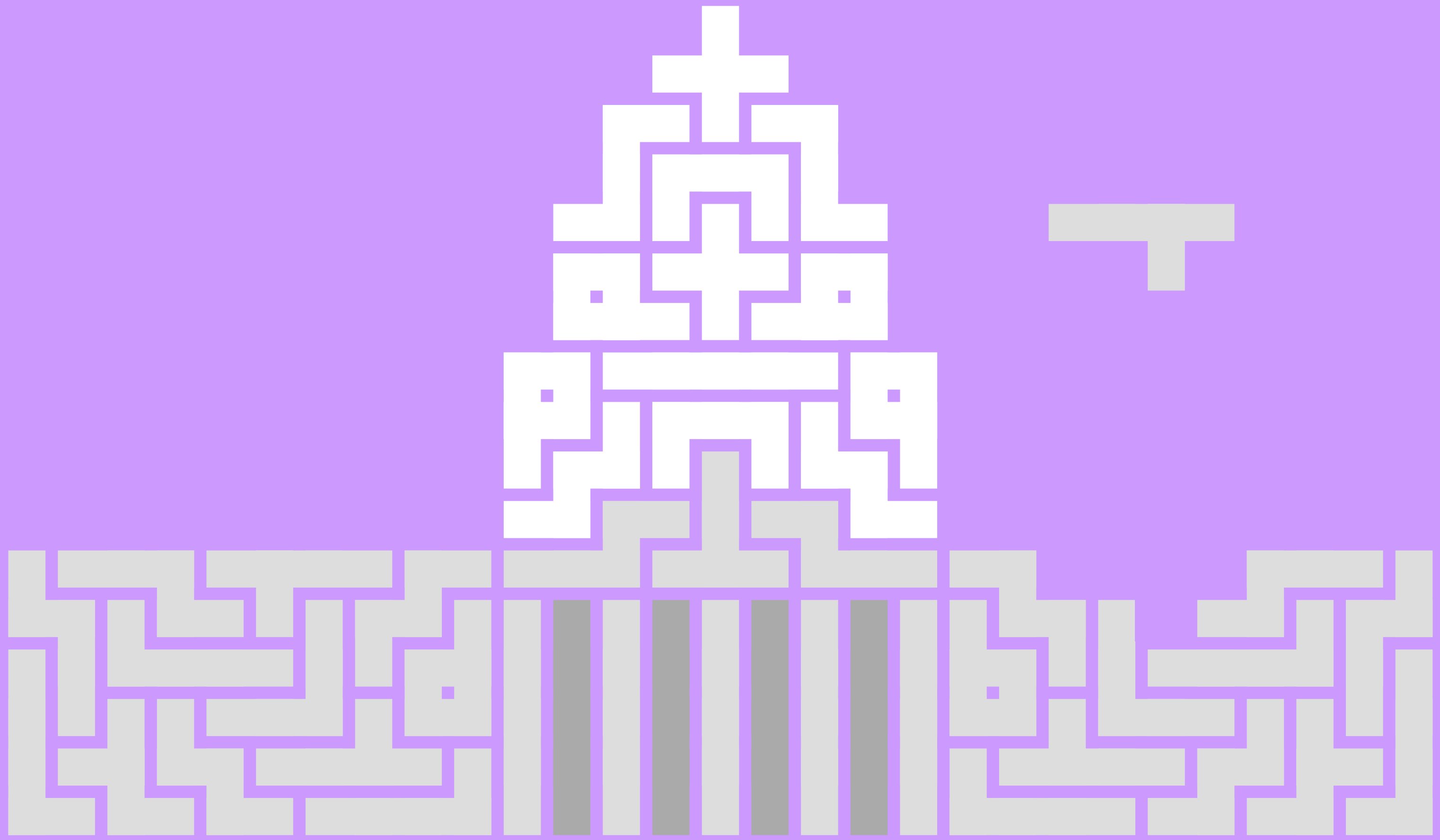

Algorithms For Democratic Decision-Making

Jamie Tucker-Foltz • Yale University • Spring 2026

Lecture 19: **Apportionment 2**

The quota axioms

An apportionment method satisfies

- *Lower quota* if the number of seats awarded to party i is at least $\lfloor q_i \rfloor$
- *Upper quota* if the number of seats awarded to party i is at most $\lceil q_i \rceil$

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▸ Which axioms does Jefferson's method satisfy?

- **Lower quota only**
- **Upper quota only**
- **Both**
- **Neither**

(Recall Jefferson's method finds an apportionment $x \in \mathbb{Z}_{\geq 0}$ and a multiplier λ such that each $x_i = \lfloor \lambda q_i \rfloor$ and $\sum_{i=1}^{n} x_i = H$, the desired house size.)

Respond at:

pollev.com/jtuckerfoltz255 or

bit.ly/jtfpoll or

text jtuckerfoltz255 to 37607

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Note: Adams' method satisfies upper but not lower quota; no divisor method can satisfy both

Population paradox

Hamilton's method satisfies upper and lower quotas but...

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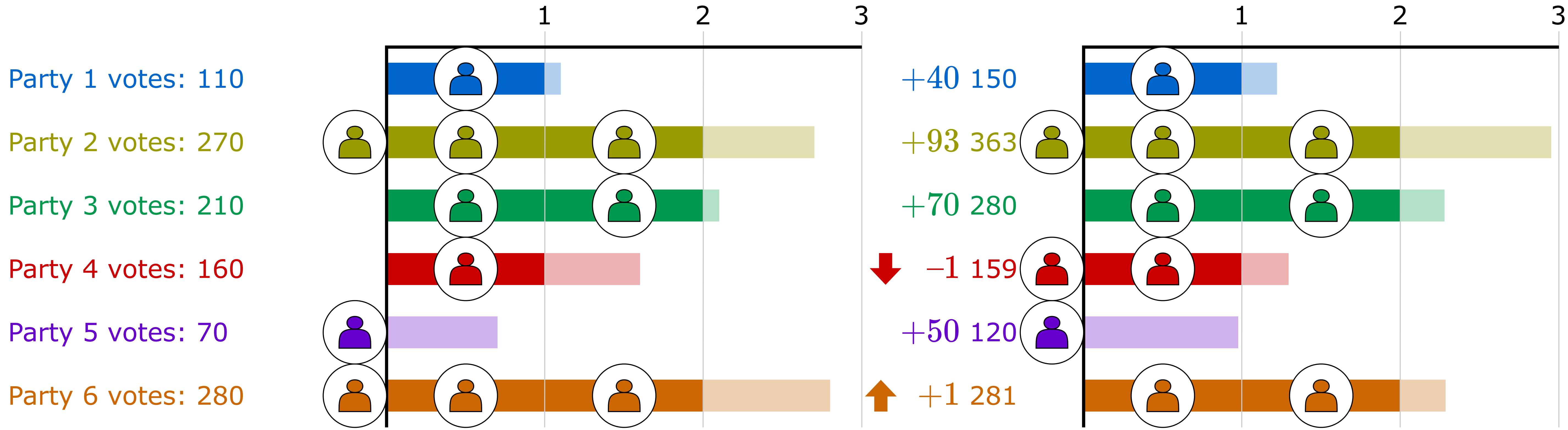

