoom again

Next Monday: Back to in-person

Next Wednesday: No class, individual meetings to discuss projects

Apportionment in the US

We the People

of the United States, in order to form a more perfect Union, establish Justice, insure domestic Tranquility, provide for the common defence, promote the general Welfare, and secure the Blessings of Liberty to ourselves and our Posterity, do ordain and establish this Constitution for the United States of America.

Article 1.

Section 1. All legislative Powers herein granted shall be vested in a Congress of the United States, which shall consist of a Senate and House of Representatives.

Section 2. The House of Representatives shall be composed of Members chosen every second Year by the People of the several States, and the Electors in each State shall have the Qualifications requisite for Electors of the most numerous Branch of the State Legislature.

No Person shall be a Representative who shall not have attained to the Age of twenty five Years, and been seven Years a Citizen of the United States, and who shall not, when elected, be an Inhabitant of that State in which he shall be chosen.

Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers, which shall be determined by adding to the whole Number of free Persons, including those bound to Service for a Term of Years, and excluding Indians not taxed, three fifths of all other Persons. The actual Enumeration shall be made within three Years after the first Meeting of the Congress of the United States, and within every subsequent Term of ten Years, in such Manner as they shall by Law direct. The Number of Representatives shall not exceed one for every thirty Thousand, but each State shall have at least one Representative; and until such Enumeration shall be made, the State of New Hampshire shall be entitled to choose three, Massachusetts eight, Rhode Island and Providence Plantations one, Connecticut five, New York six, New Jersey four, Pennsylvania

seven, Delaware three, Virginia five, North Carolina five, South Carolina three, and Georgia three.

When vacancies happen in the Representation from any State, the Executive Authority thereof shall issue Writs of Election to fill such Vacancies.

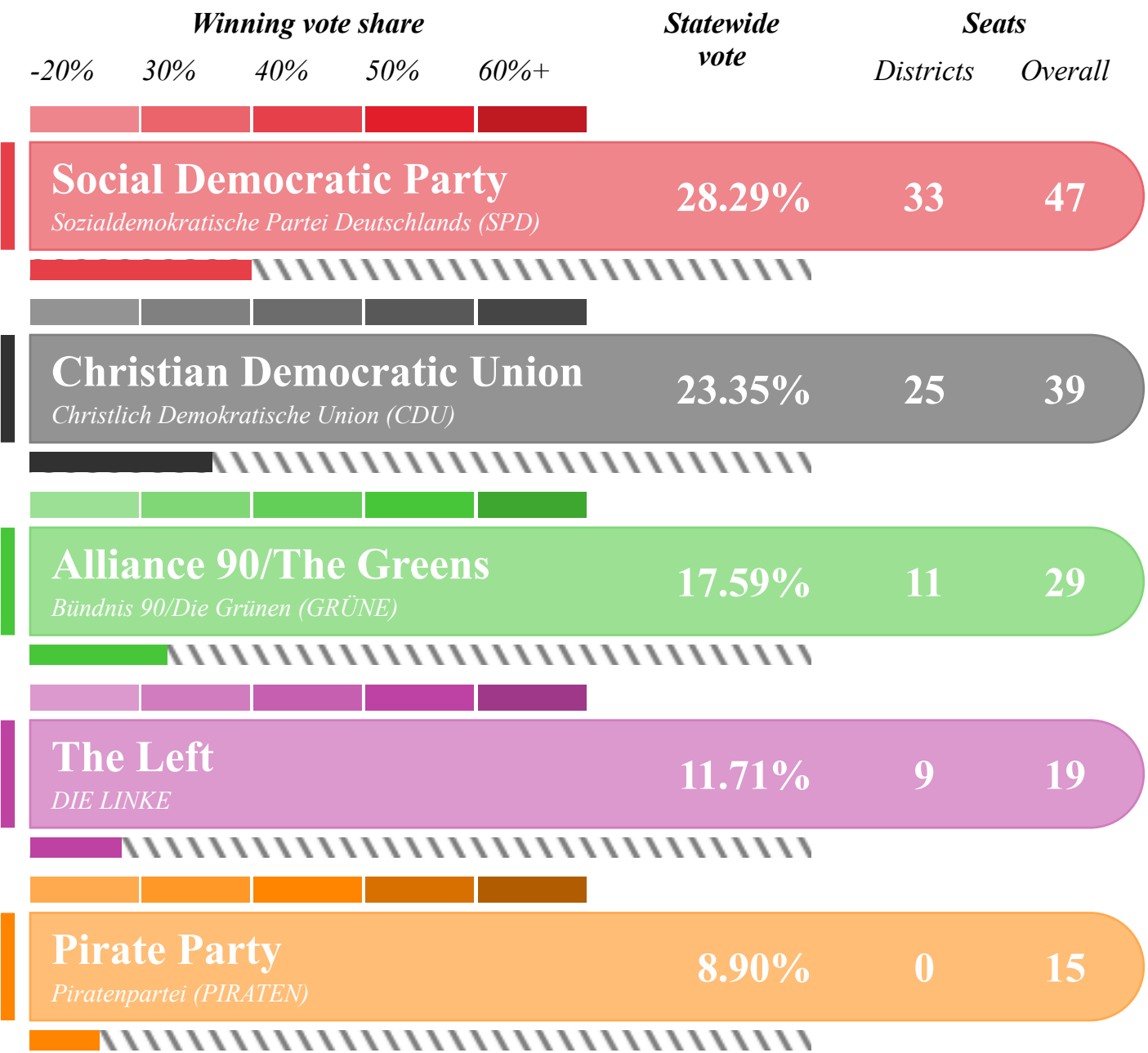
The House of Representatives shall choose their Speaker and other Officers; and shall have the sole Power of Impeachment.

Section 3. The Senate of the United States shall be composed of two Senators from each State, chosen by the Legislature thereof, for six Years, and each Senator shall have one Vote.

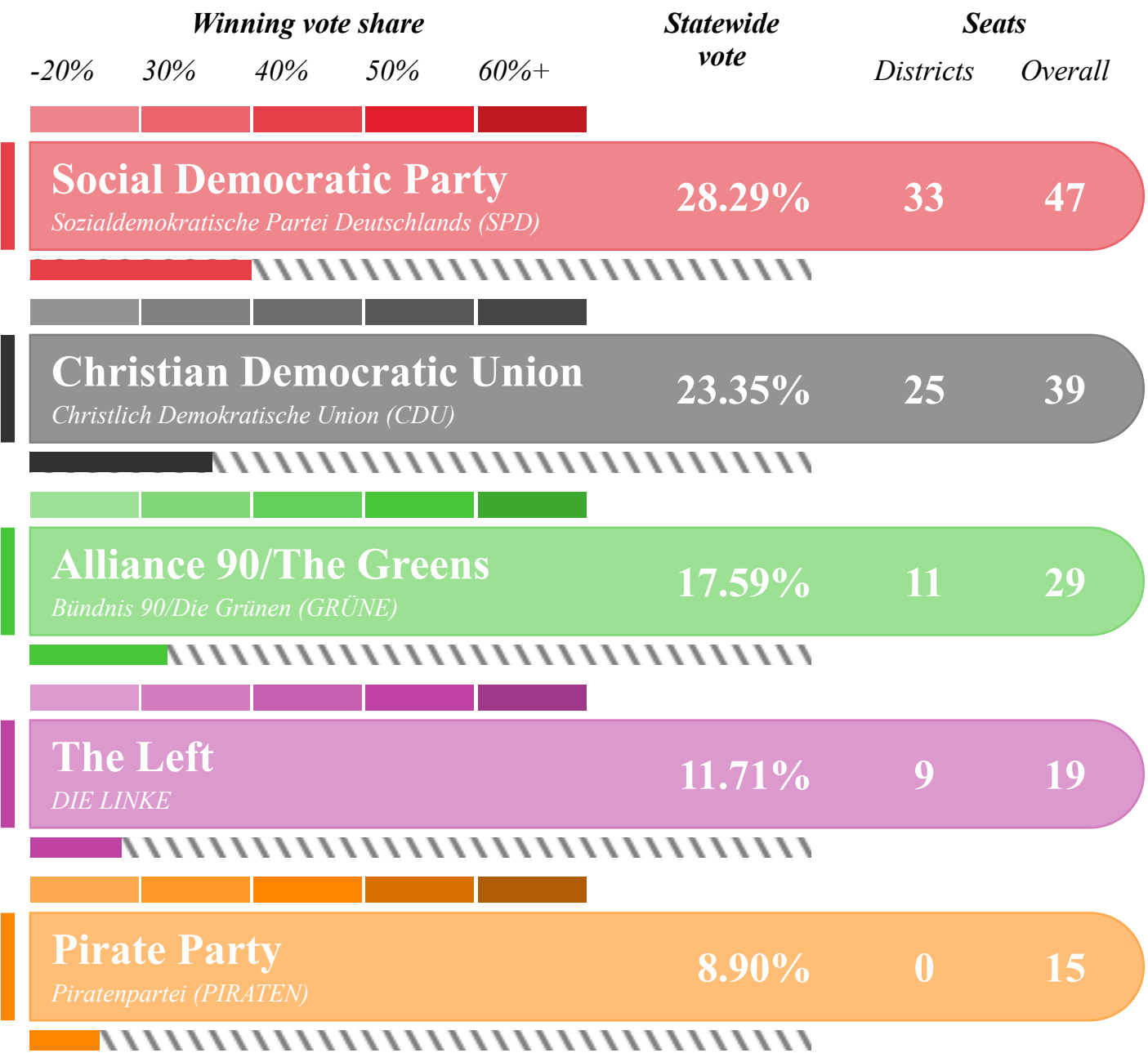
Immediately after they are assembled in each Year, they shall choose one of their Number to be President, who shall hold Office for one Year, and whose Electors shall have the Qualifications requisite for Electors of the most numerous Branch of the State Legislature.

Representatives... shall be apportioned among the several States which may be included within this Union, according to their respective Numbers... within every subsequent Term of ten Years

Apportionment in parliamentary democracies



Apportionment in parliamentary democracies



2023 Dutch general election

← 2021 **22 November 2023** Next →

All 150 seats in the [House of Representatives](#)
76 seats needed for a majority

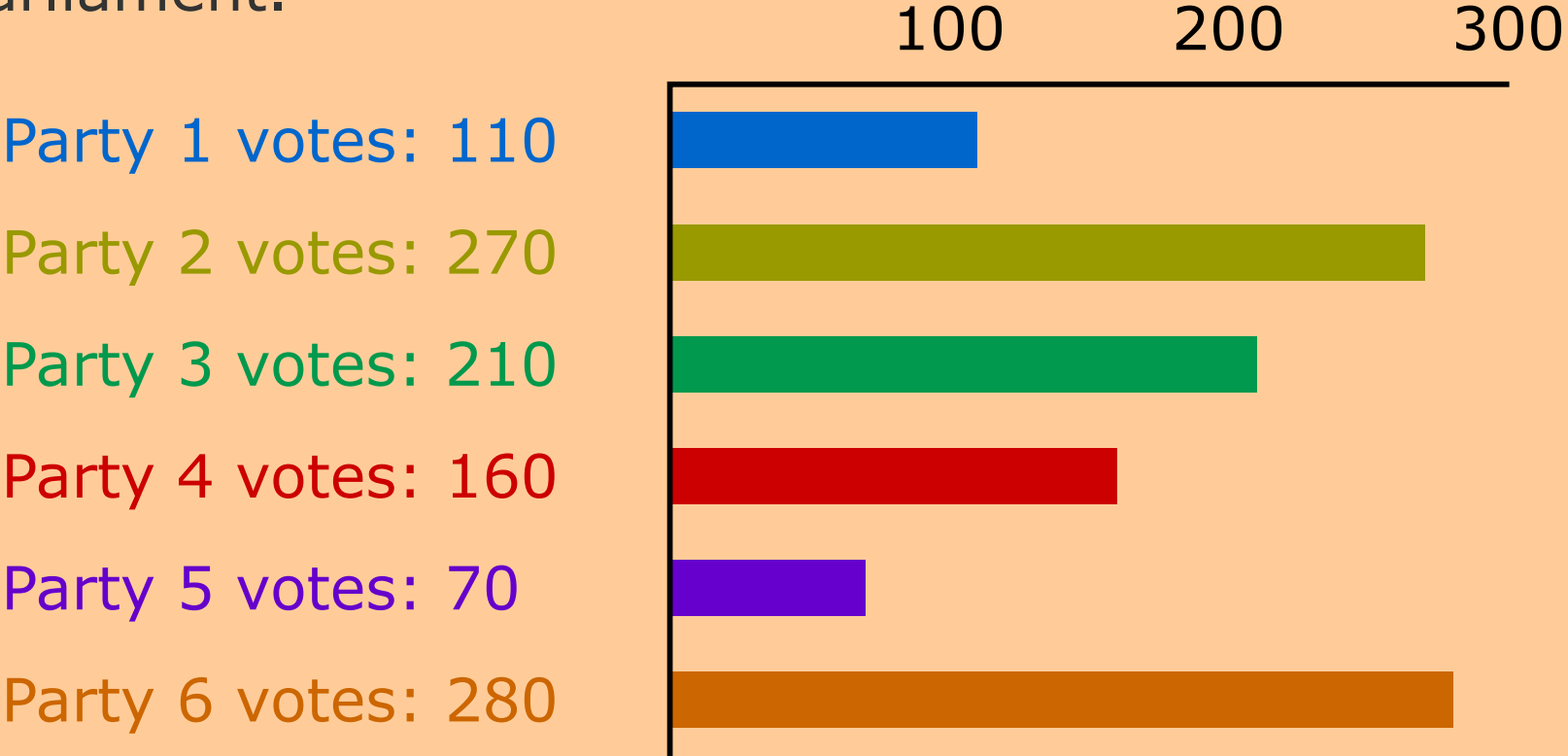
Turnout 77.75% (▼ 0.96pp)

Party	Leader	%	Seats	+/-
PVV	Geert Wilders	23.49	37	+20
GL/PvdA	Frans Timmermans	15.75	25	+8
VVD	Dilan Yeşilgöz	15.24	24	-10
NSC	Pieter Omtzigt	12.88	20	New
D66	Rob Jetten	6.29	9	-15
BBB	Caroline van der Plas	4.65	7	+6
CDA	Henri Bontenbal	3.31	5	-10
SP	Lilian Marijnissen	3.15	5	-4
Denk	Stephan van Baarle	2.37	3	0
PvdD	Esther Ouwehand	2.25	3	-3
FvD	Thierry Baudet	2.23	3	-5
SGP	Chris Stoffer	2.08	3	0
CU	Mirjam Bikker	2.04	3	-2
Volt	Laurens Dassen	1.71	2	-1
JA21	Joost Eerdmans	0.68	1	-2

Running example

Example

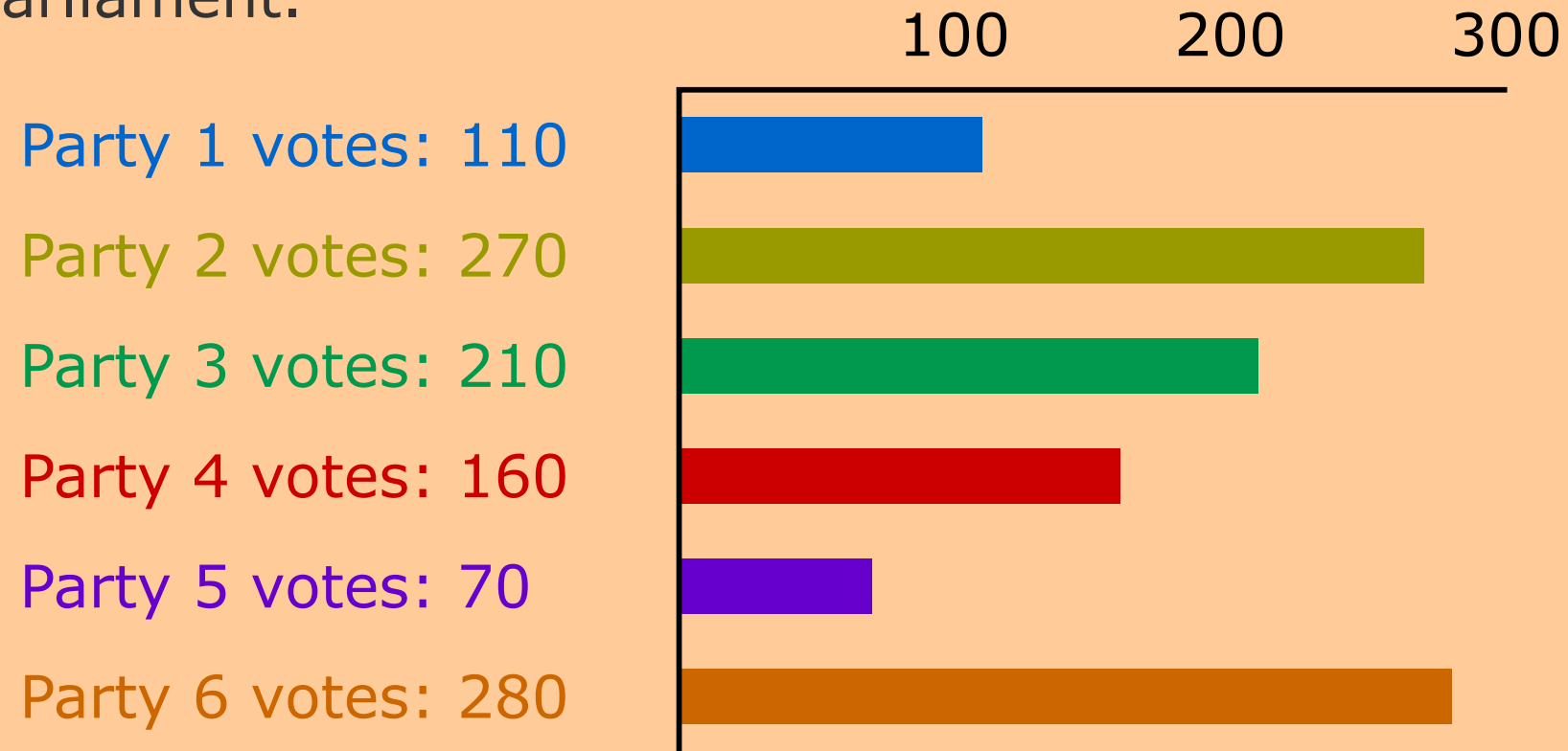
Suppose there are 1100 voters and 11 seats in parliament.



Running example

Example

Suppose there are 1100 voters and 11 seats in parliament.



► **How many seats should each party get?**
(No right answer!)