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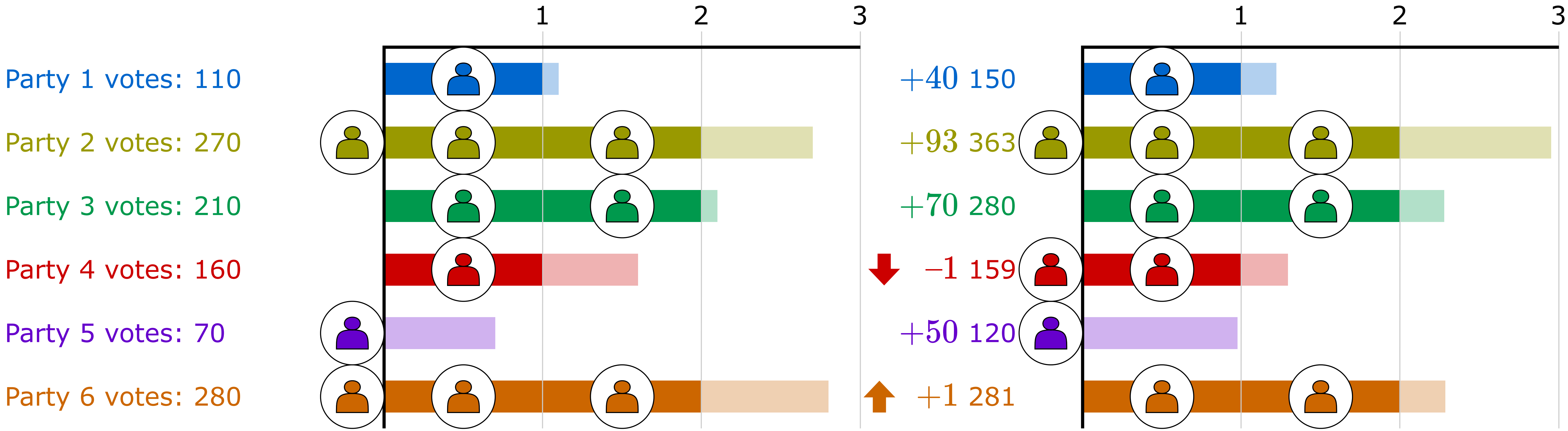
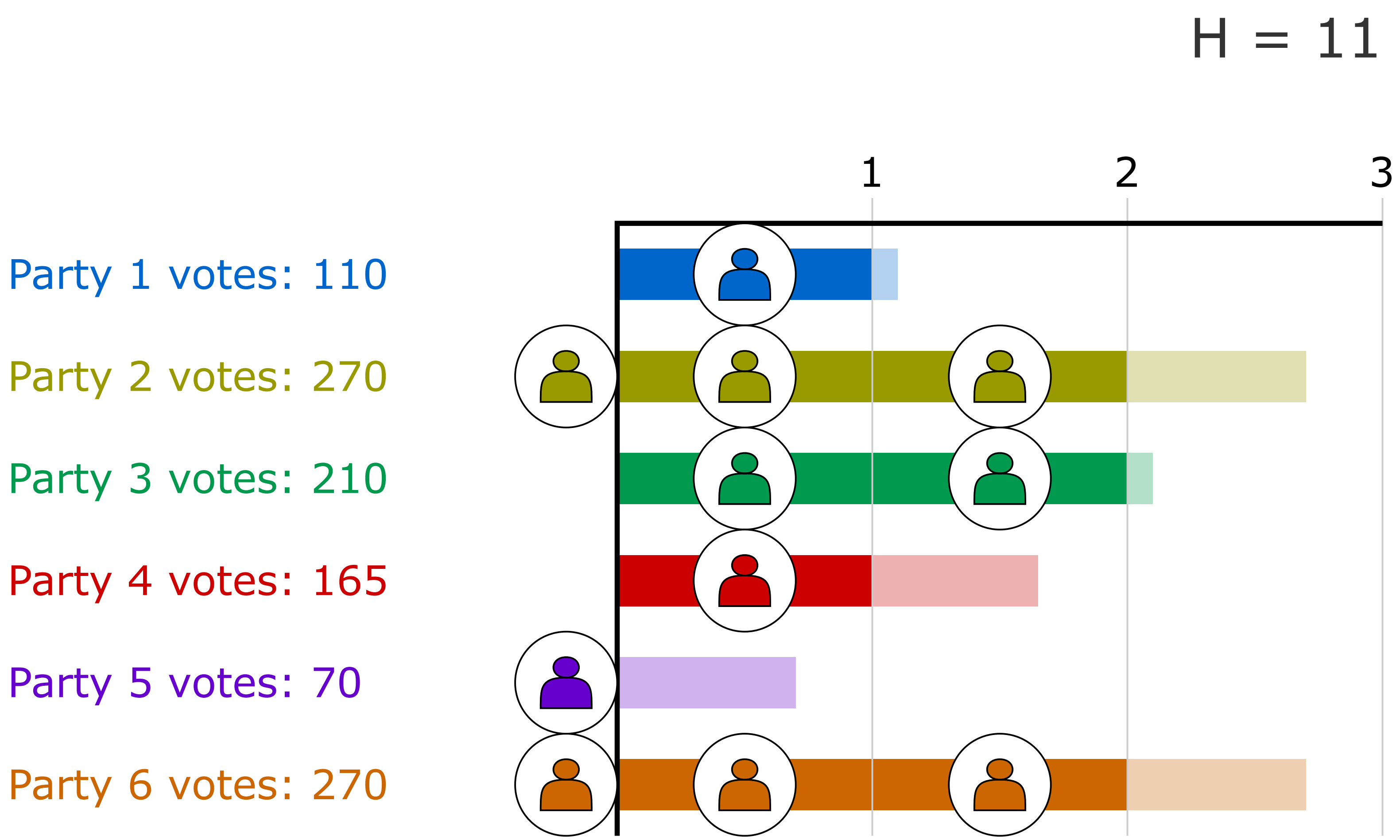


...it fails *population monotonicity*: It should not be the case that state i weakly grows in population and strictly loses seats, while simultaneously, state j weakly shrinks in population and strictly gains seats.

Alabama paradox



Hamilton's method also fails *house monotonicity*: Increasing the house size should simply award one more seat, without taking away seats from any state.

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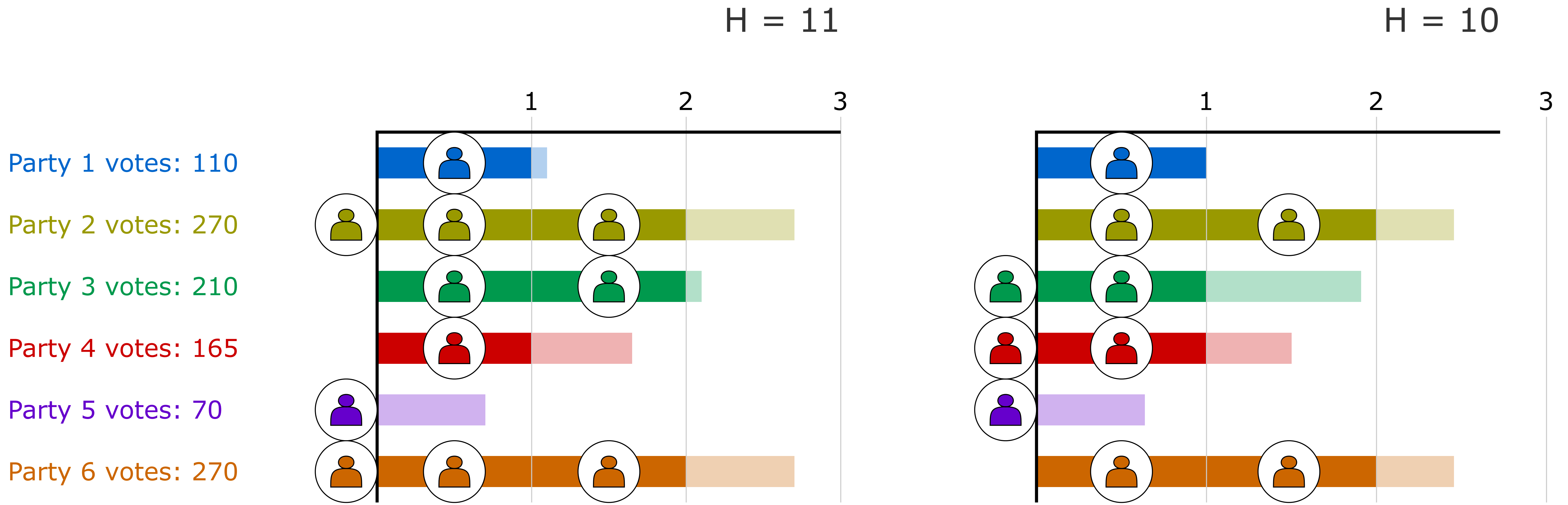

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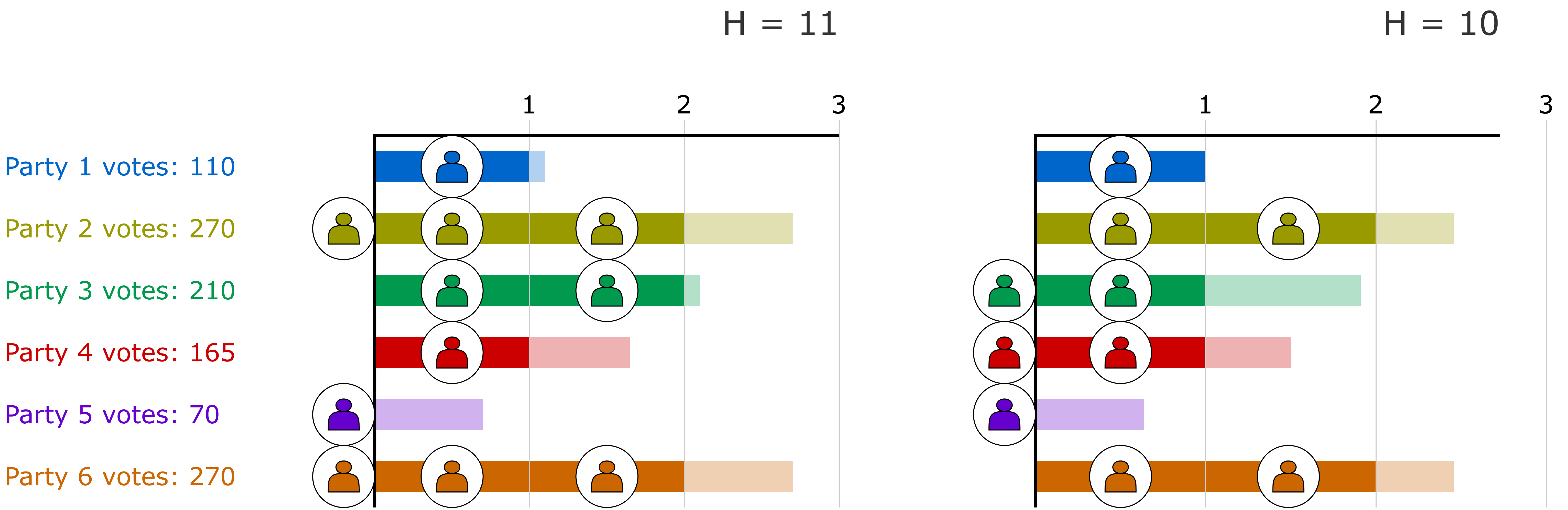

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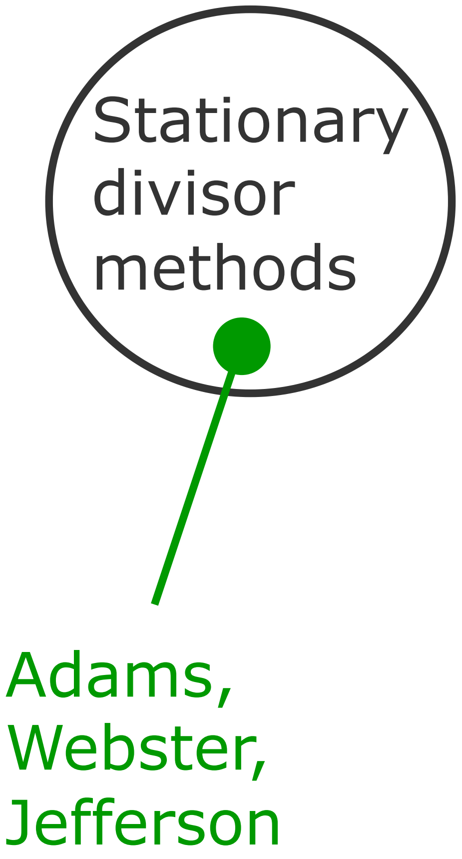
Following the 1880 census, the chief clerk of the census office noticed Alabama would lose a seat in going from 299 seats in the U.S. House to 300.