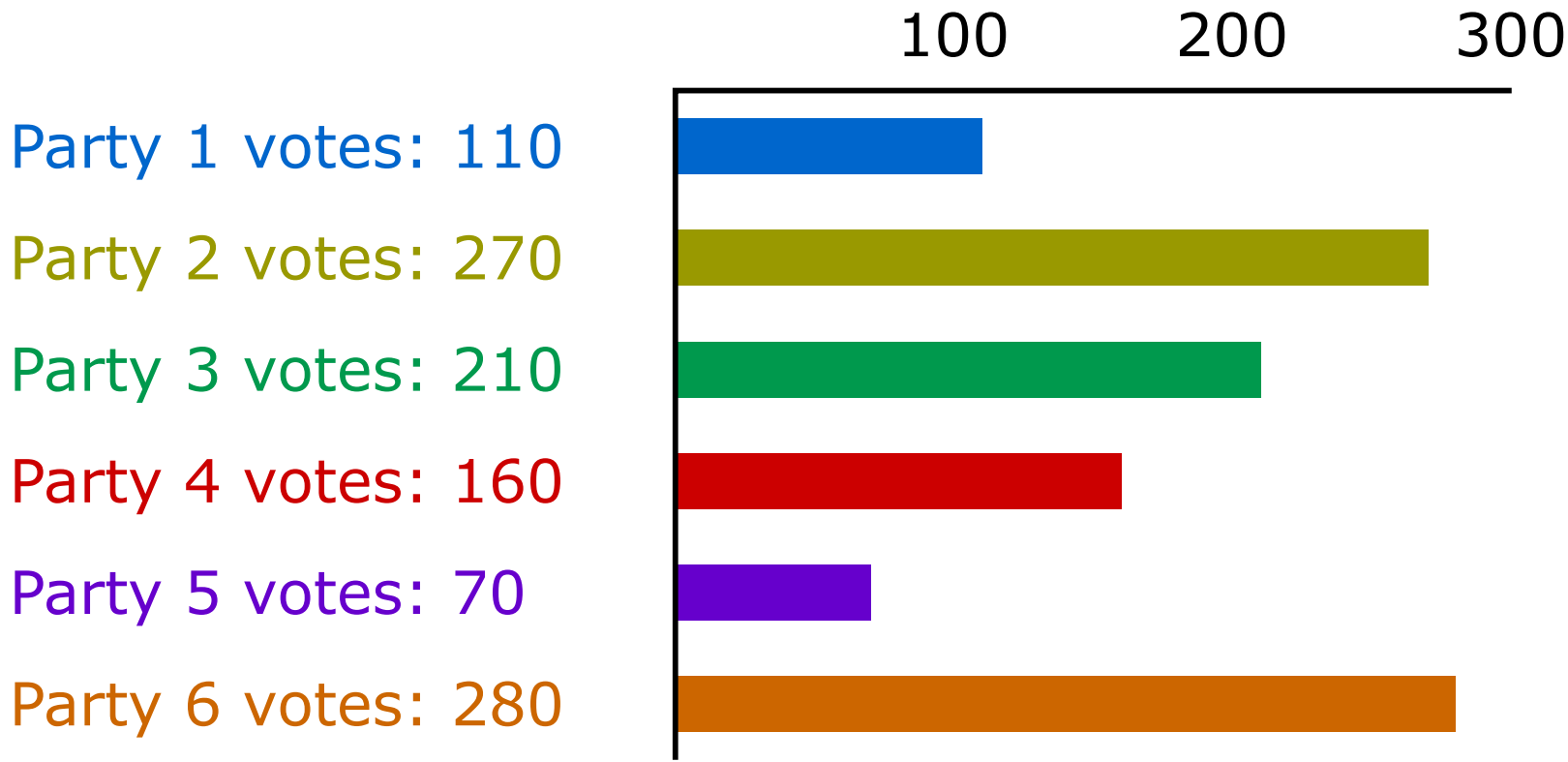
Respond at:

pollev.com/jtuckerfoltz255 or

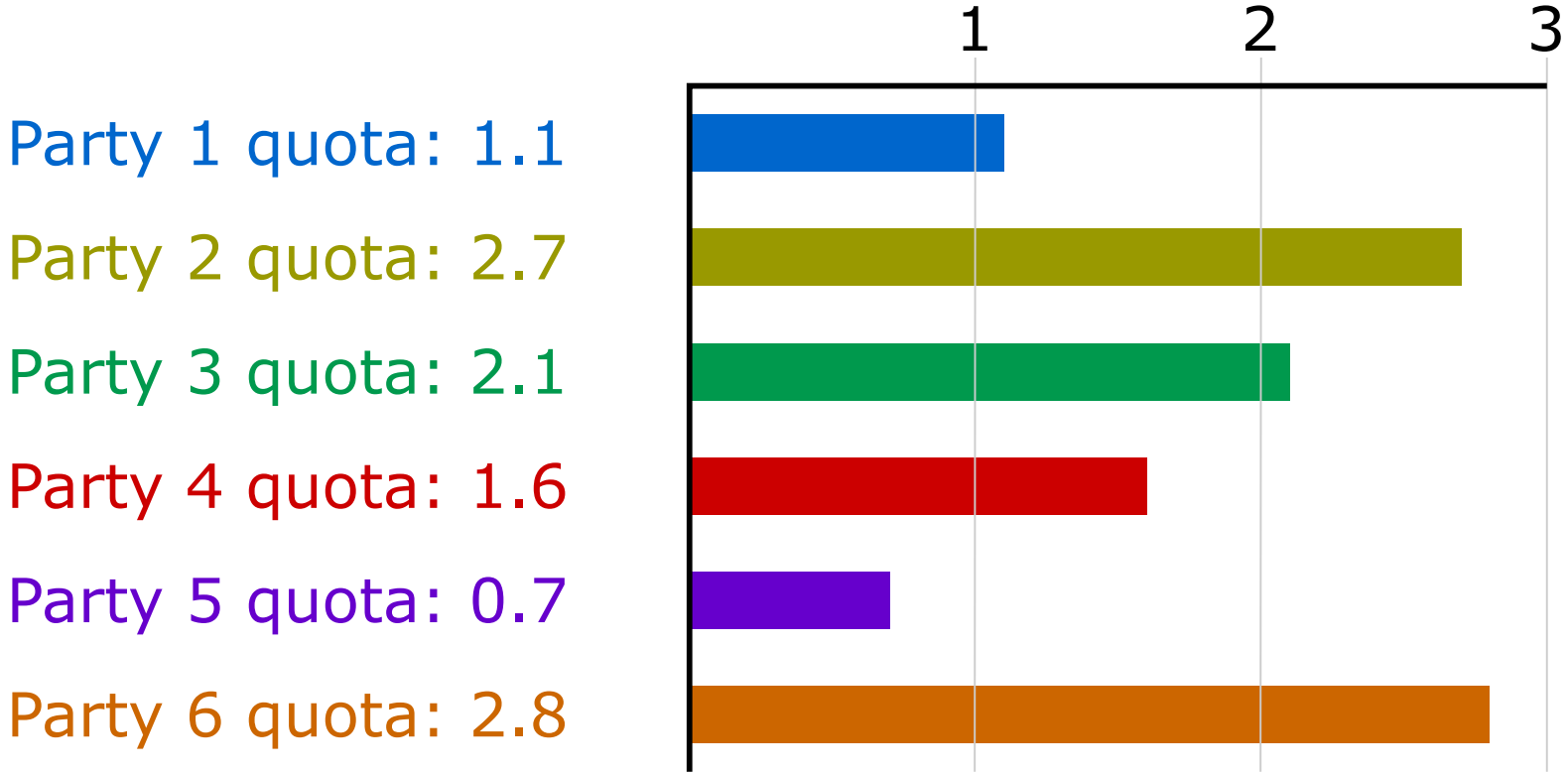
bit.ly/jtfpoll or

text jtuckerfoltz255 to 37607

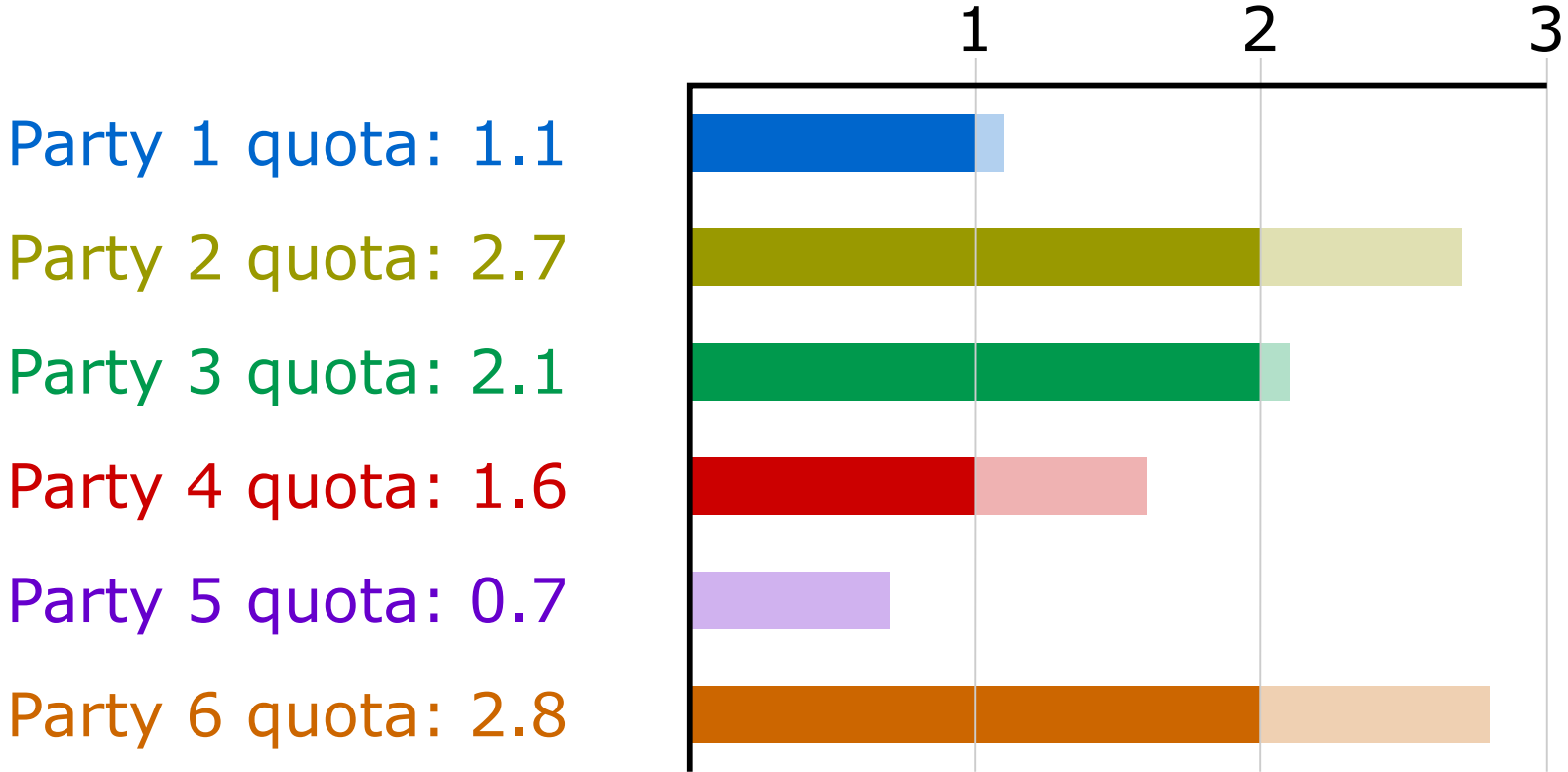
Hamilton's method



Hamilton's method



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Party 1 quota: 1.1

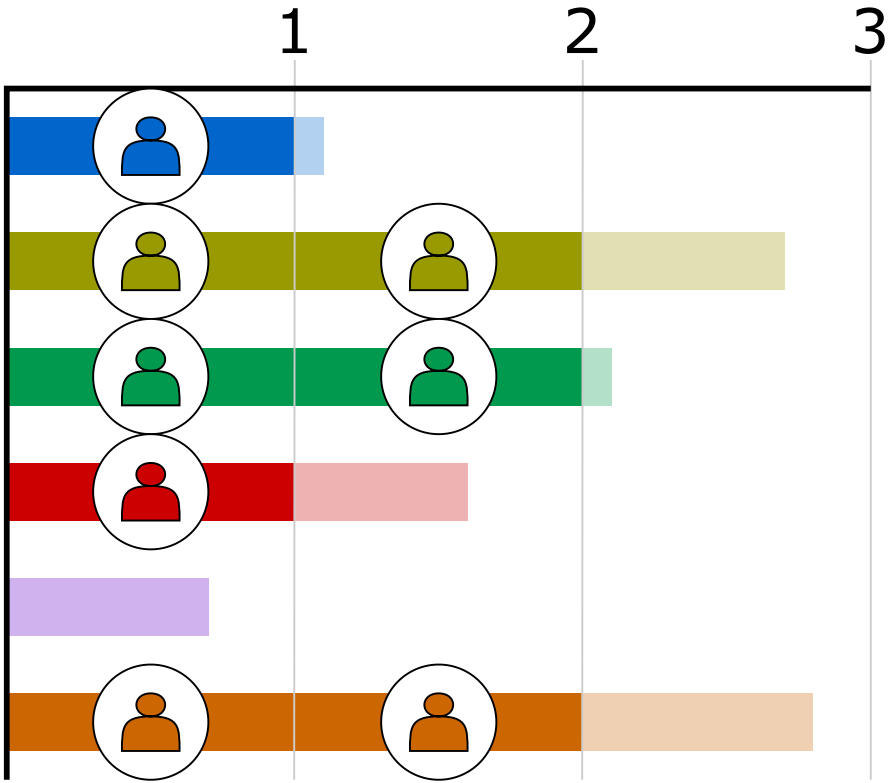
Party 2 quota: 2.7

Party 3 quota: 2.1

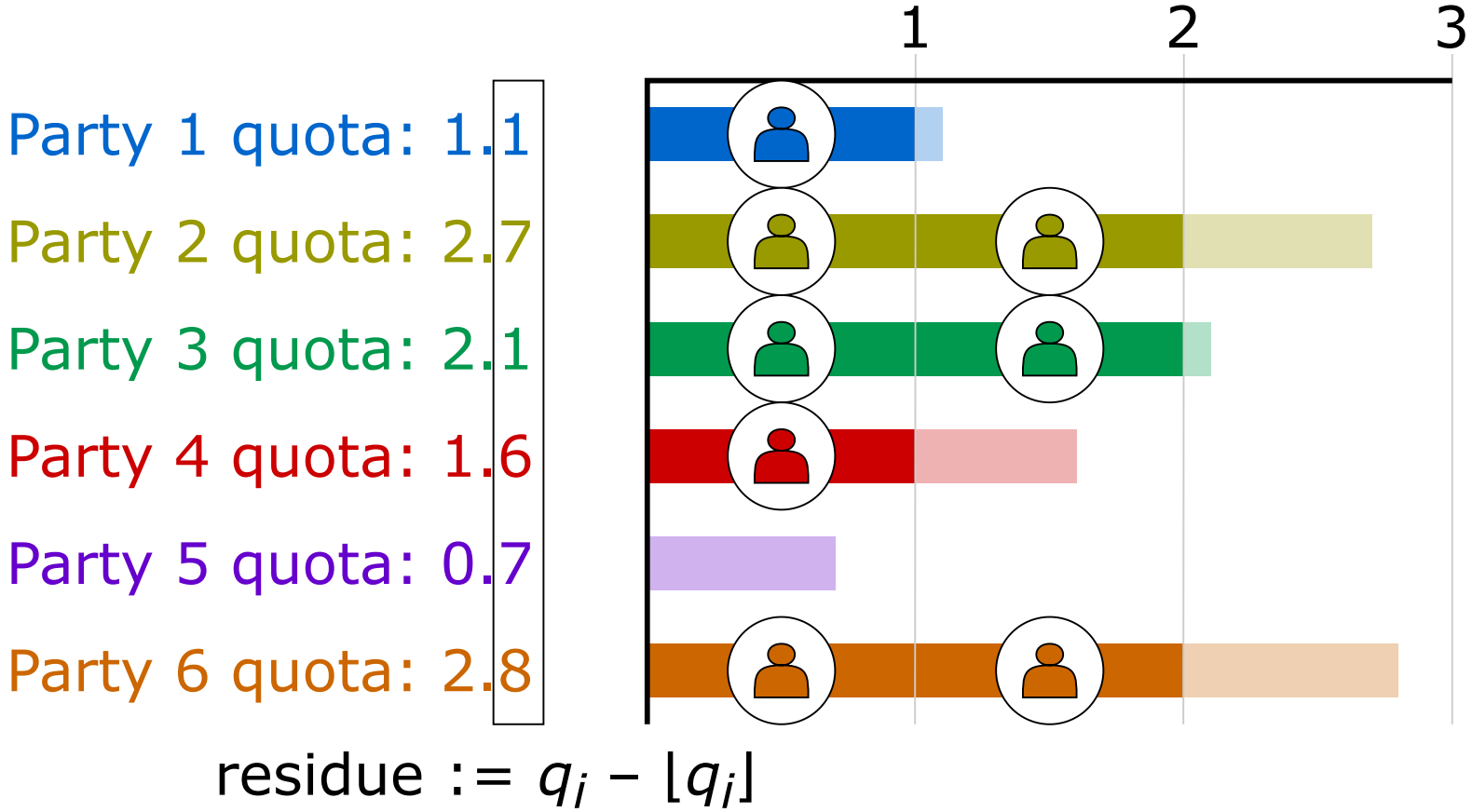
Party 4 quota: 1.6

Party 5 quota: 0.7

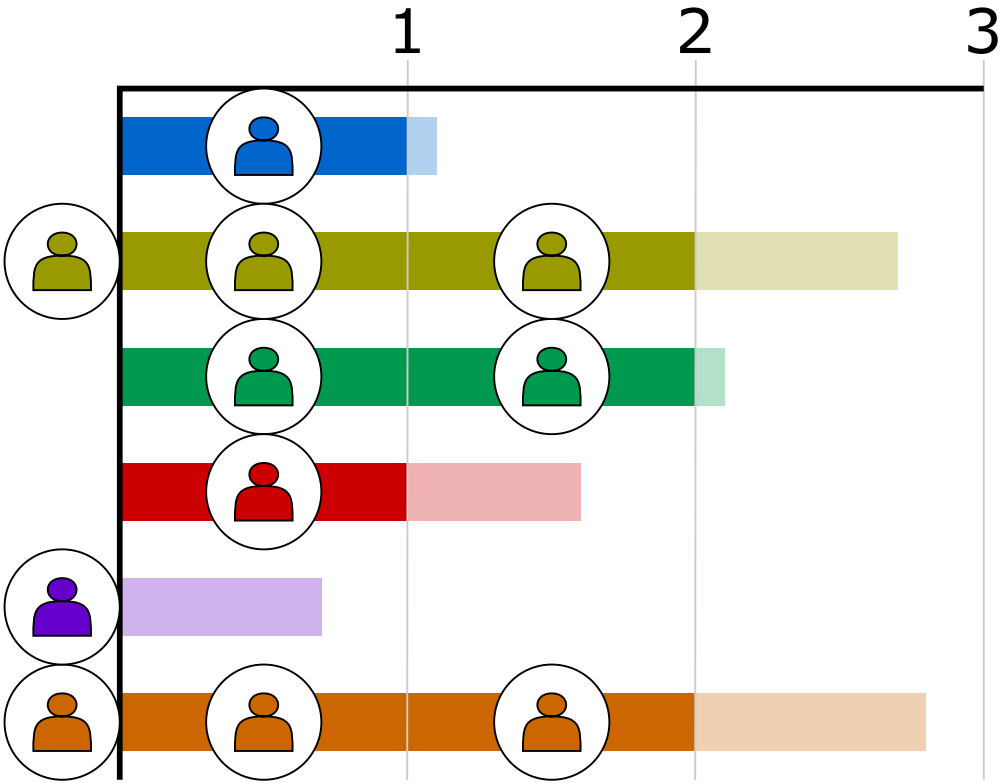
Party 6 quota: 2.8



Hamilton's method



Hamilton's method



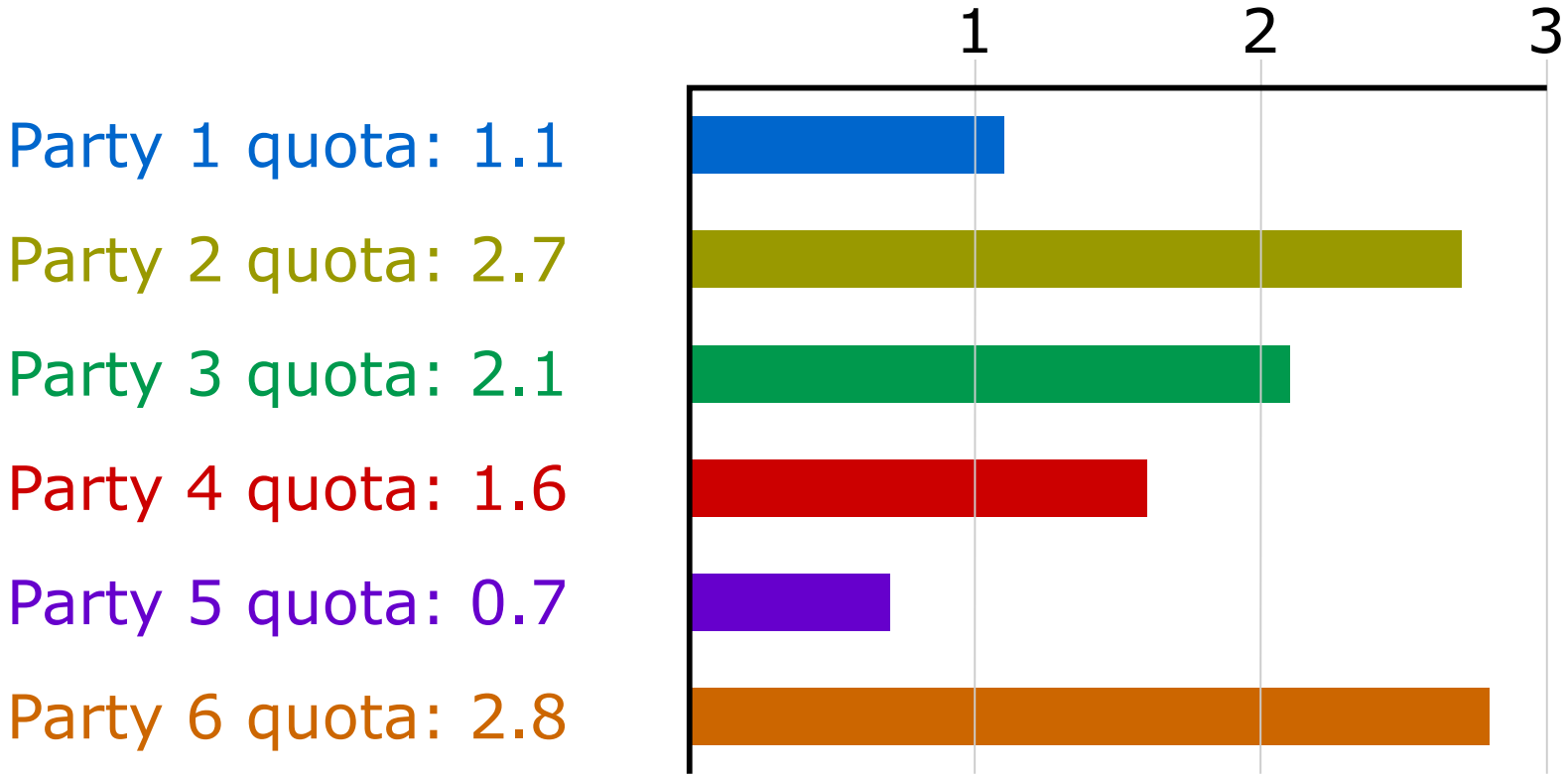
#Seats	Ham
Party 1	1
Party 2	3
Party 3	2
Party 4	1
Party 5	1
Party 6	3

Jefferson's method

Divisor methods: Fix a rounding rule, then scale populations to get the right num seats.

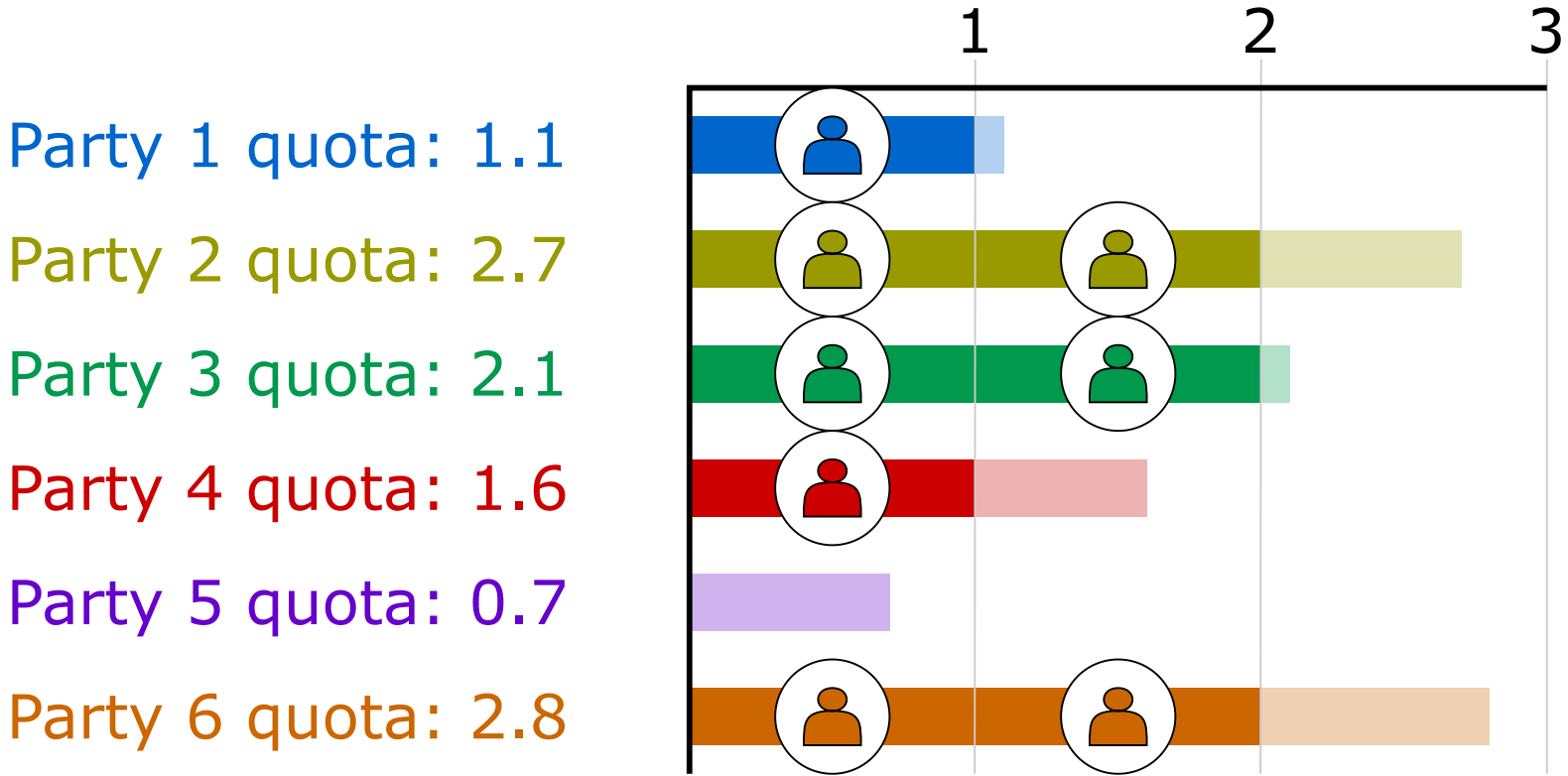
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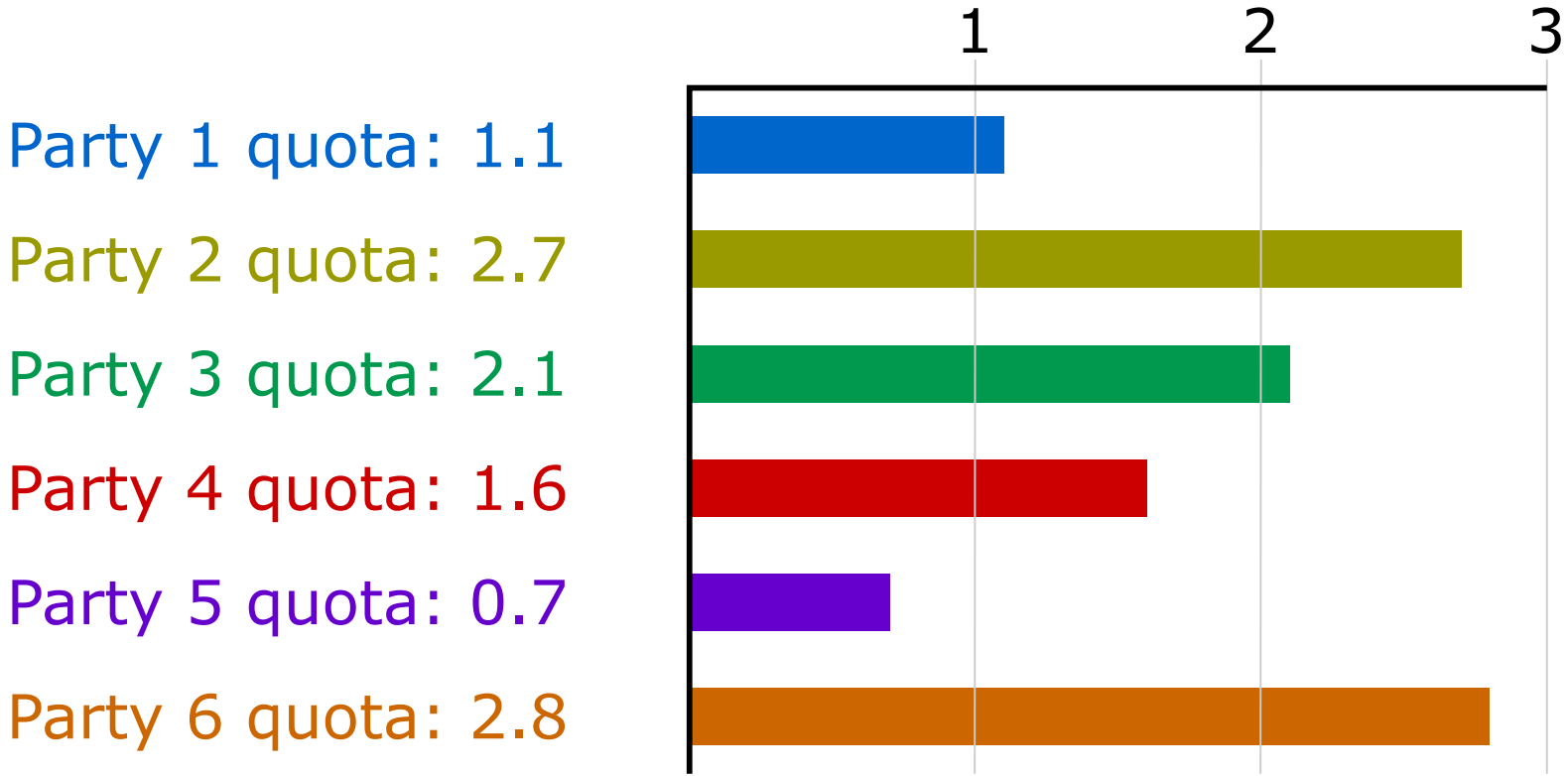
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Jefferson's method: Round down

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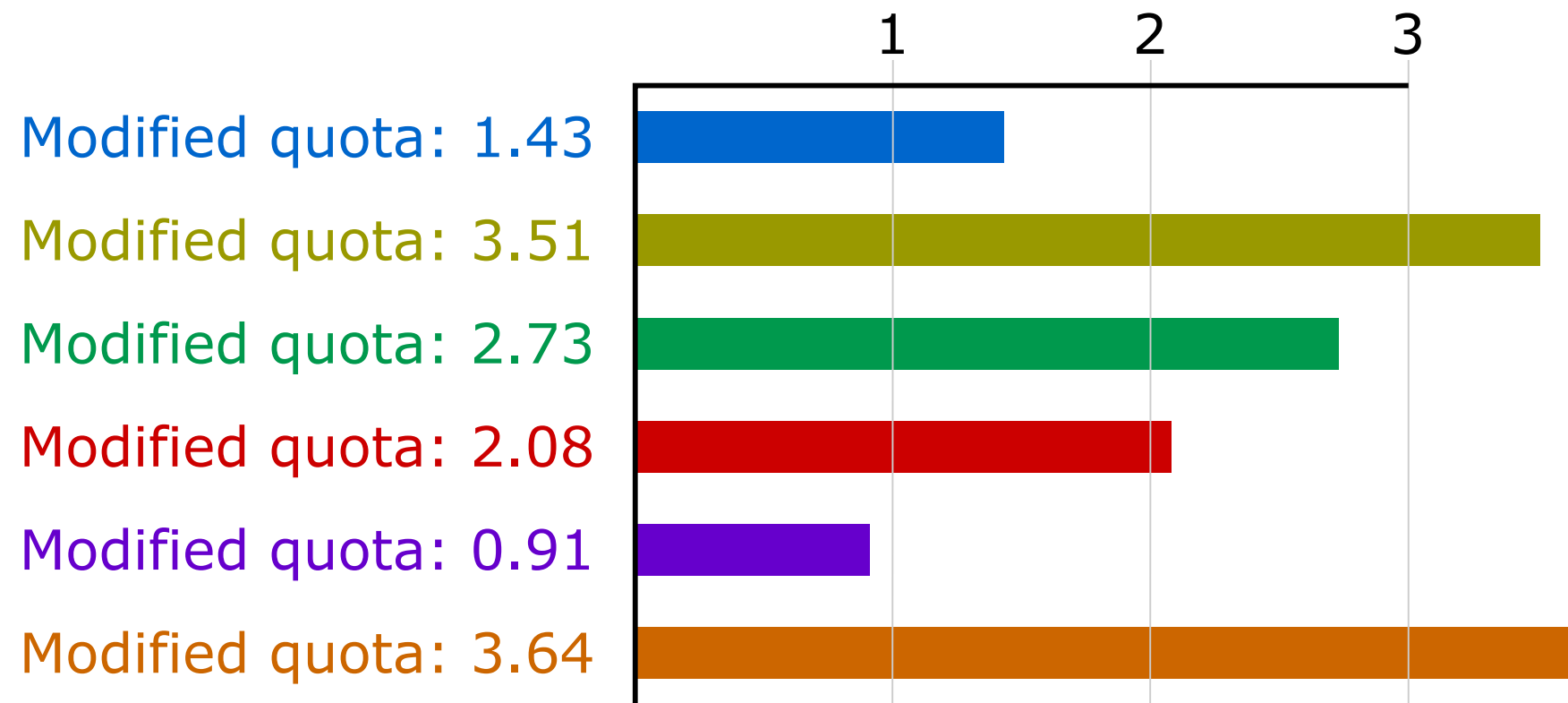
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Jefferson's method: Round down - need to scale up first

Jefferson's method

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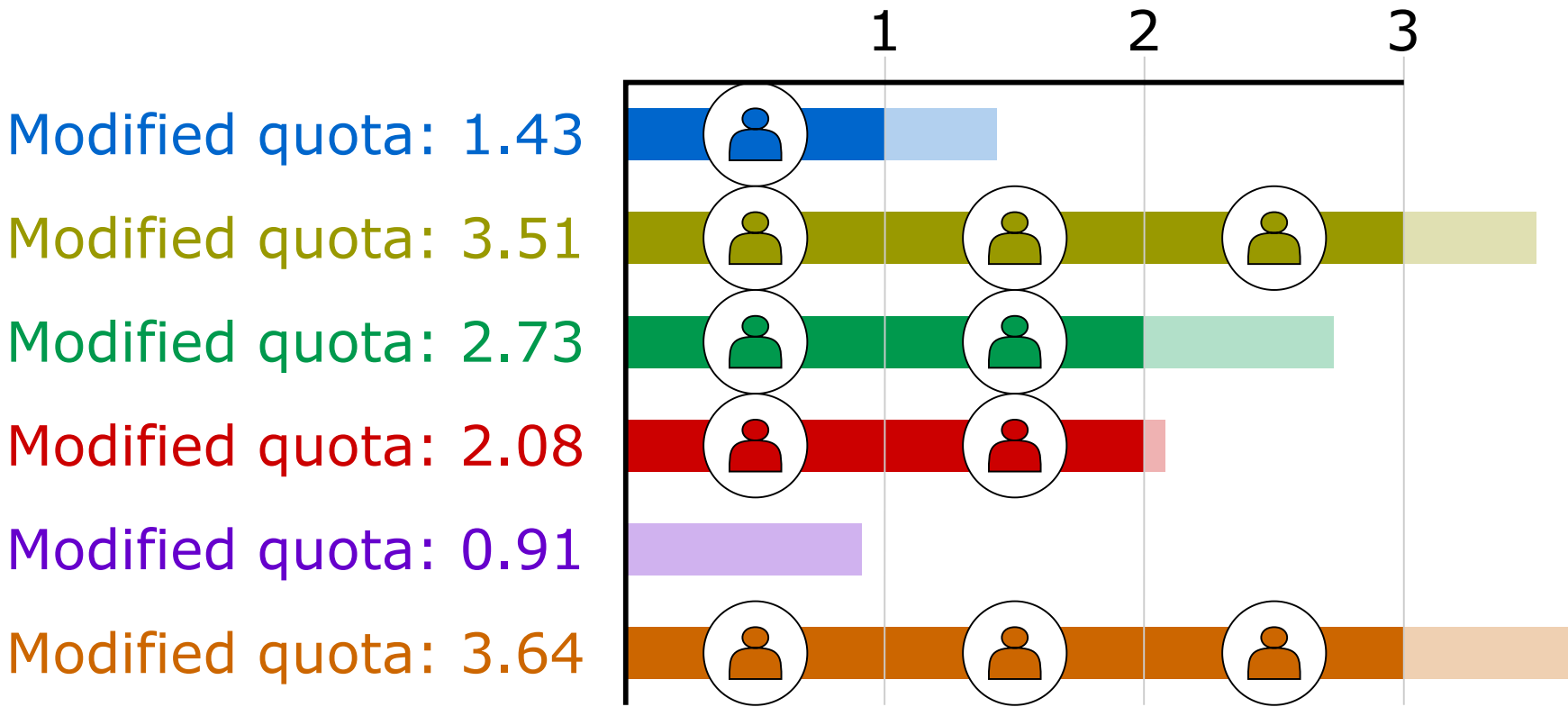