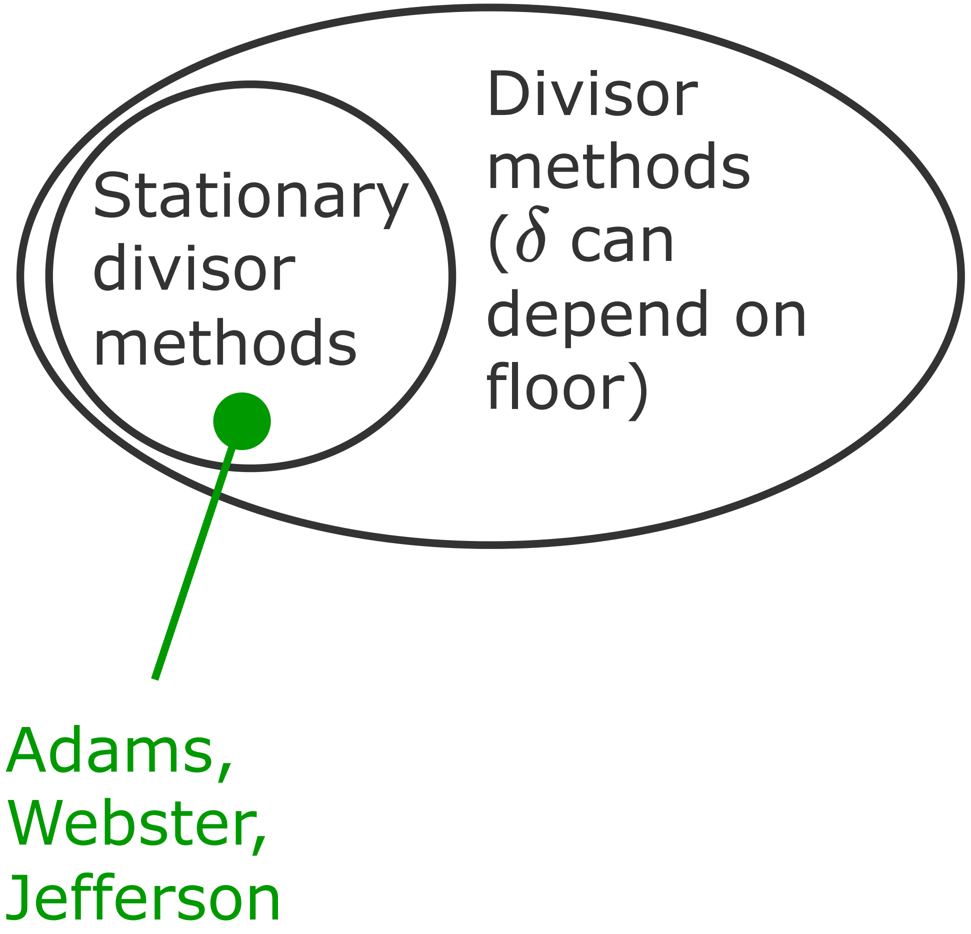
Axioms for deterministic apportionment

Stationary
divisor
methods

Axioms for deterministic apportionment



Axioms for deterministic apportionment



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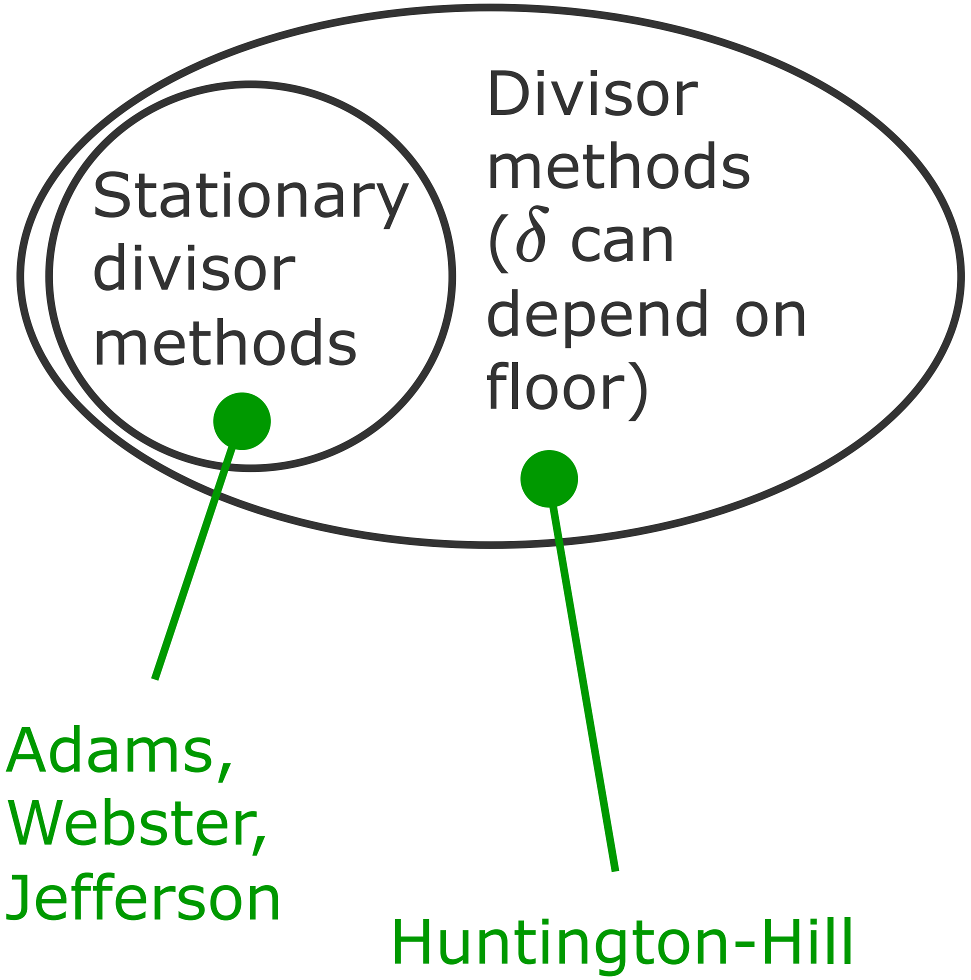

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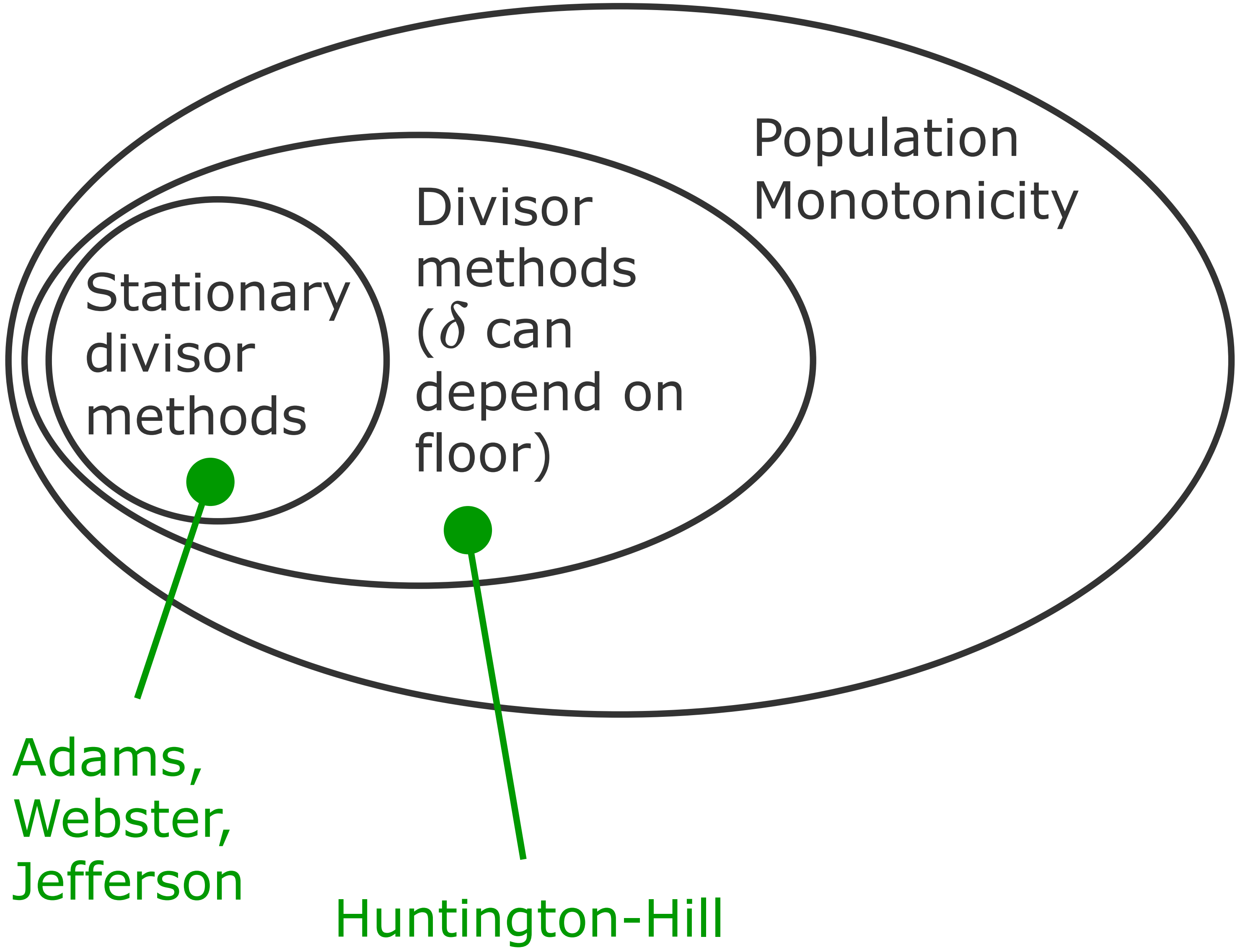

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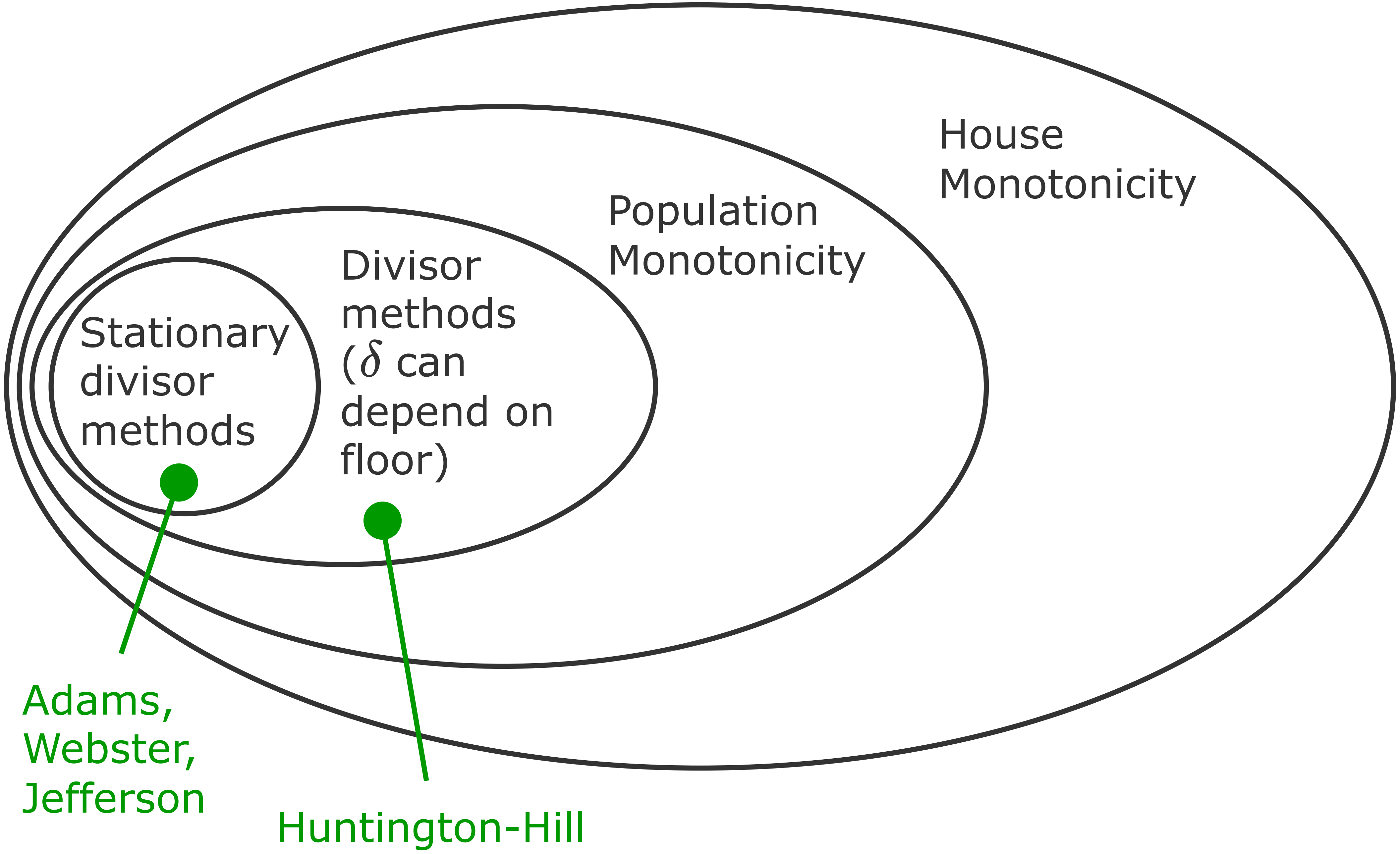

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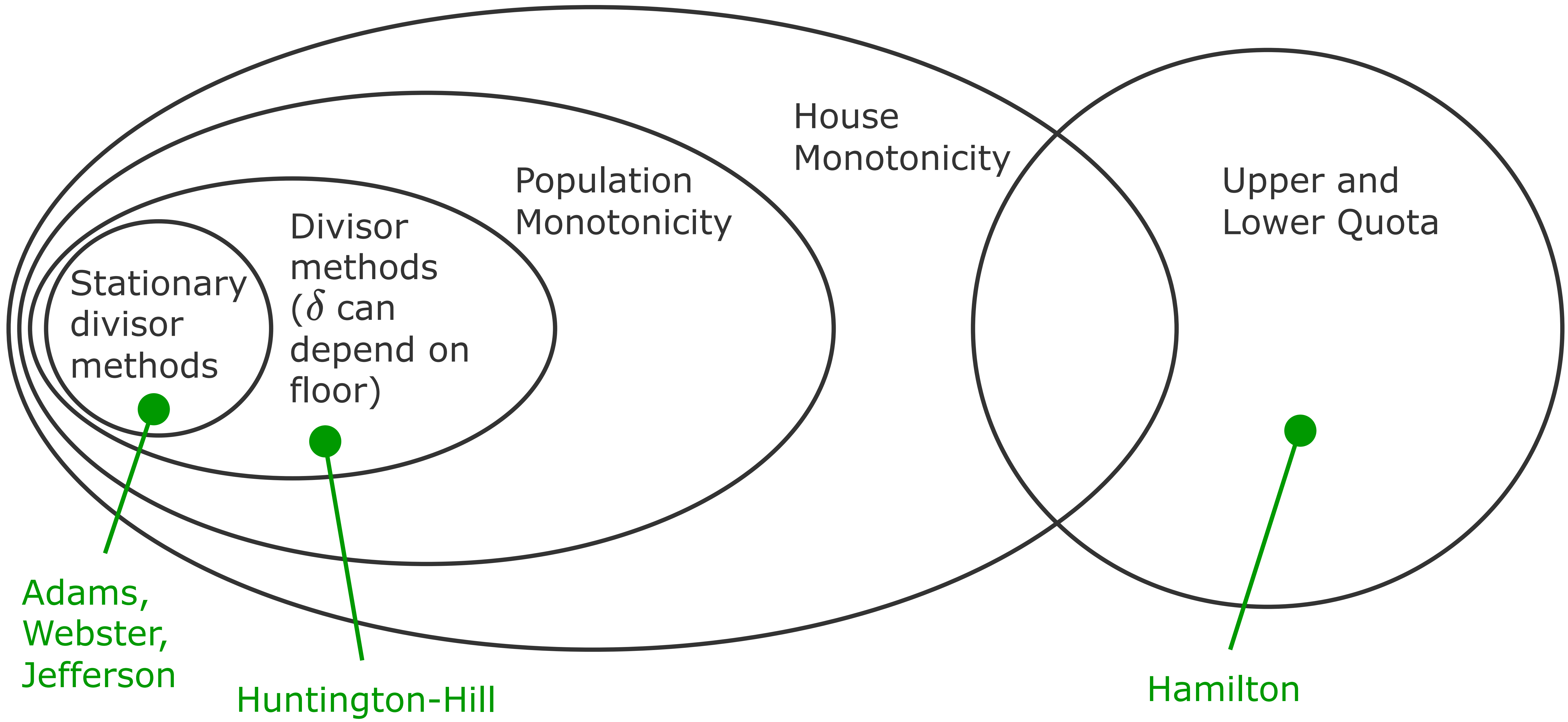