Jefferson's method: Round down - need to scale up first

The correct scale factor turns out to be 1.3

Jefferson's method

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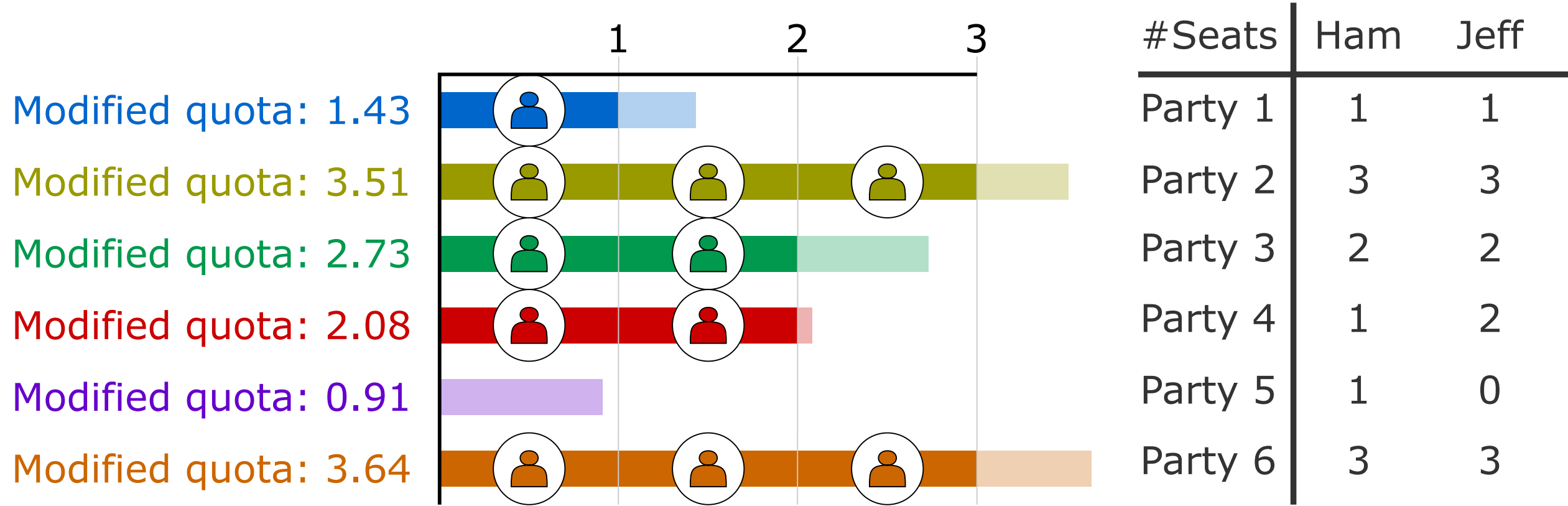


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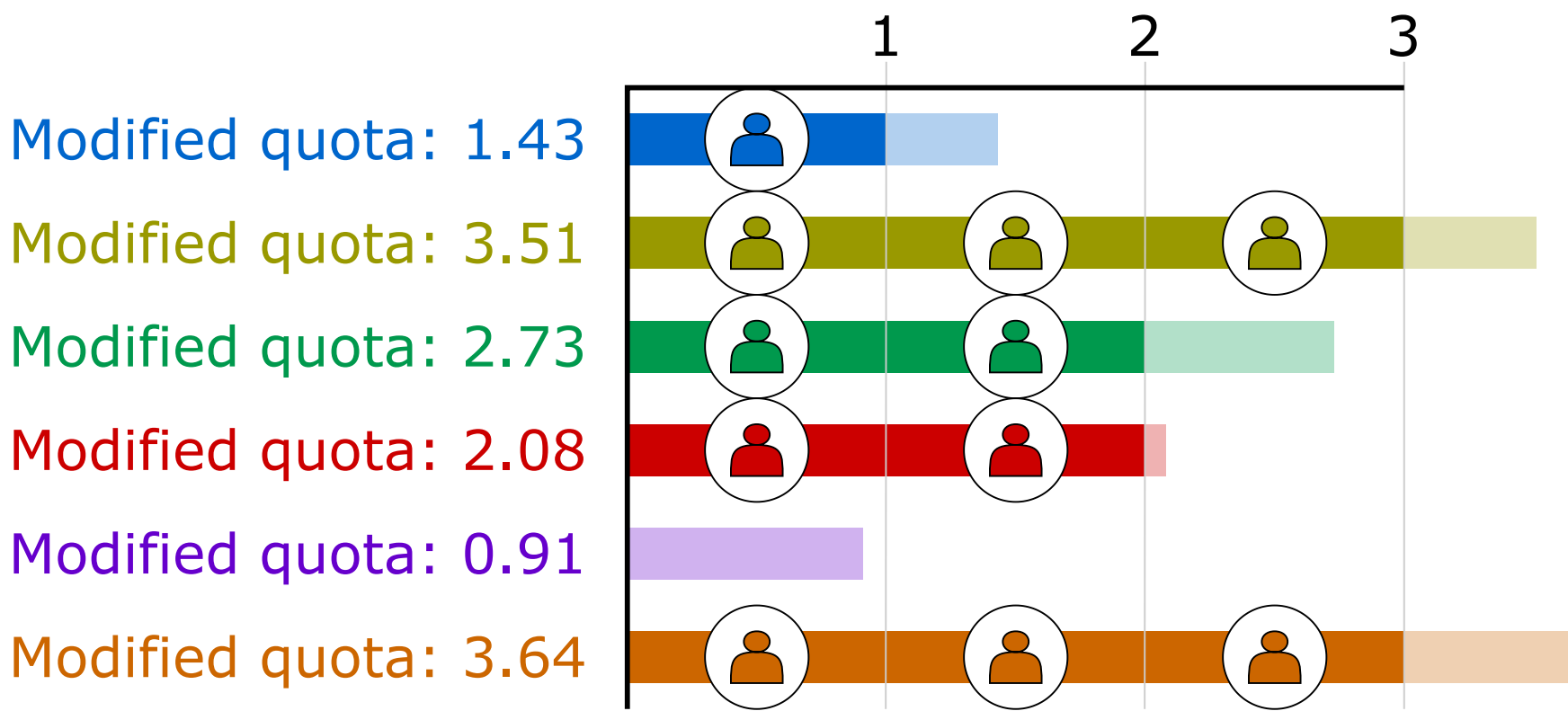


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Modified quota: 1.43

Modified quota: 3.51

Modified quota: 2.73

Modified quota: 2.08

Modified quota: 0.91

Modified quota: 3.64

#Seats	Ham	Jeff
Party 1	1	1
Party 2	3	3
Party 3	2	2
Party 4	1	2
Party 5	1	0
Party 6	3	3

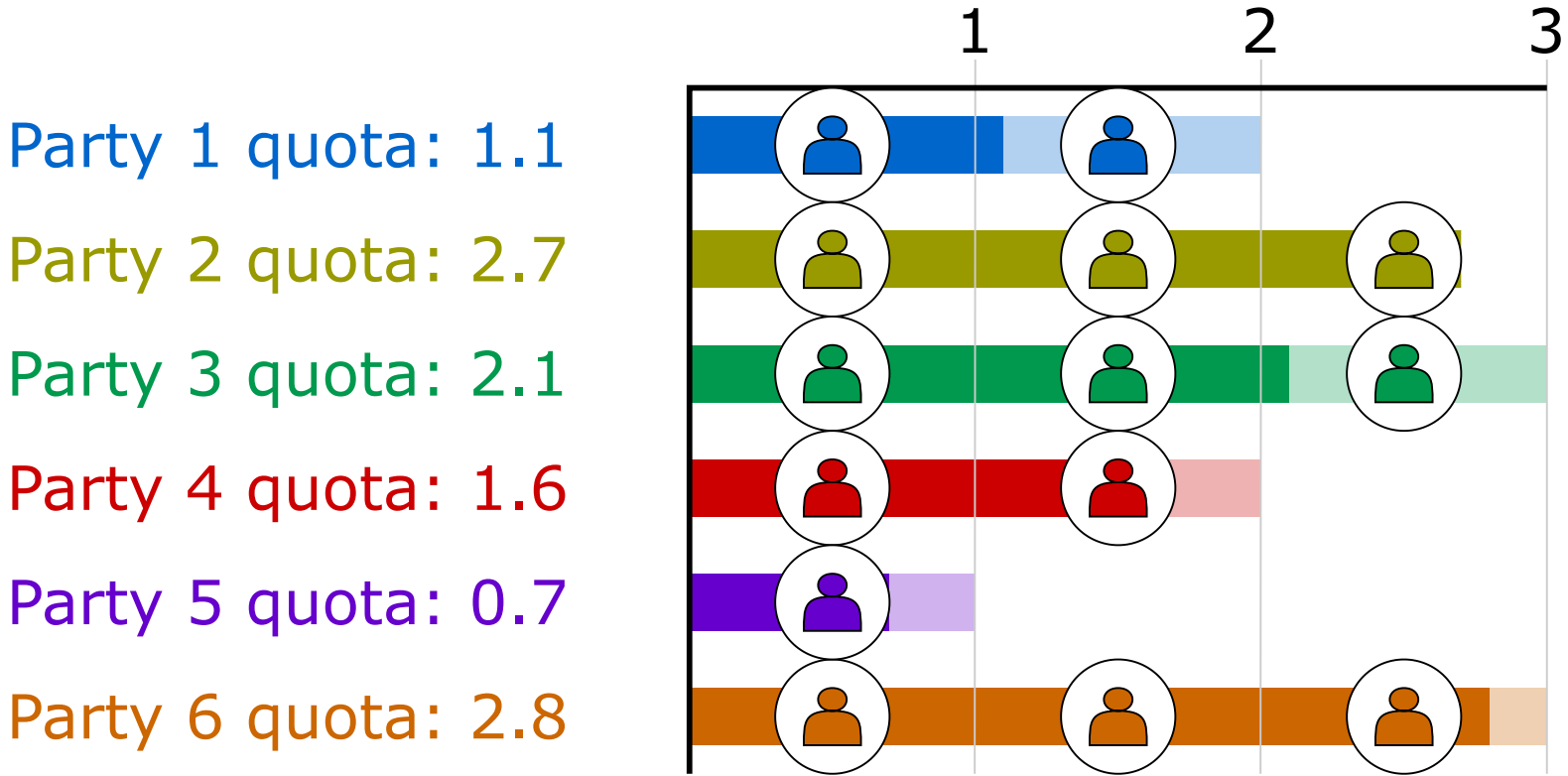
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Jefferson's has large state bias!

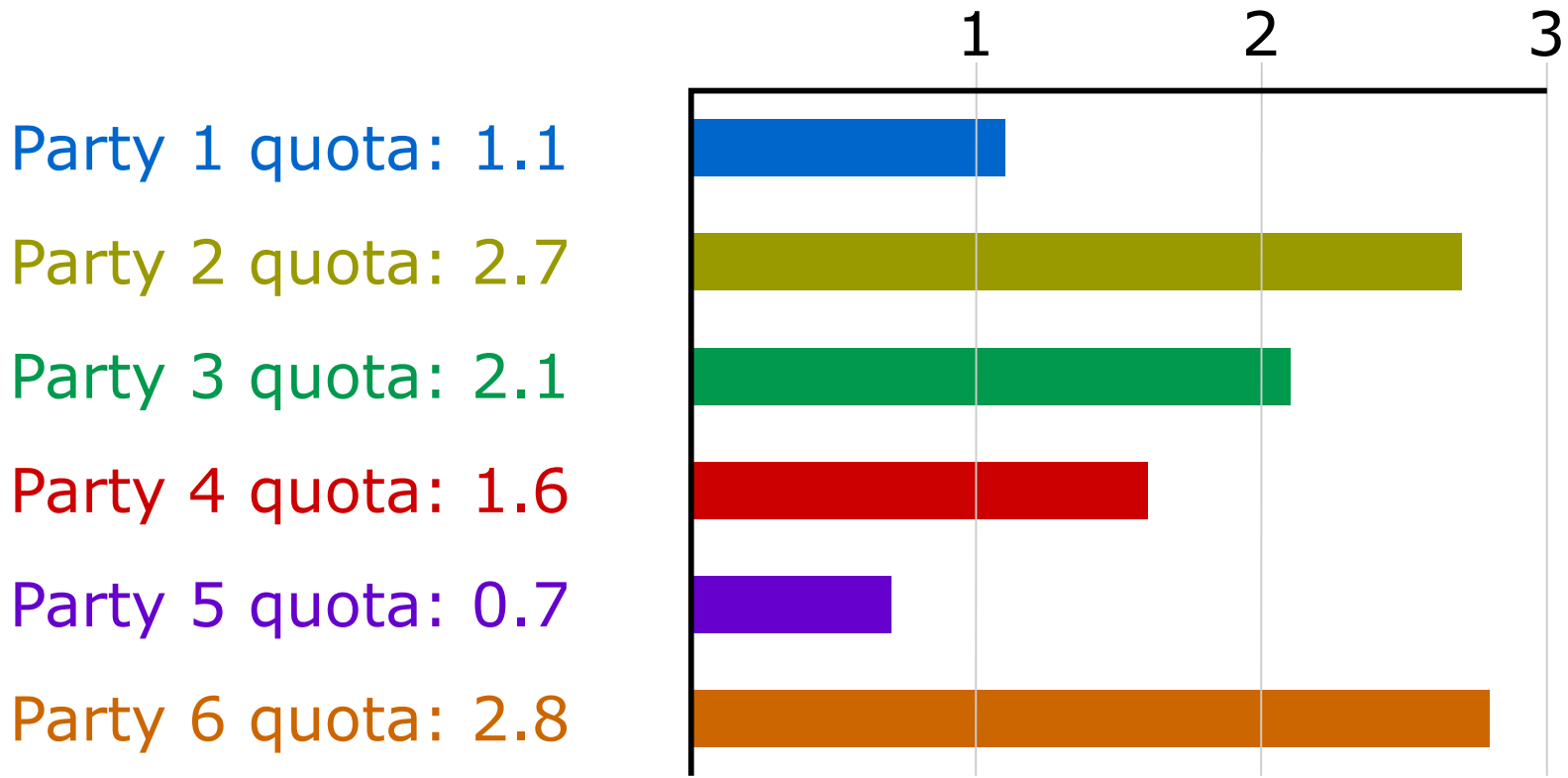
Adams' method

Round up instead of down



Adams' method

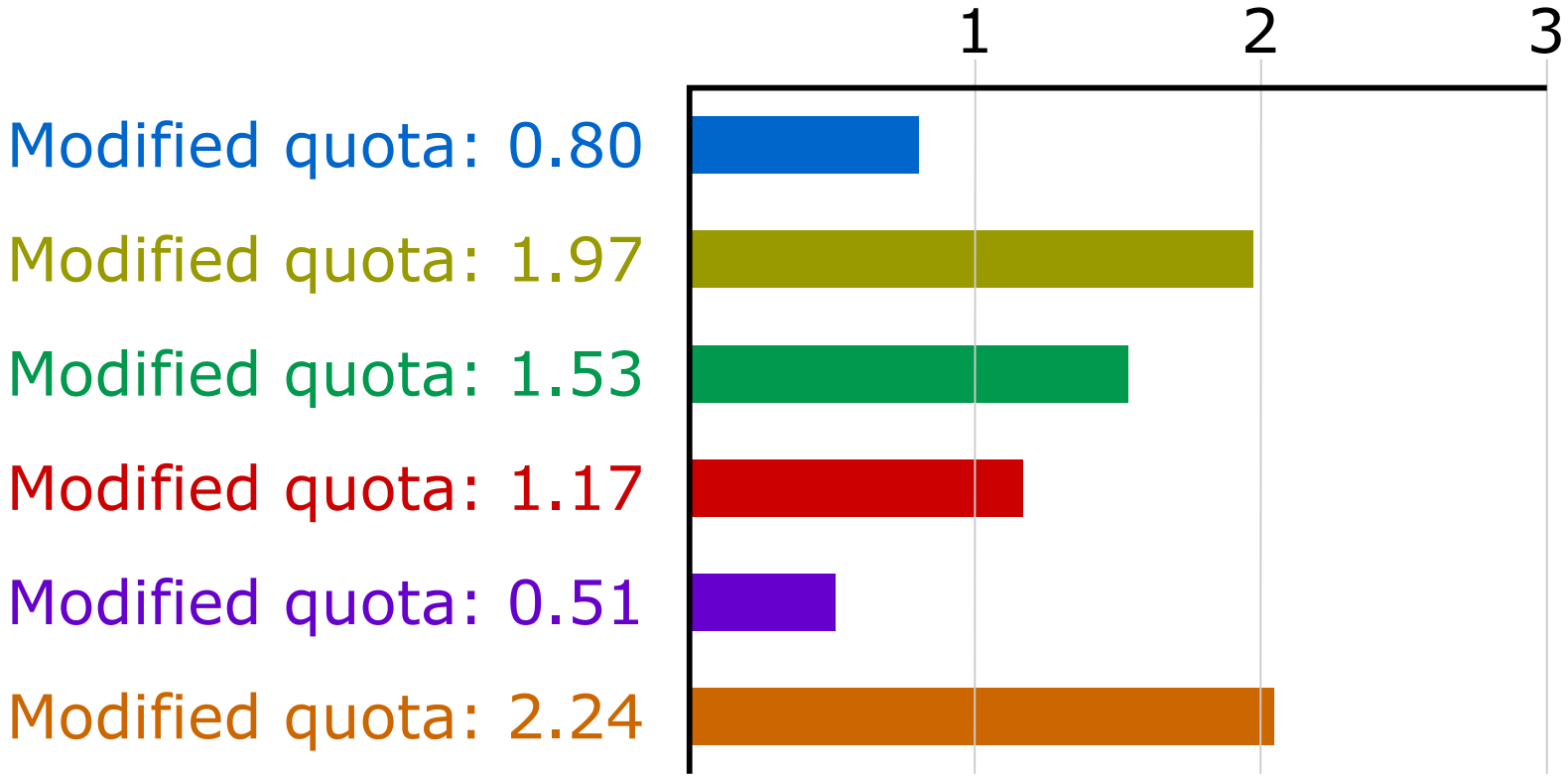
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So we have to scale *down* first