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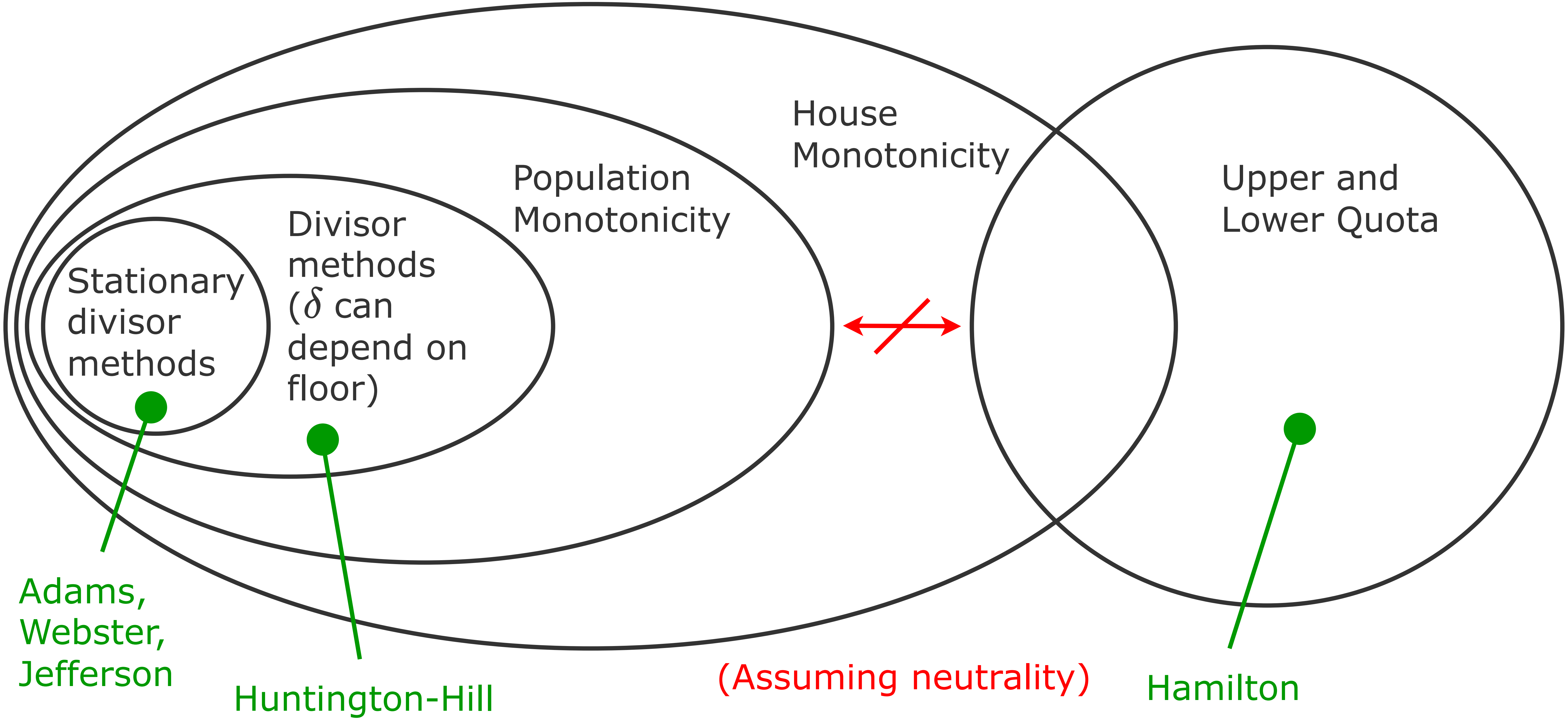

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So either i gains or j loses; in other words, we do not have both states changing in the non-monotone way. ■

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Let's prove the contrapositive: A house monotonicity violation, for some state i implies a population monotonicity violation for some pair (i, j).

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Thus, we have a population monotonicity violation.

Population monotonicity is incompatible with upper/lower quotas

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Lemma

Under an apportionment rule satisfying neutrality and population monotonicity, for any pair of parties i *and* j*,* $p_i > p_j \implies x_i \geq x_j$*.*

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Now consider the following pair of instances, with $H = 10$:

	p_i	q_i	x_i
State 1	699	6.99	?
State 2	52	0.52	?
State 3	50	0.50	?
State 4	199	1.99	?

	p_i	q_i	x_i
State 1	680	8.02	?
State 2	55	0.65	?
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State 2	52	0.52	?	} ≥ 1 (States 2 and 3 together)
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(States 2–4 together: ≤ 2)

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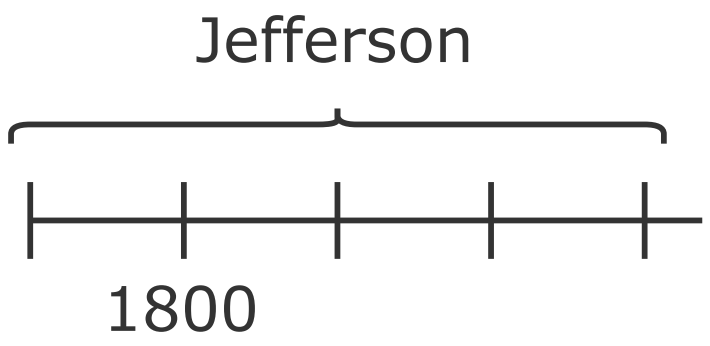
History of apportionment methods

According to law:



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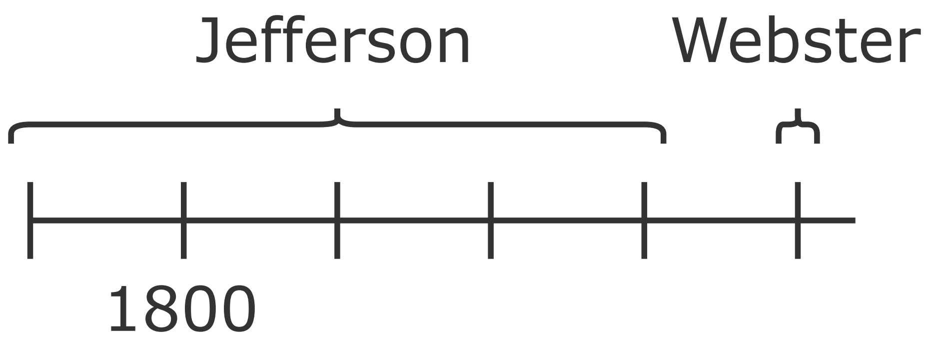

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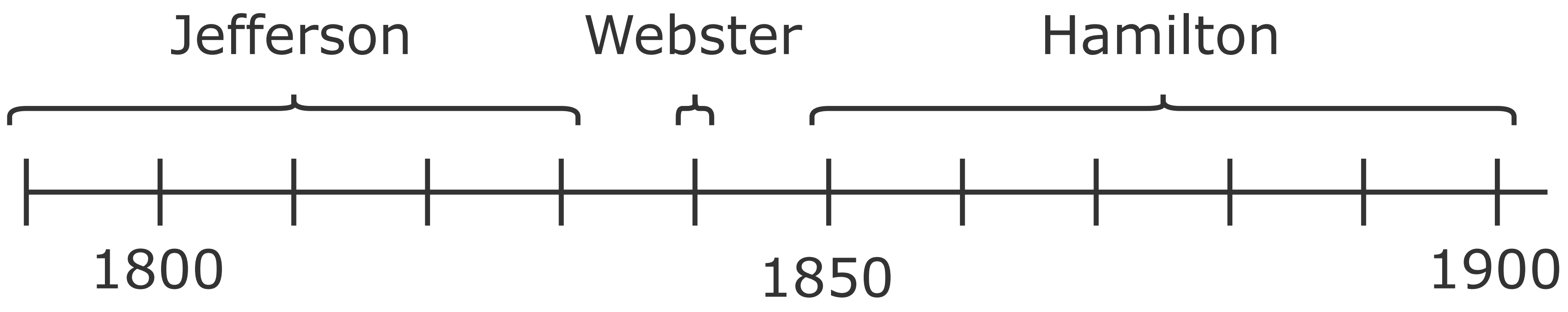

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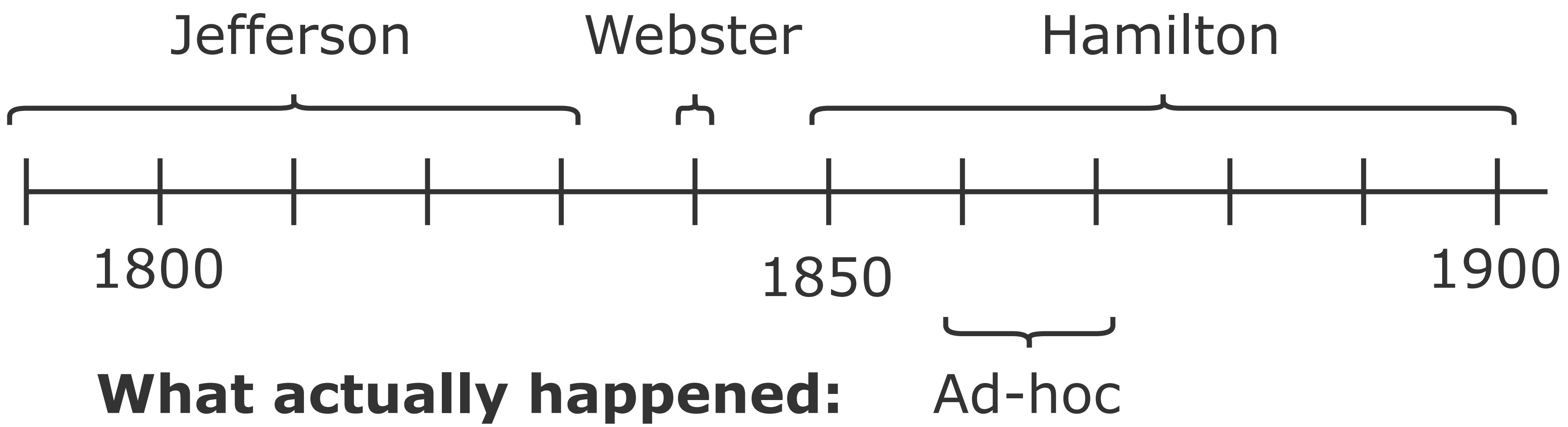

What actually happened:

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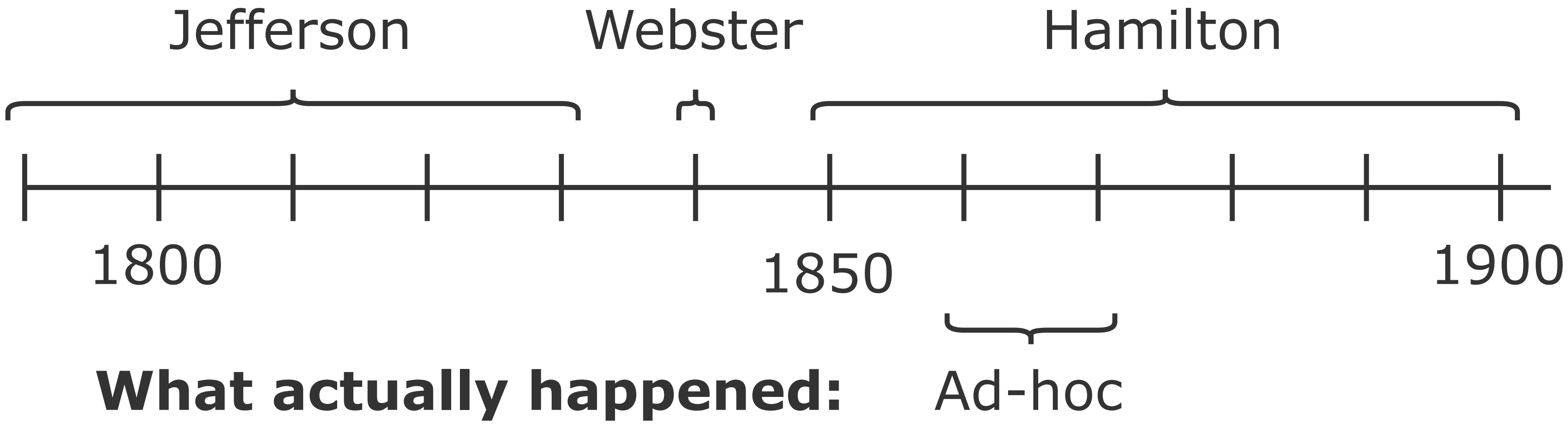

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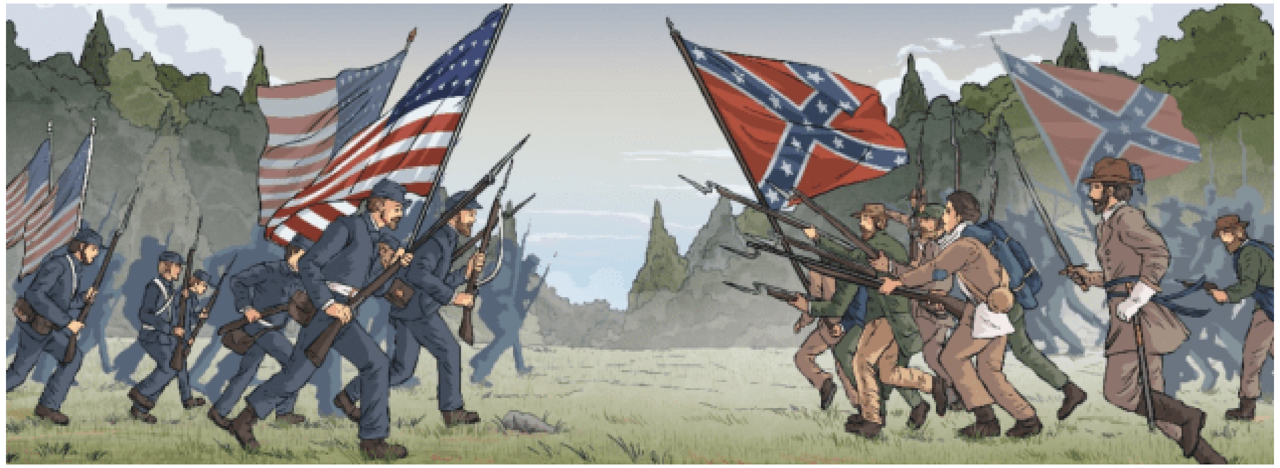

Image credit: https://www.twinkl.fr/teaching-wiki/american-civil-war

History of apportionment methods

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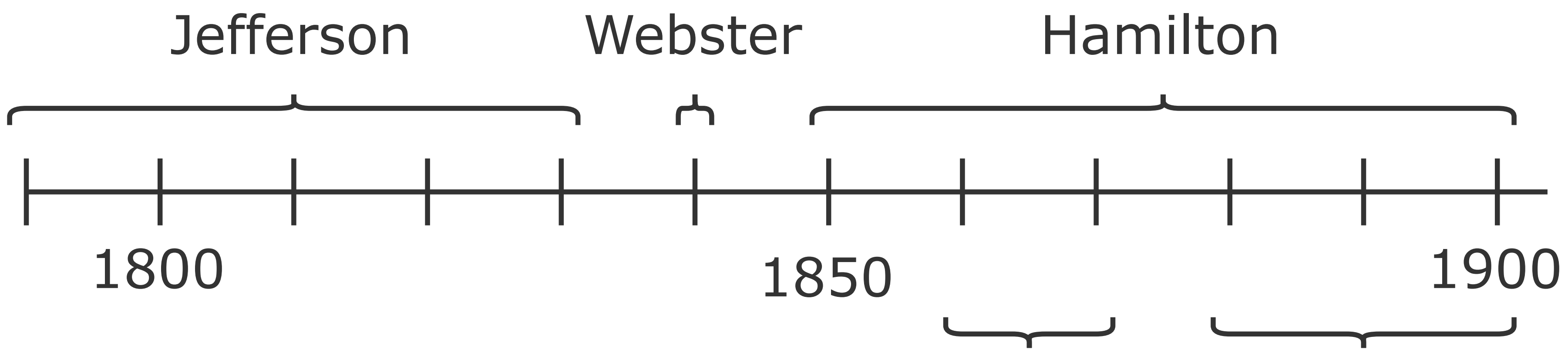

What actually happened: Ad-hoc Hamster

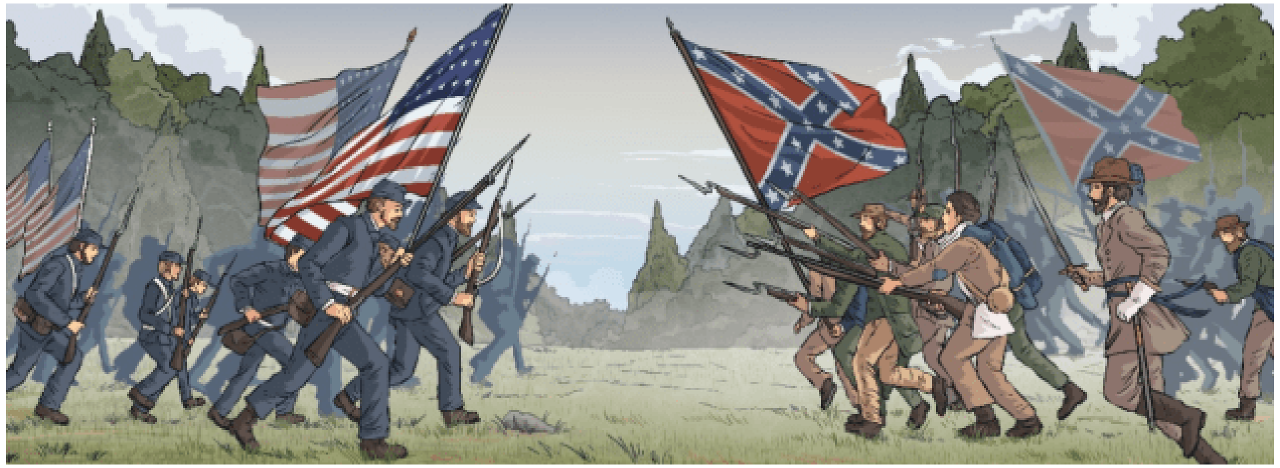

Image credit: https://www.twinkl.fr/teaching-wiki/american-civil-war

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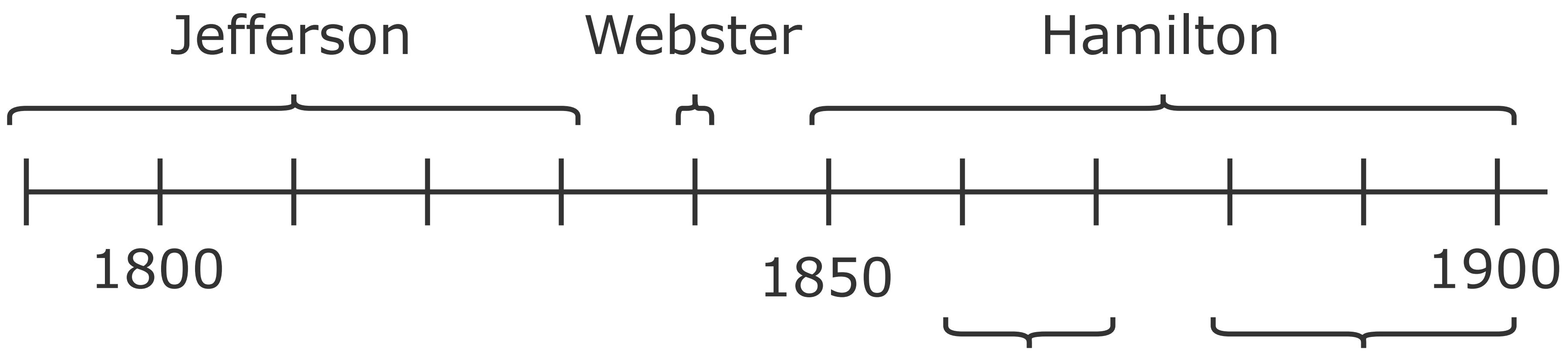