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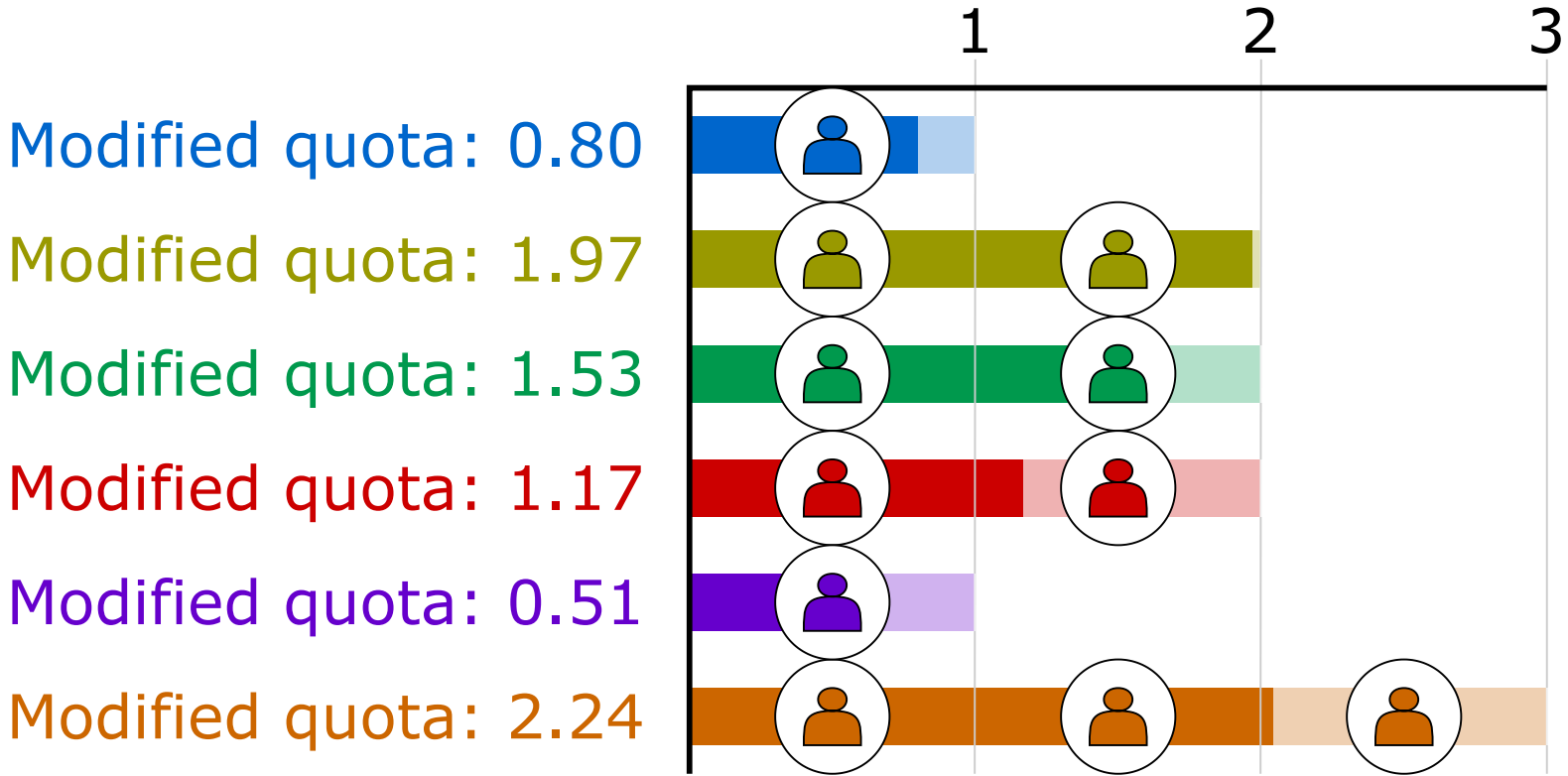


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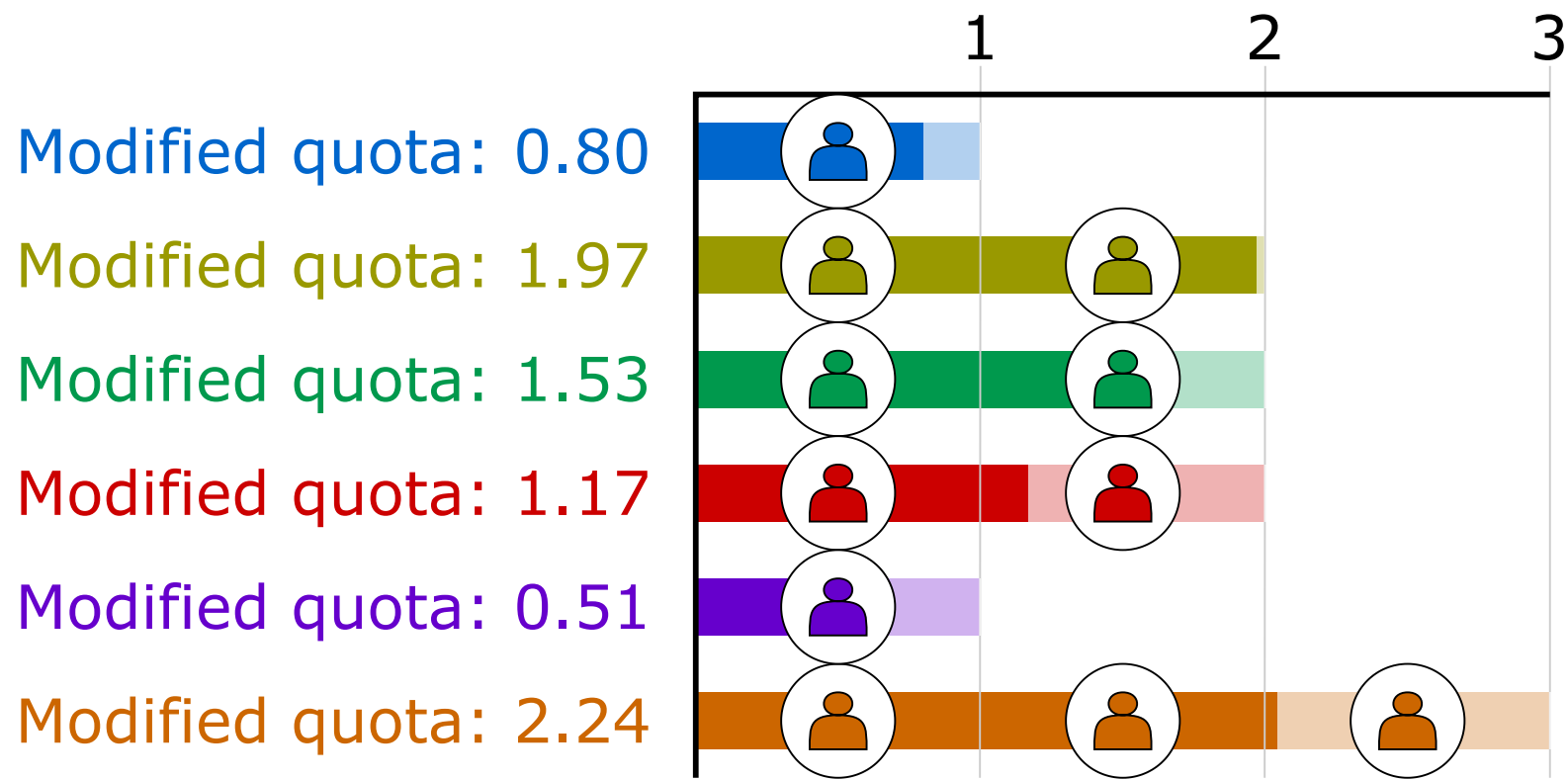


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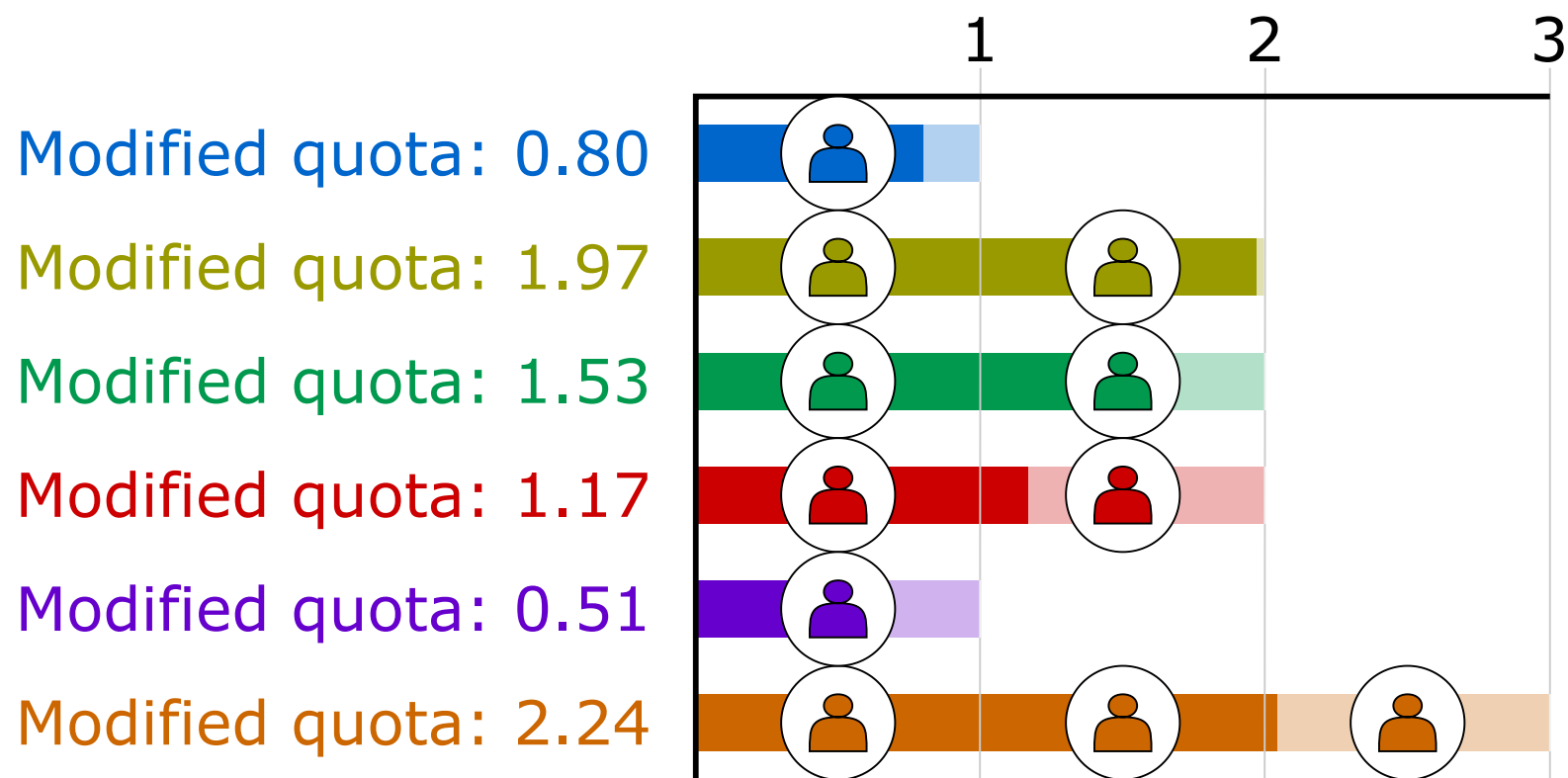
#Seats	Ham	Jeff	Adams
Party 1	1	1	1
Party 2	3	3	2
Party 3	2	2	2
Party 4	1	2	2
Party 5	1	0	1
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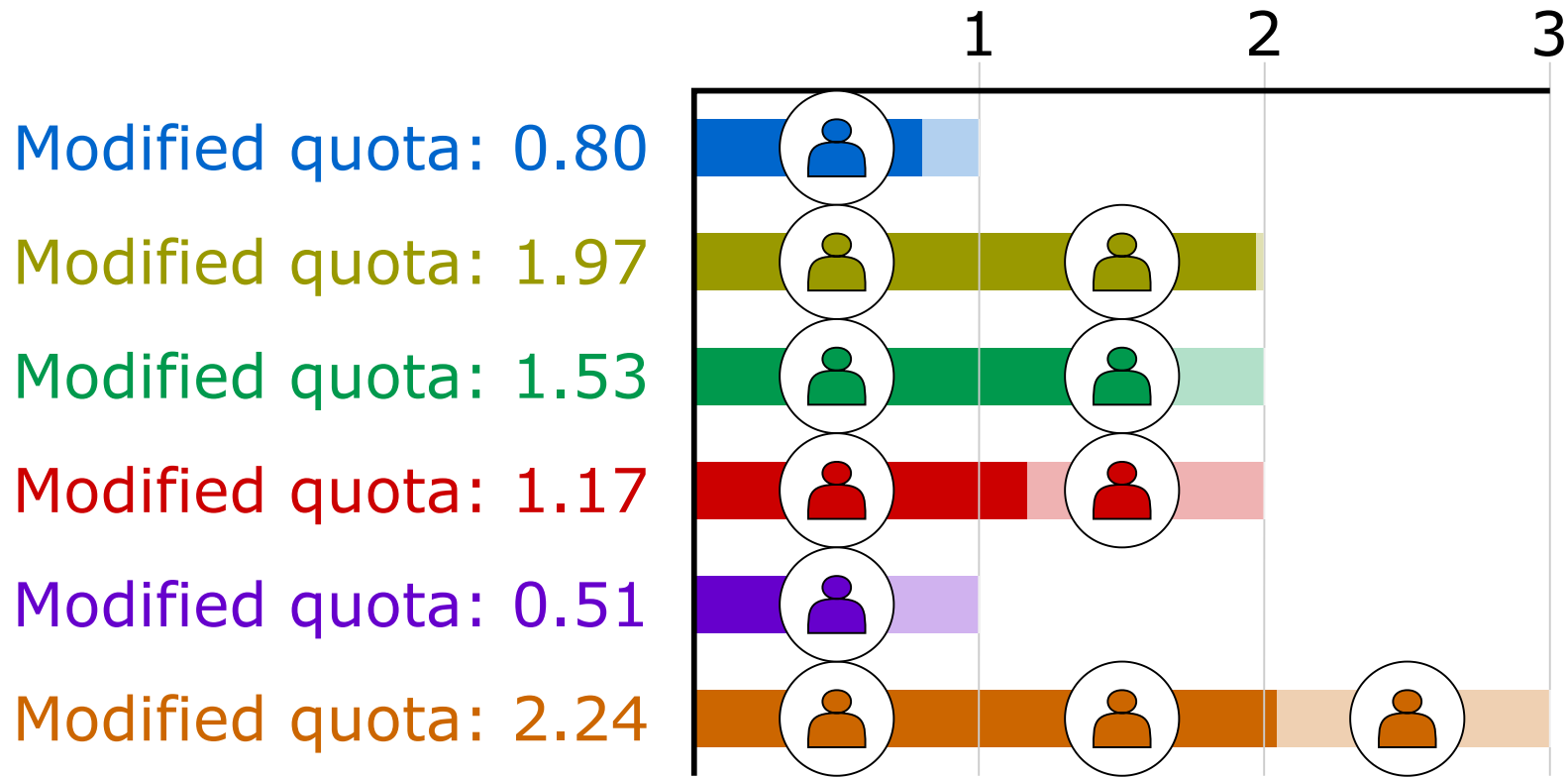
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Adams' has
small state bias!

Adams' method

Round up instead of down



So we have to scale *down* first

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Party 1	1	1	1
Party 2	3	3	2
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Also coincides with *Webster's method*: round to nearest.

Stationary divisor methods

Let $[r]_\delta$ denote the rounding of r , where we round r down to $\lfloor r \rfloor$ if $r - \lfloor r \rfloor < \delta$ and up to $\lceil r \rceil$ if $r - \lfloor r \rfloor > \delta$. Examples: $[2.7]_{0.5} = 3$, $[2.7]_{0.9} = 2$.

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Given input populations $p \in \mathbb{Z}_{>0}^n$ and house size $H \in \mathbb{Z}_{>0}$, the δ -divisor method returns any vector $x \in \mathbb{Z}_{\geq 0}$ such that

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Questions:

1. Is this even well-defined? Does x always exist and is it unique?
2. Where do the large-state and small-state biases come from? Does larger δ necessarily mean bias toward larger states?
3. How many different apportionments x can we get by choosing different δ ?

Associated line arrangement

The following ideas are all from:

(Cembrano, Correa, Tsigonias-Dimitriadis, Schmidt-Kraepelin, Verdugo, 2025)

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$$\mathcal{L}(p, H) = \left\{ \ell_{i,t}(\delta) = \frac{t}{p_i} + \frac{\delta}{p_i} \mid i \in [n], t \in \{0, 1, 2, \dots, H-1\} \right\}.$$

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$$\iff \text{The point } (\delta, \lambda) \text{ is above exactly } \sum_{i=1}^n x_i = H \text{ lines from } \mathcal{L}, \text{ namely above } x_i \text{ of each type } i.$$

The $(H - 1)$ -level defines all δ -divisor rules

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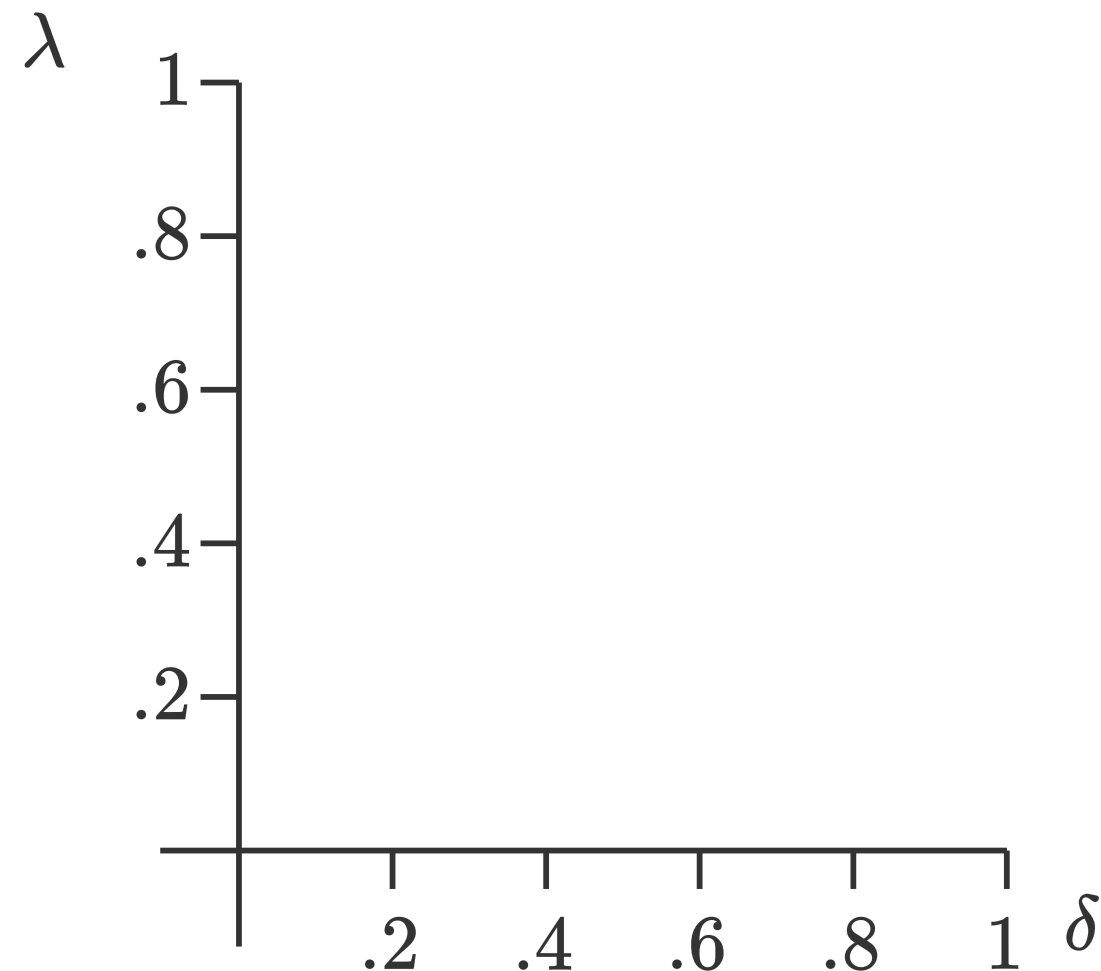
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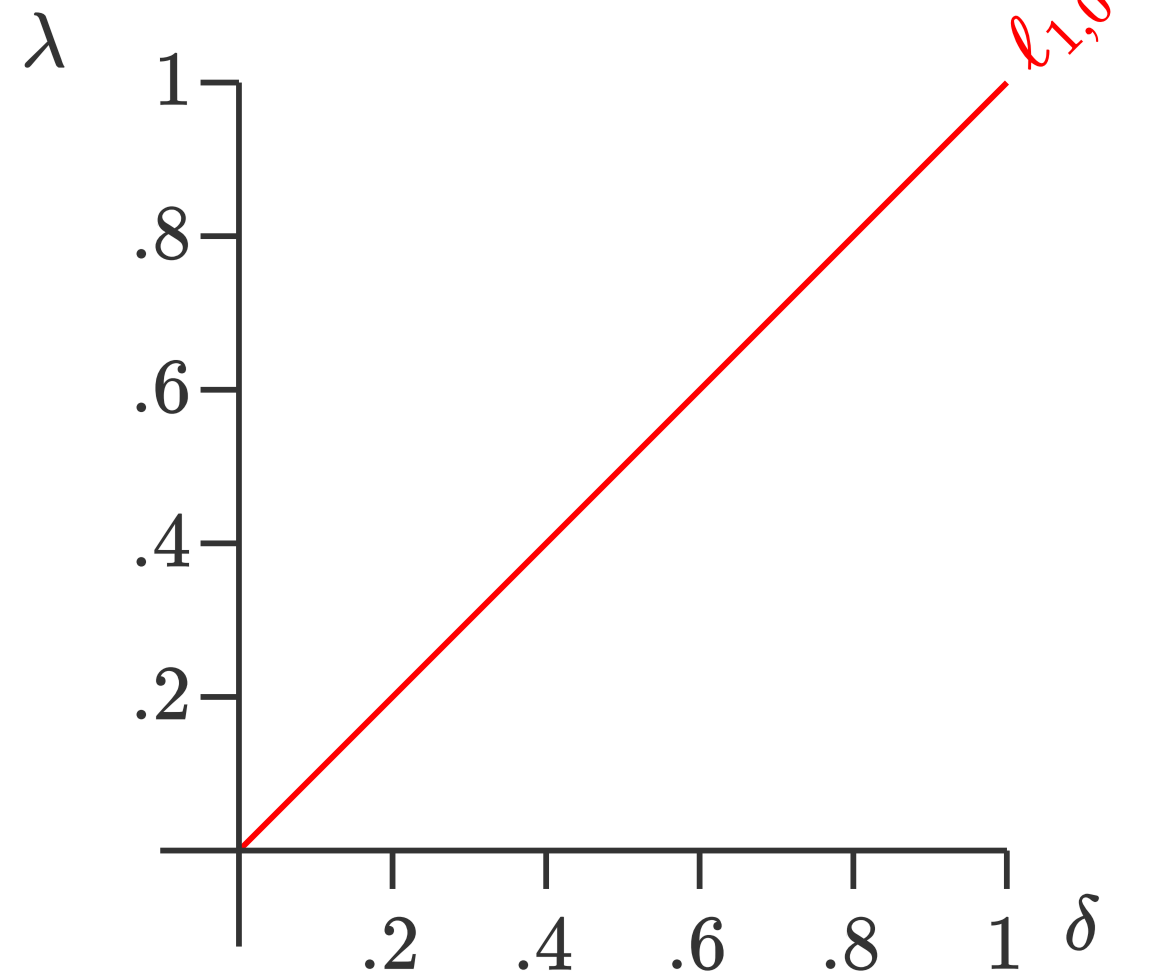


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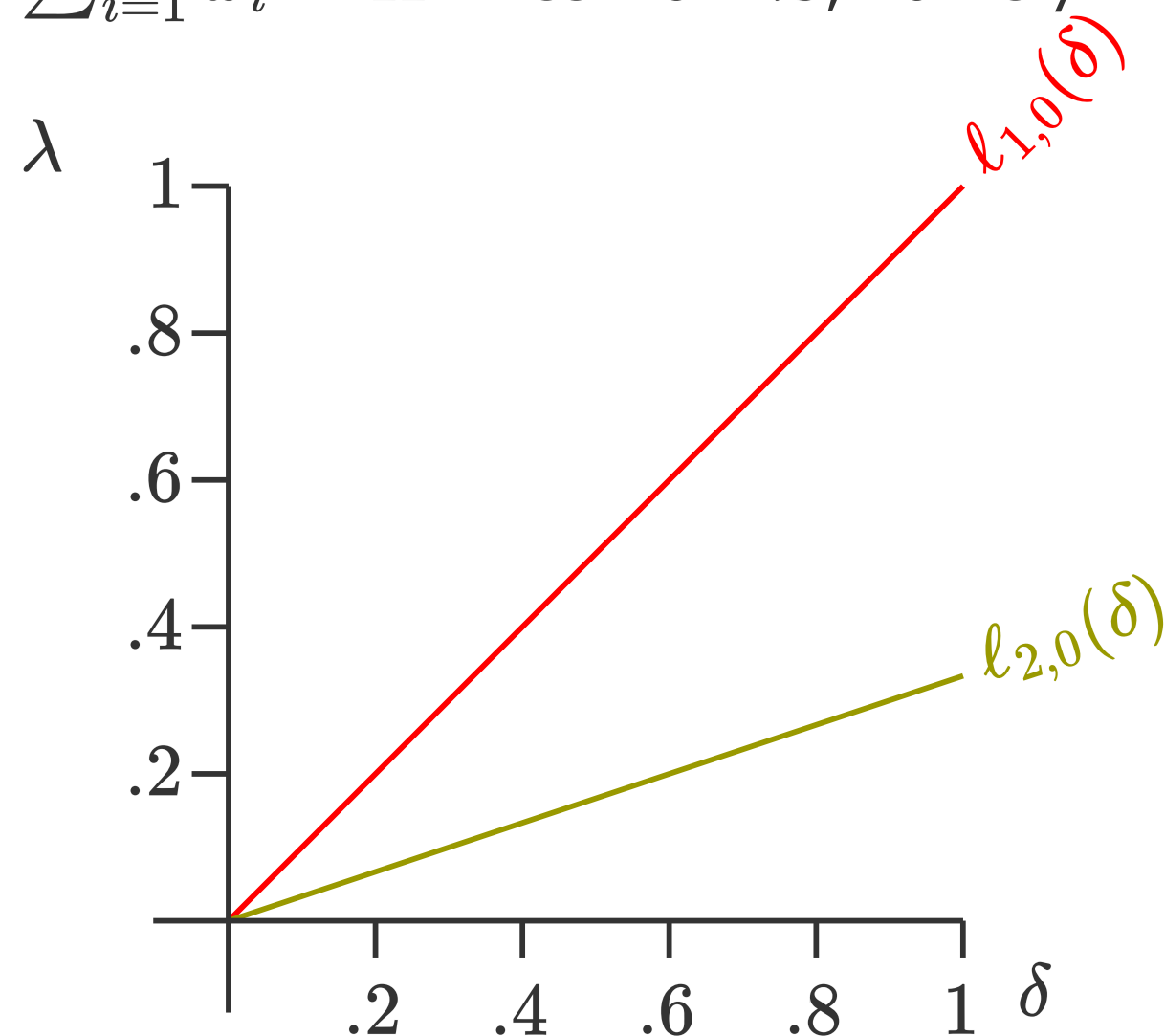


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Example: Consider $p = (1, 3, 8)$ and $H = 6$.

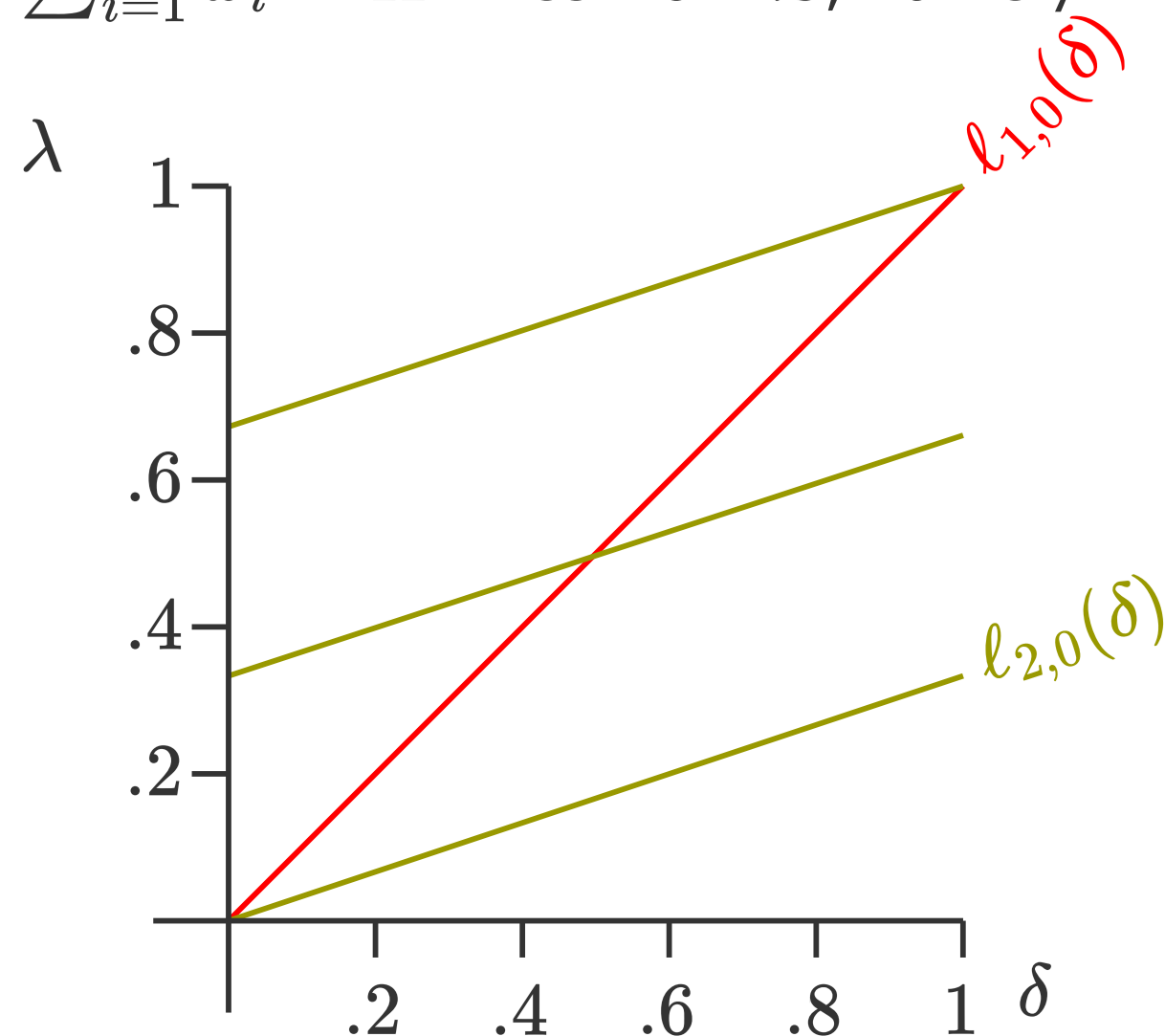


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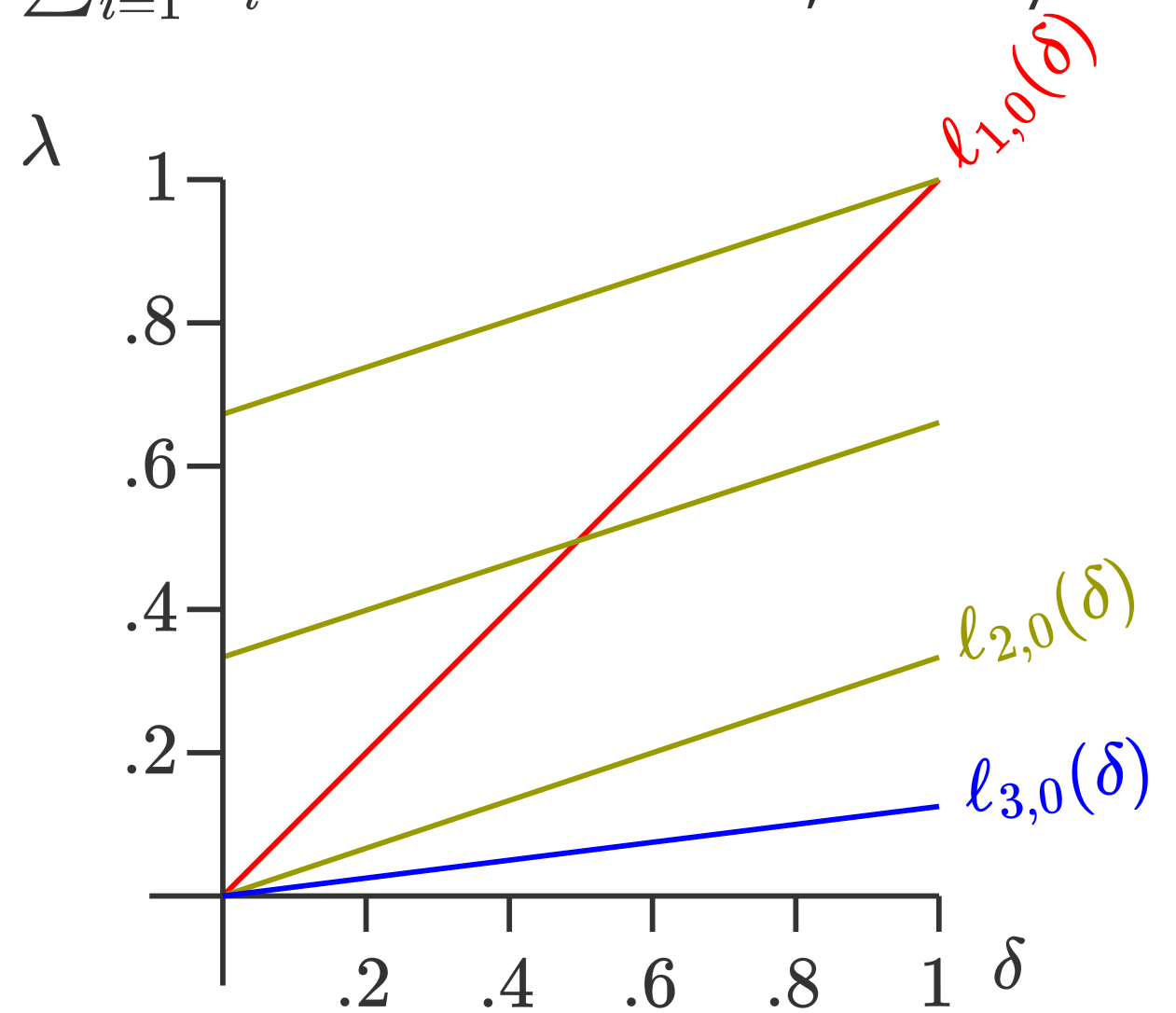