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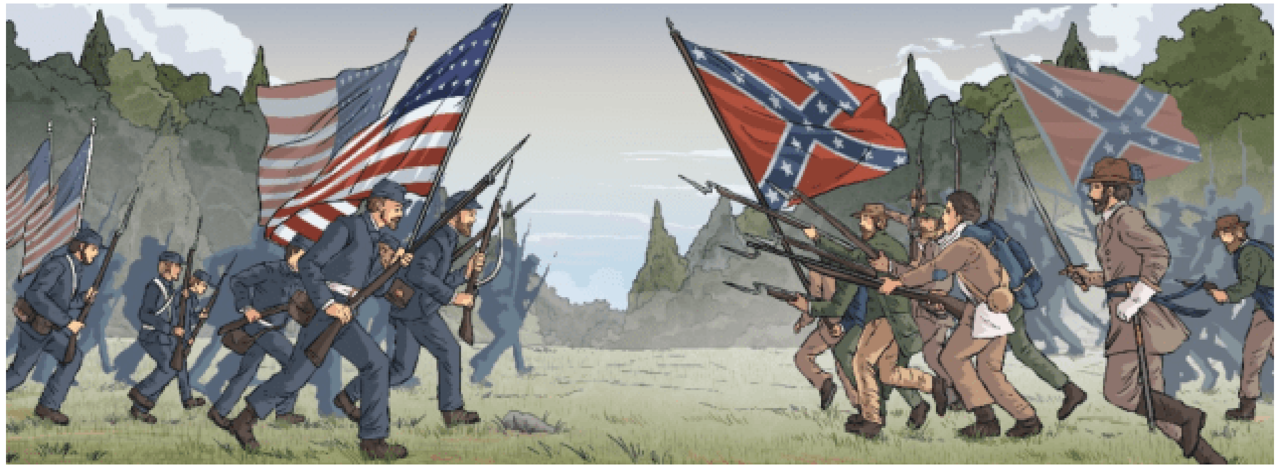

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(Increase house until **Ham**ilton = Web**ster**)

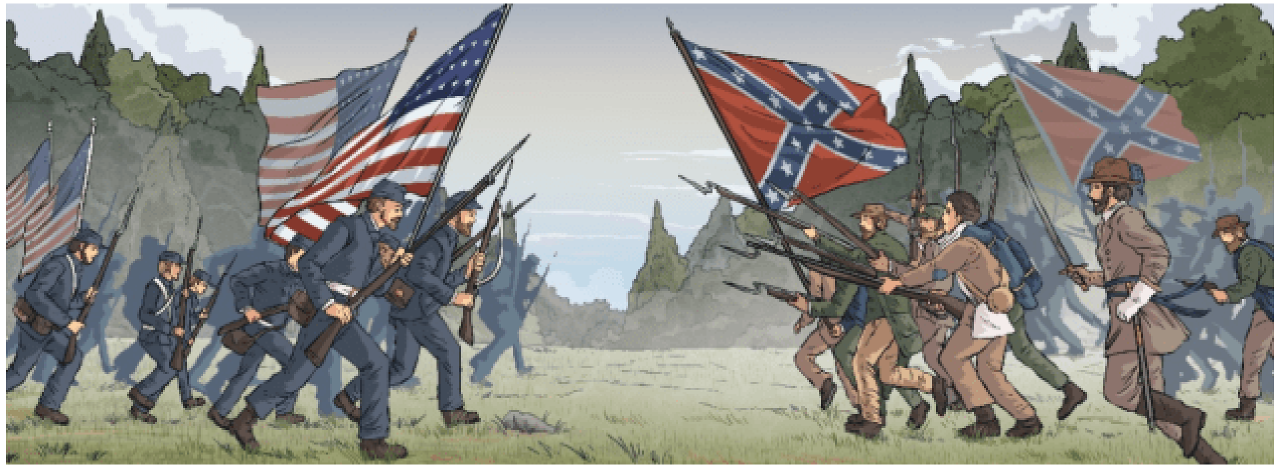
History of apportionment methods

According to law:

Jefferson — Webster — Hamilton — Webster

1800 — 1850 — 1900

What actually happened: Ad-hoc — Hamster



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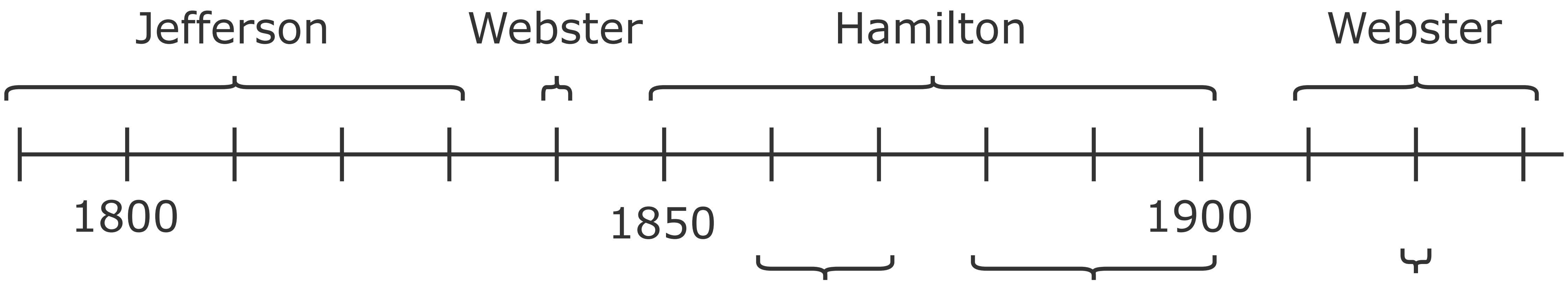
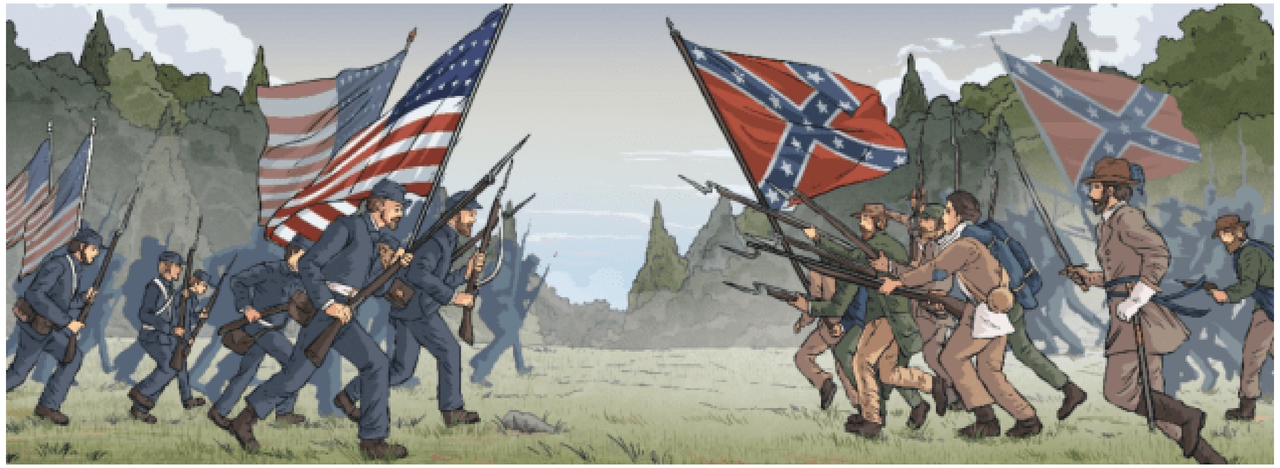


What actually happened: Ad-hoc Hamster Nothing!!



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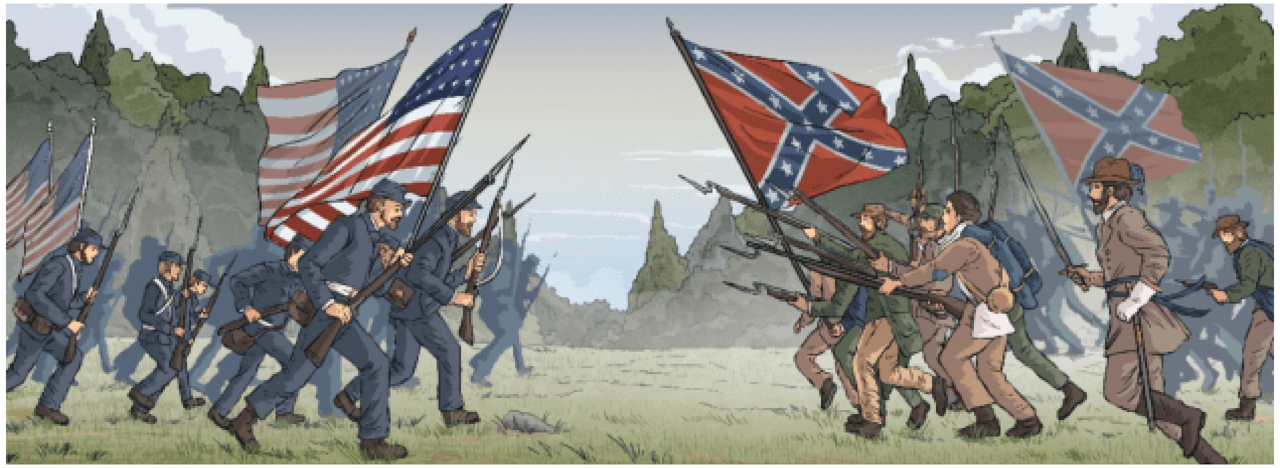
History of apportionment methods

According to law:

Jefferson · Webster · Hamilton · Webster

1800 · 1850 · 1900

What actually happened: Ad-hoc · Hamster · Nothing!!