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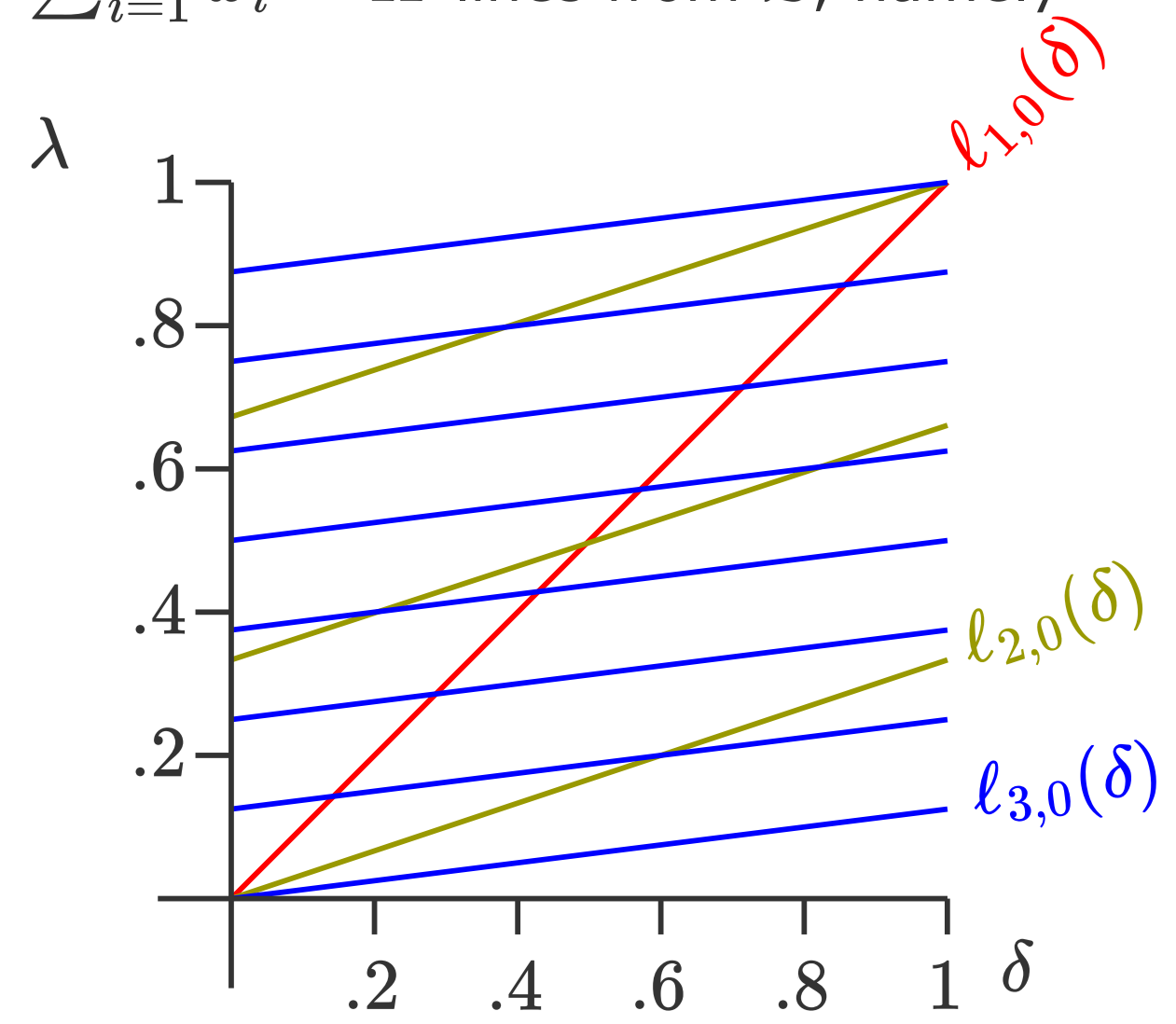


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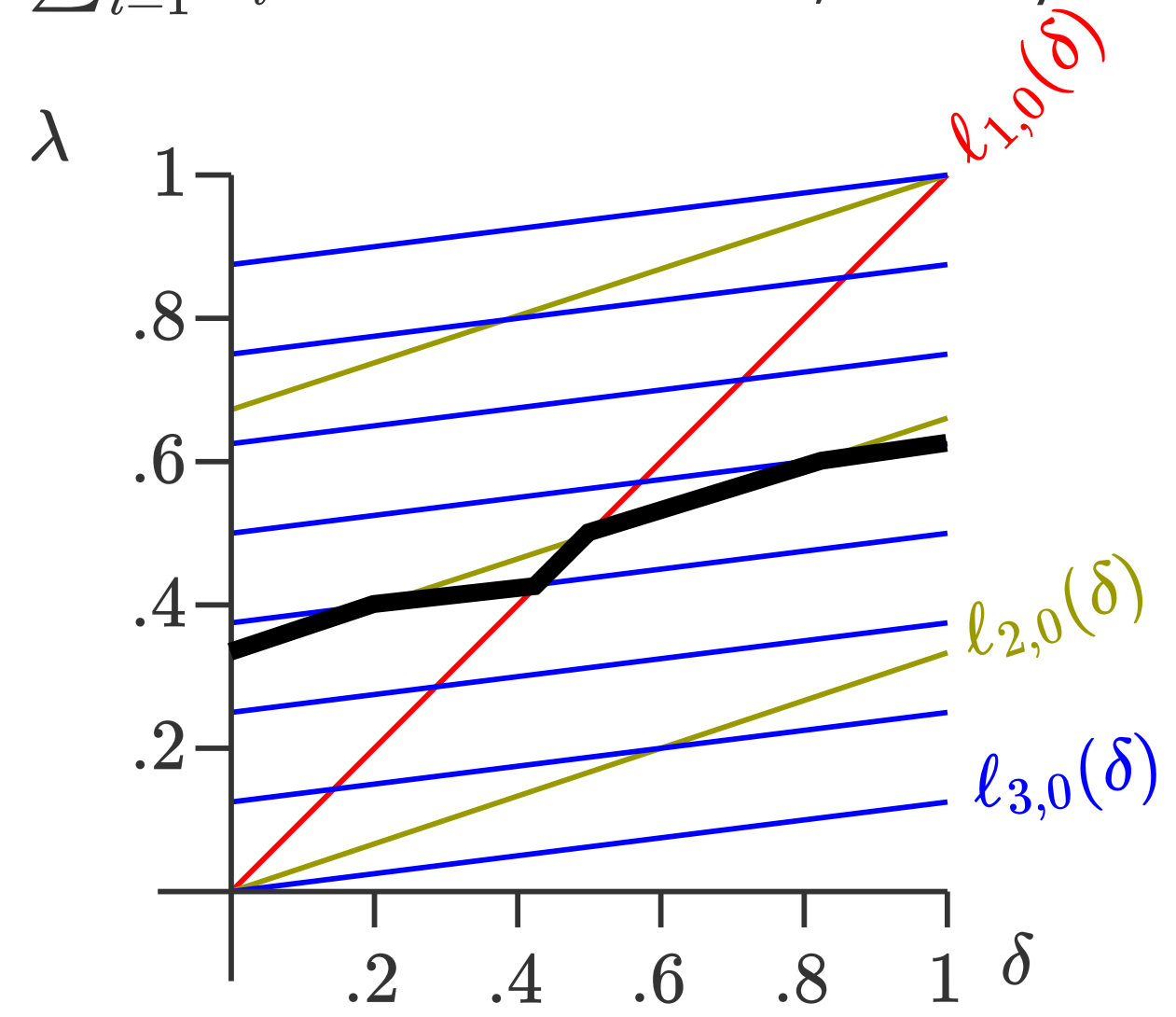
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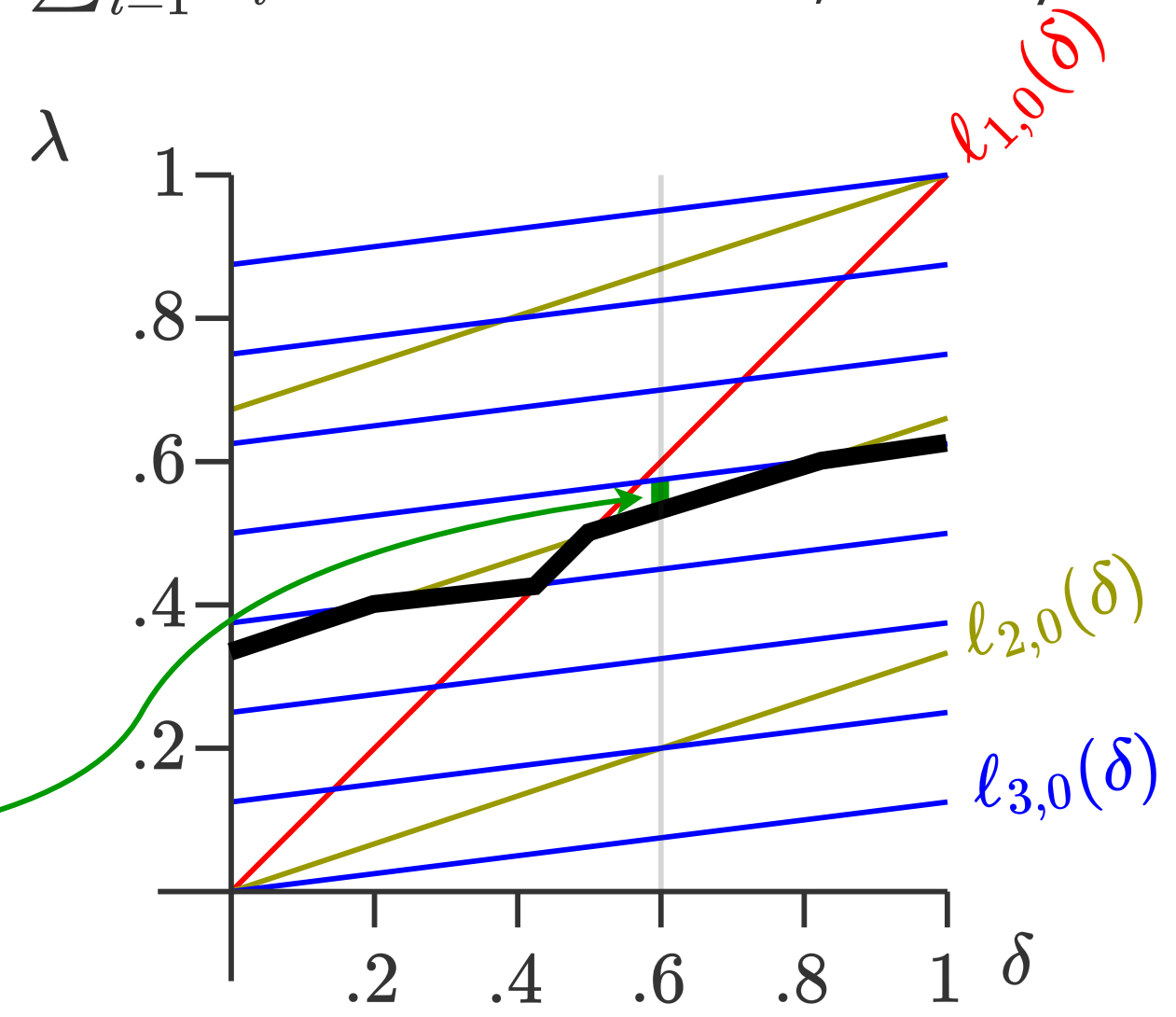
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Generically, the i index of lines at or below the $H - 1$ uniquely determines the allocation of seats, even if λ is not unique.



Small vs large states

Theorem (Marshall, 2002)

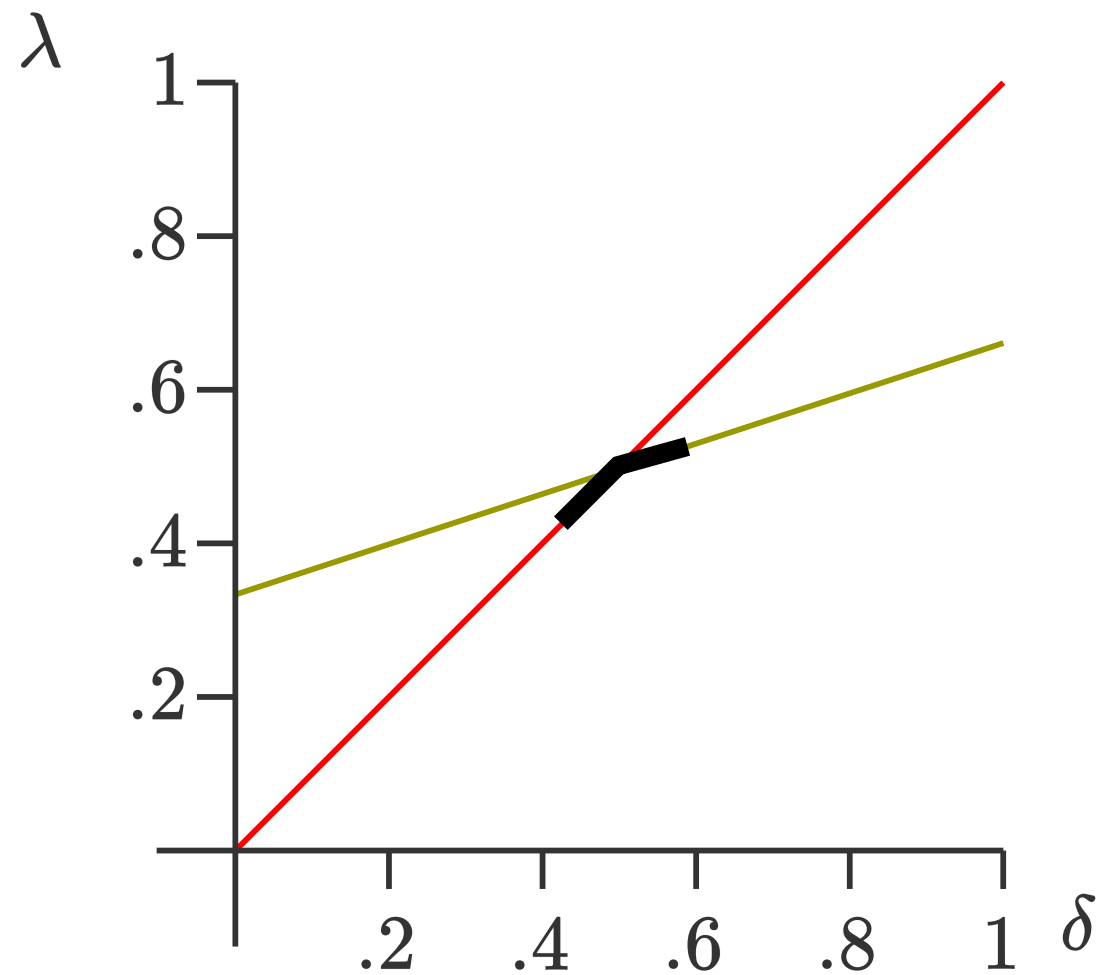
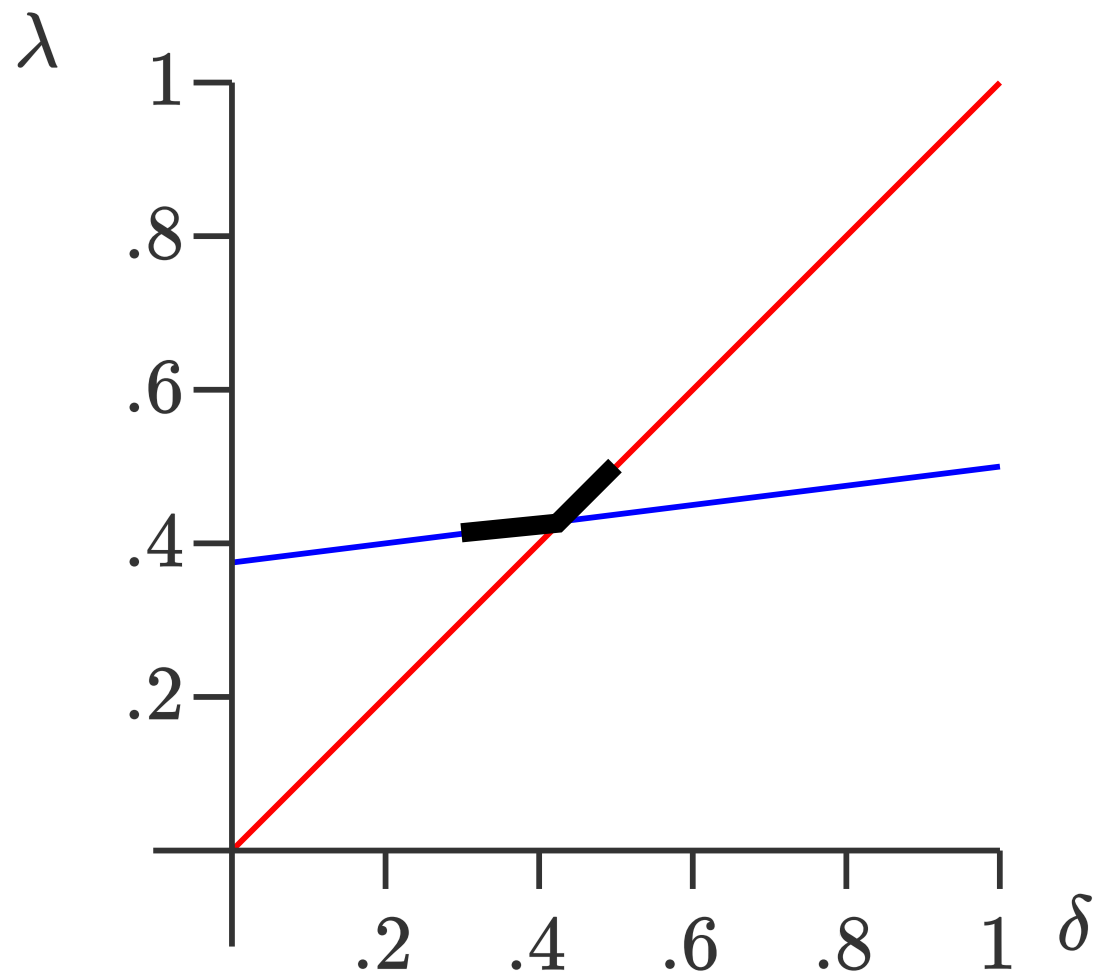
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Proof. As we traverse the $(H - 1)$ -level, two possible things can happen:

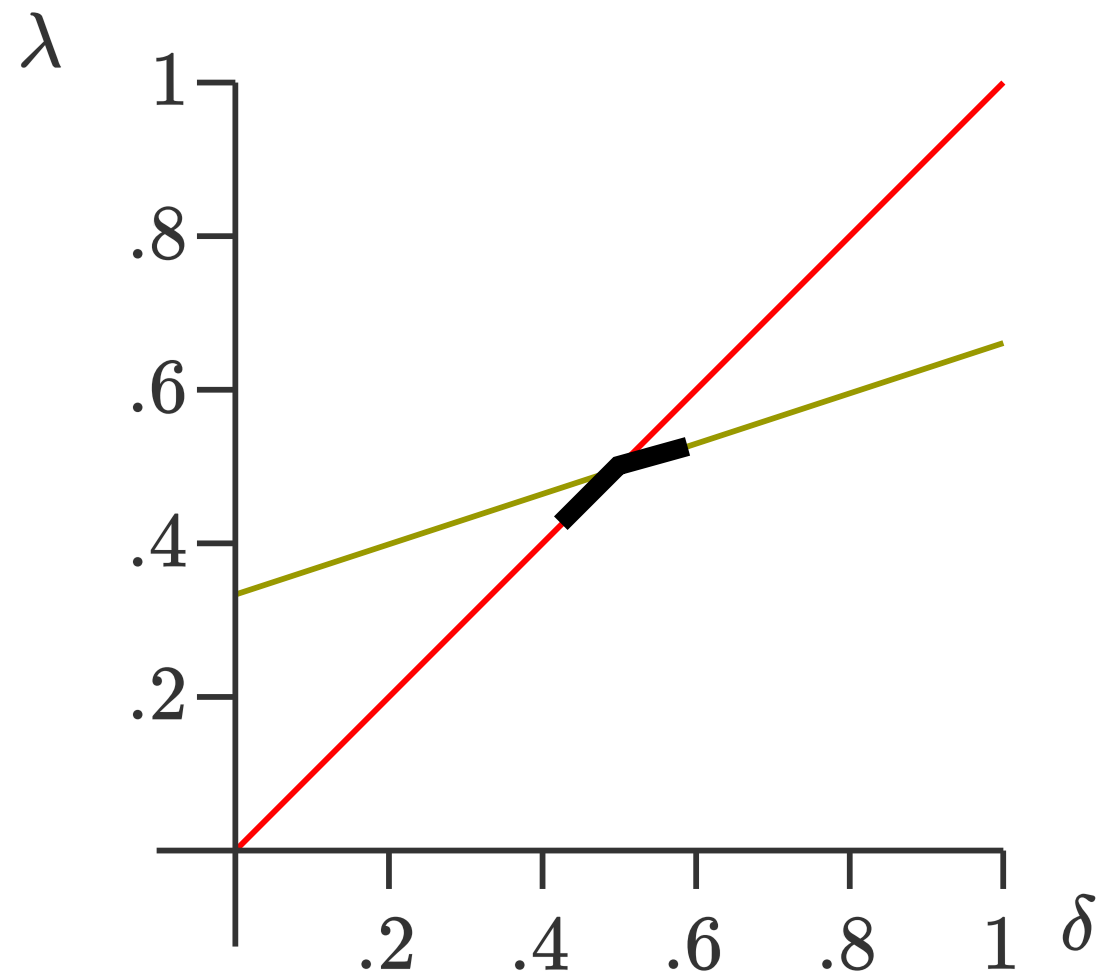
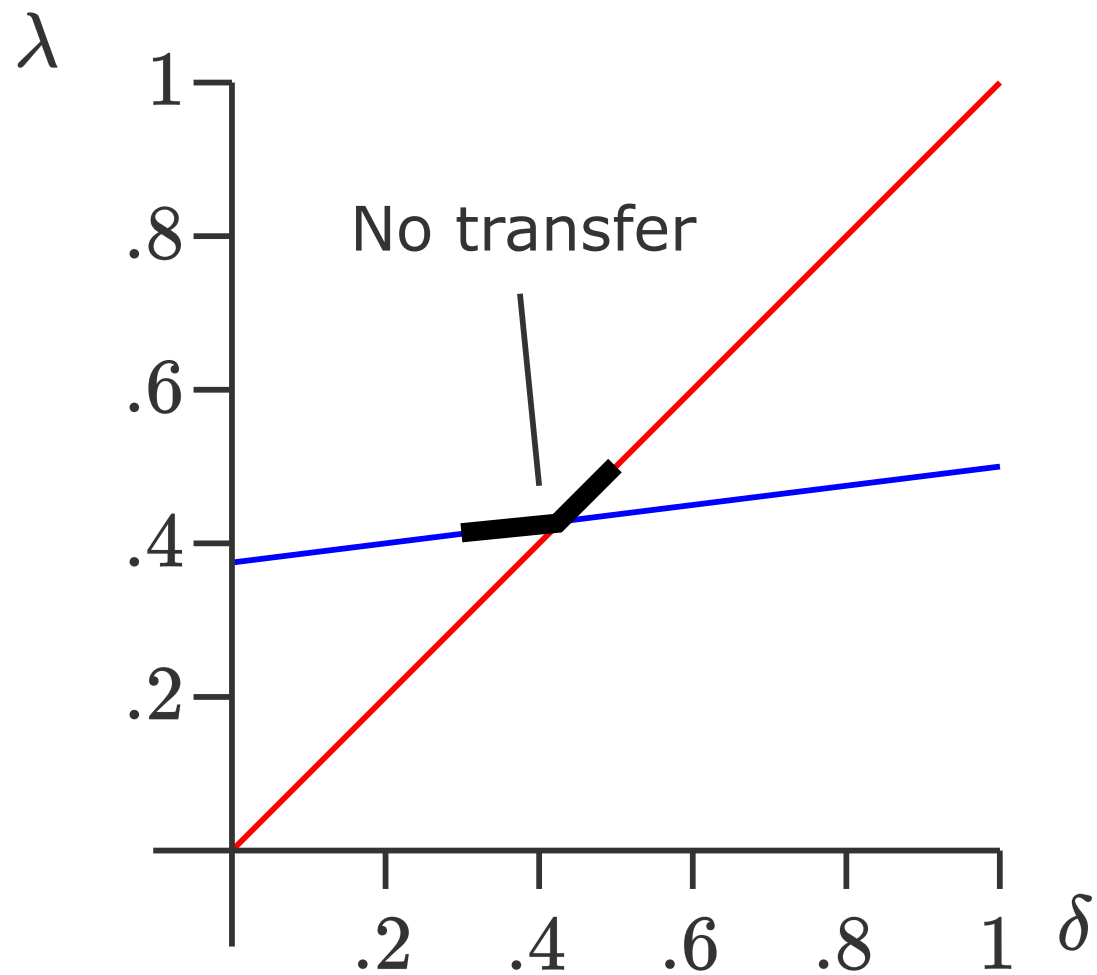


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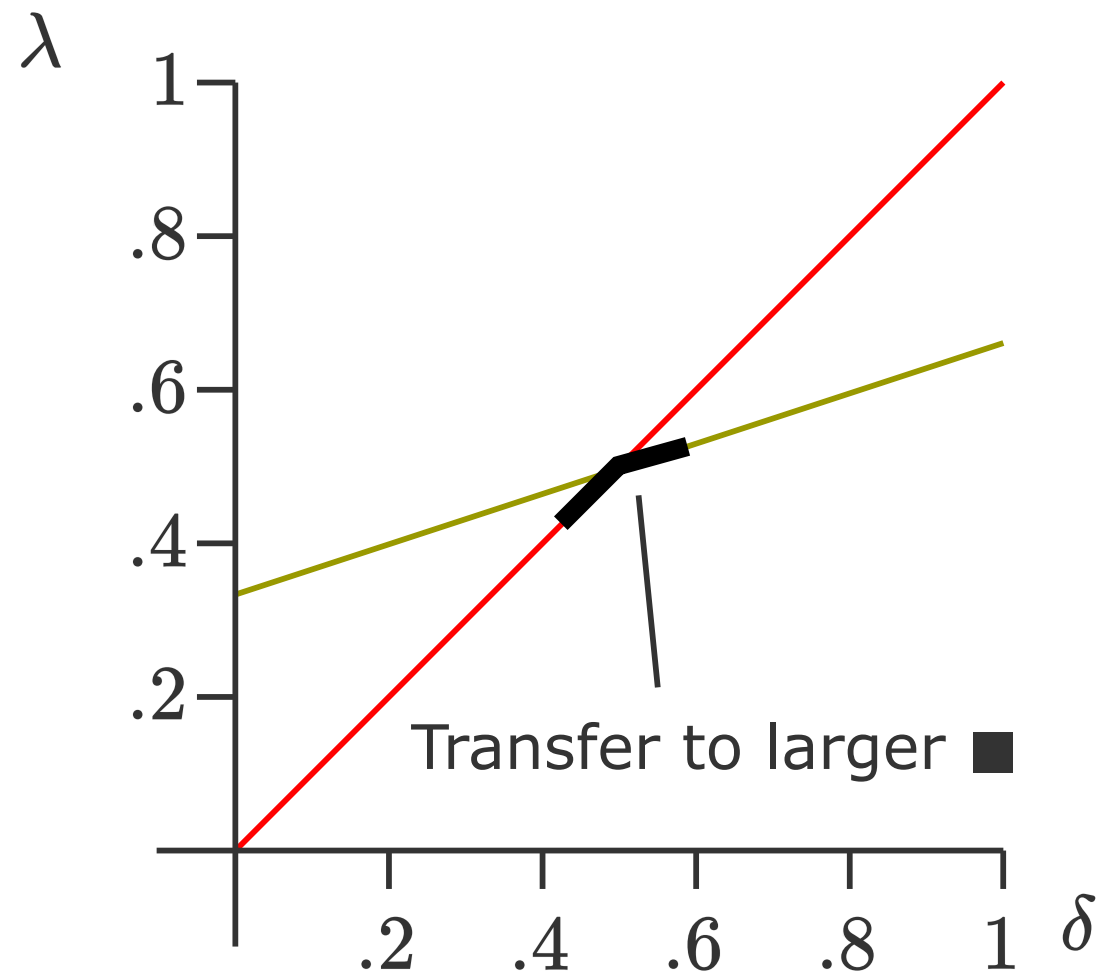
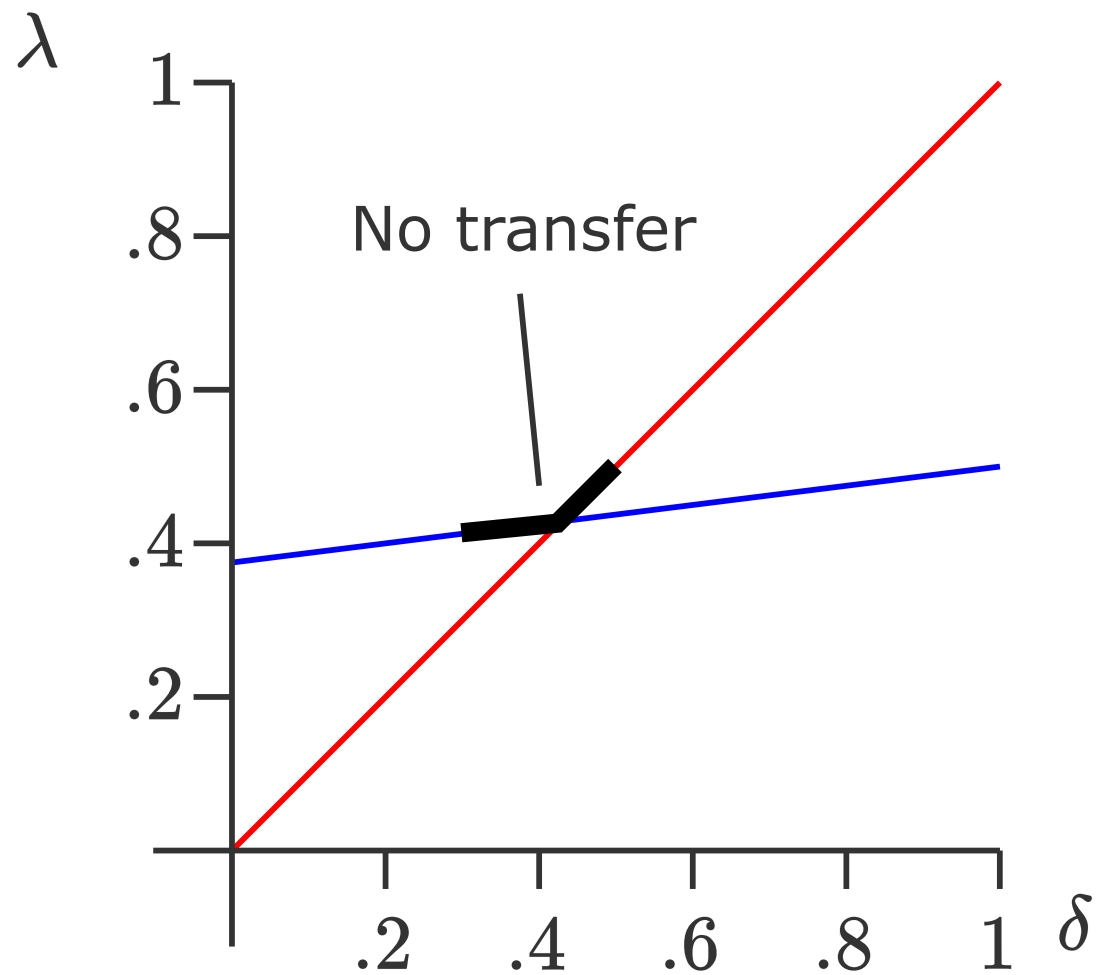


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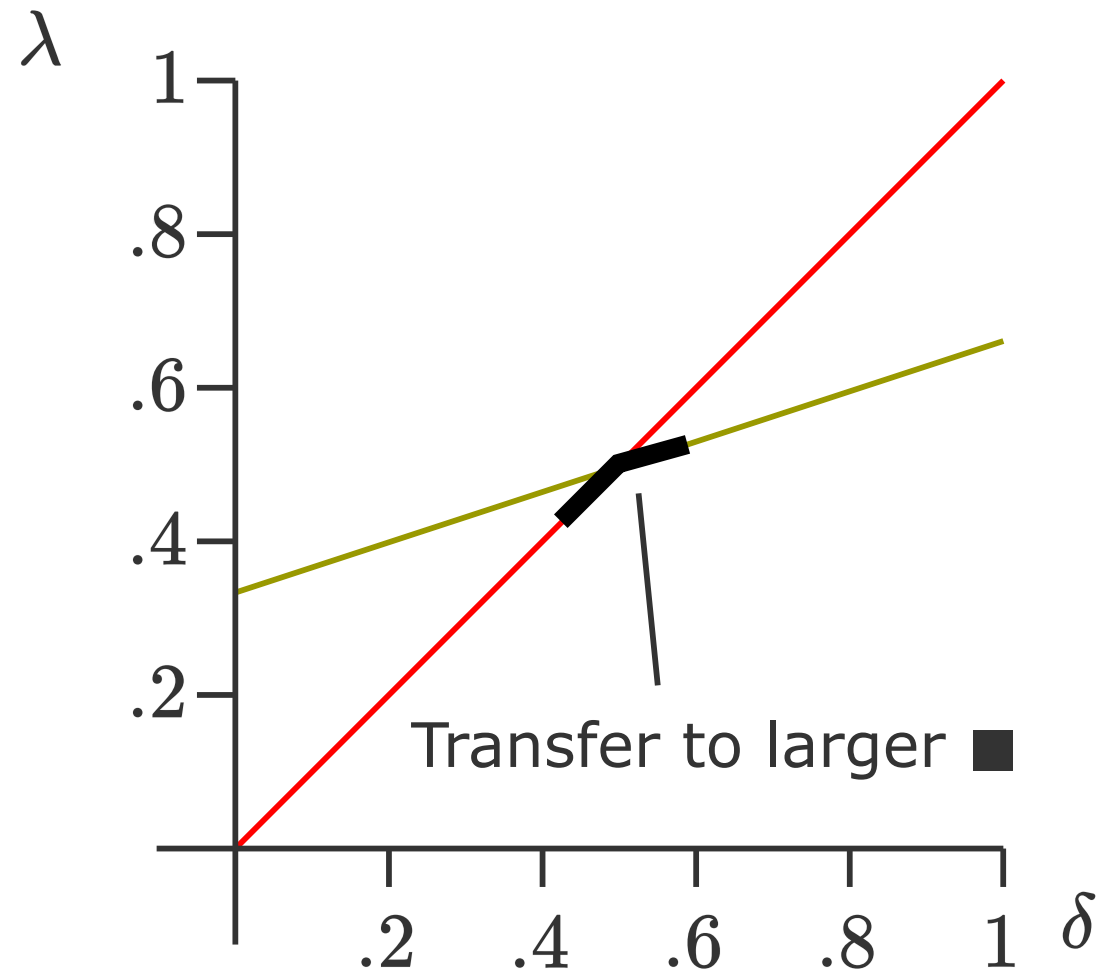
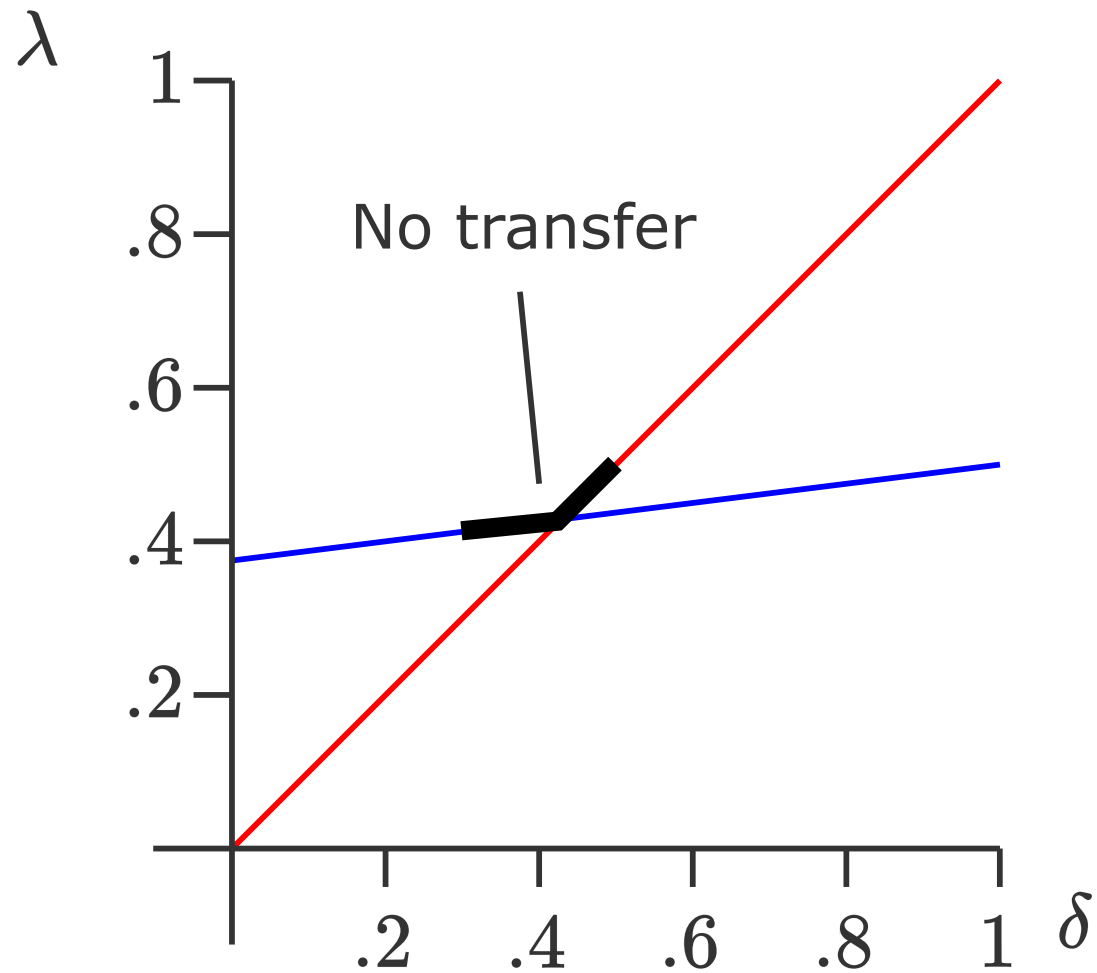


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Theorem (Cembrano et al., 2025)

Whatever the answer to the above question is, the maximum number of possible distinct outputs of stationary divider methods is the same up to constant factors.