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(Increase house until **Ham**ilton = Web**ster**)

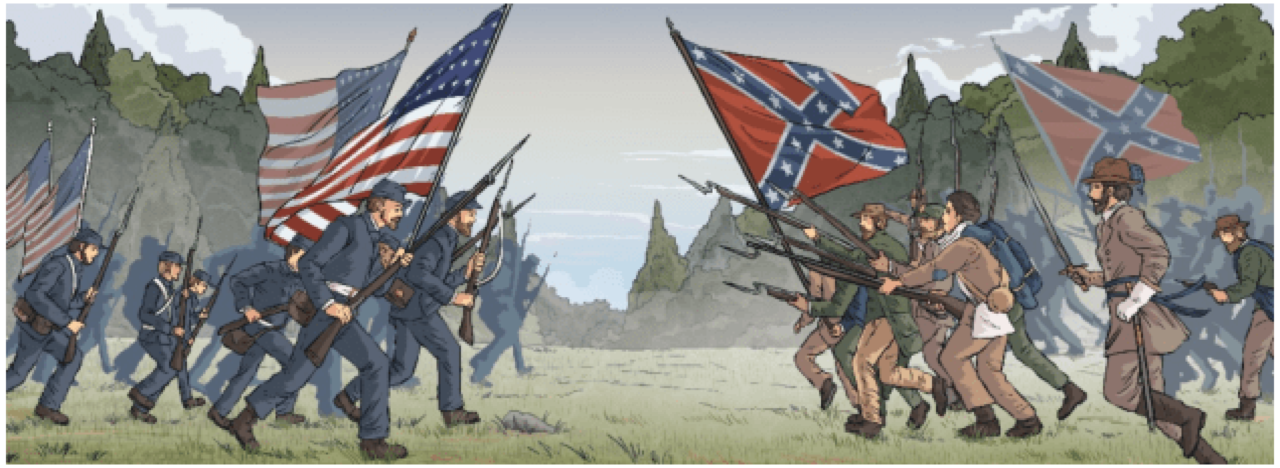
History of apportionment methods

According to law:

Jefferson | Webster | Hamilton | Webster | Huntington-Hill

1800 | 1850 | 1900 | ... | 2020

What actually happened: Ad-hoc | Hamster | Nothing!!



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(Increase house until **Ham**ilton = Web**ster**)

History of apportionment methods

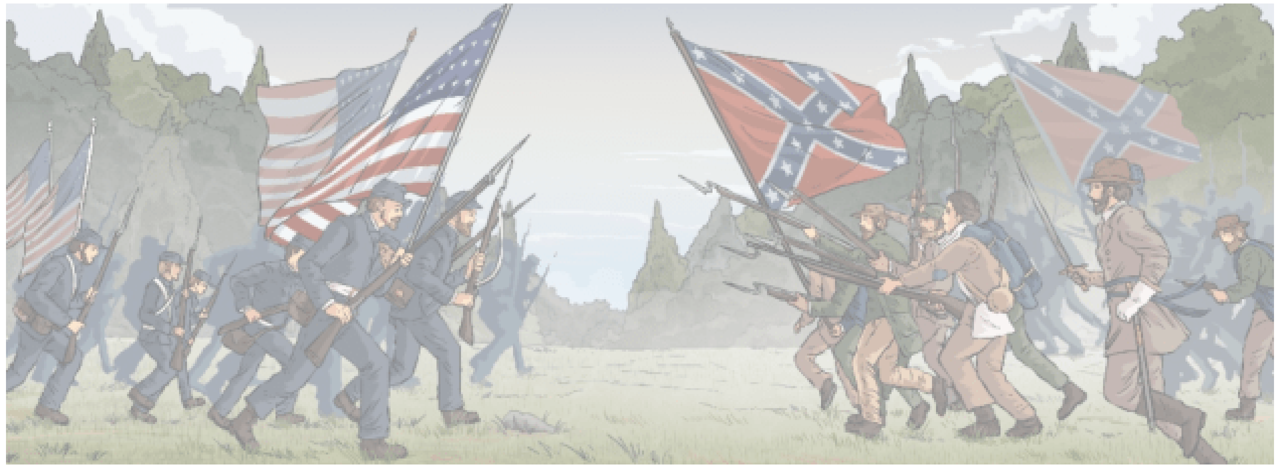
According to law:

When will the debate be re-opened??

Jefferson | Webster | Hamilton | Webster | Huntington-Hill

1800 | 1850 | 1900 | … | 2020 | … | ????

What actually happened: Ad-hoc | Hamster | Nothing!!



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(Increase house until **Ham**ilton = Web**ster**)

The Huntington-Hill method

Find an apportionment $x \in \mathbb{Z}_{\geq 0}$ summing to the house size H and a multiplier λ such that, for each $i \in [n]$,

$$x_i = \begin{cases} \lfloor \lambda q_i \rfloor & \text{if } \lambda q_i < \sqrt{\lfloor \lambda q_i \rfloor \cdot \lceil \lambda q_i \rceil} \\ \lceil \lambda q_i \rceil & \text{if } \lambda q_i \geq \sqrt{\lfloor \lambda q_i \rfloor \cdot \lceil \lambda q_i \rceil} \end{cases}.$$

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This is a *non-stationary divisor method*, i.e., a δ-divisor method where we pick a different value of δ depending on the lower quotas:

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This is a *non-stationary divisor method*, i.e., a δ-divisor method where we pick a different value of δ depending on the lower quotas:

- For rounding modified quotas between 0 and 1, use $\delta = 0$ (like Adams' method)

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- For rounding modified quotas between 0 and 1, use $\delta = 0$ (like Adams' method)
- For rounding modified quotas between 1 and 2, use $\delta = 0.41$

The Huntington-Hill method

Find an apportionment $x \in \mathbb{Z}_{\geq 0}$ summing to the house size H and a multiplier λ such that, for each $i \in [n]$,

$$x_i = \begin{cases} \lfloor \lambda q_i \rfloor & \text{if } \lambda q_i < \sqrt{\lfloor \lambda q_i \rfloor \cdot \lceil \lambda q_i \rceil} \\ \lceil \lambda q_i \rceil & \text{if } \lambda q_i \geq \sqrt{\lfloor \lambda q_i \rfloor \cdot \lceil \lambda q_i \rceil} \end{cases}.$$

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- For rounding modified quotas between 1 and 2, use $\delta = 0.41$
- For rounding modified quotas between 2 and 3, use $\delta = 0.45$

...

The Huntington-Hill method

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...

Theorem

For any pair of states i and j, under the Huntington-Hill method it is never possible to equalize the ratio of the population-per-seat of state i to the population-per-seat of state j by transferring a seat from i to j or vice versa.

Huntington-Hill is maximally fair

Proof. Let $(x_1, x_2, \ldots, x_n)$ be the apportionment for $(q_1, q_2, \ldots, q_n)$ using multiplier λ.
For each $i \in [n]$, there are two possibilities:

Huntington-Hill is maximally fair

Proof. Let $(x_1, x_2, \dots, x_n)$ be the apportionment for $(q_1, q_2, \dots, q_n)$ using multiplier λ. For each $i \in [n]$, there are two possibilities:

- λq_i was within $[x_i, x_i + 1]$ and rounded down
- λq_i was within $[x_i - 1, x_i]$ and rounded up

Huntington-Hill is maximally fair

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In either case, $\sqrt{(x_i-1)x_i} \leq \lambda q_i < \sqrt{x_i(x_i+1)} \implies \frac{(x_i-1)x_i}{q_i^2} \leq \lambda^2 < \frac{x_i(x_i+1)}{q_i^2}$.

Huntington-Hill is maximally fair

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$$\implies \frac{(x_i-1)x_i}{q_i^2} < \frac{x_j(x_j+1)}{q_j^2} \text{ for all } i, j$$

Huntington-Hill is maximally fair

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$$\implies \frac{(x_i-1)x_i}{q_i^2} < \frac{x_j(x_j+1)}{q_j^2} \text{ for all } i, j$$

Rearranging, we have $$\frac{q_i/x_i}{q_j/x_j} > \frac{q_j/(x_j+1)}{q_i/(x_i-1)}.$$

Huntington-Hill is maximally fair, proof continued

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Huntington-Hill is maximally fair, proof continued

$$\frac{q_i/x_i}{q_j/x_j} > \frac{q_j/(x_j+1)}{q_i/(x_i-1)}$$

Consider an arbitrary pair of states i and j, and suppose the population-per-seat for state i is smaller than for state j, i.e., $q_i/x_i \leq q_j/x_j$. Then:

Huntington-Hill is maximally fair, proof continued

$$\frac{q_i/x_i}{q_j/x_j} > \frac{q_j/(x_j+1)}{q_i/(x_i-1)}$$

Consider an arbitrary pair of states i and j, and suppose the population-per-seat for state i is smaller than for state j, i.e., $q_i/x_i \leq q_j/x_j$. Then:

- Increasing x_i and decreasing x_j will only make the inequality worse.

Huntington-Hill is maximally fair, proof continued

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Consider an arbitrary pair of states i and j, and suppose the population-per-seat for state i is smaller than for state j, i.e., $q_i/x_i \le q_j/x_j$. Then:

- Increasing x_i and decreasing x_j will only make the inequality worse.
- Decreasing x_i and increasing x_j by one will make the inequality worse in the opposite direction.

Huntington-Hill is maximally fair, proof continued

$$\frac{q_i/x_i}{q_j/x_j} > \frac{q_j/(x_j+1)}{q_i/(x_i-1)}$$

Consider an arbitrary pair of states i and j, and suppose the population-per-seat for state i is smaller than for state j, i.e., $q_i/x_i \leq q_j/x_j$. Then:

- Increasing x_i and decreasing x_j will only make the inequality worse.
- Decreasing x_i and increasing x_j by one will make the inequality worse in the opposite direction.
- Decreasing x_i and increasing x_j further will only exacerbate the inequality. ■

Grimmett's method - randomize!

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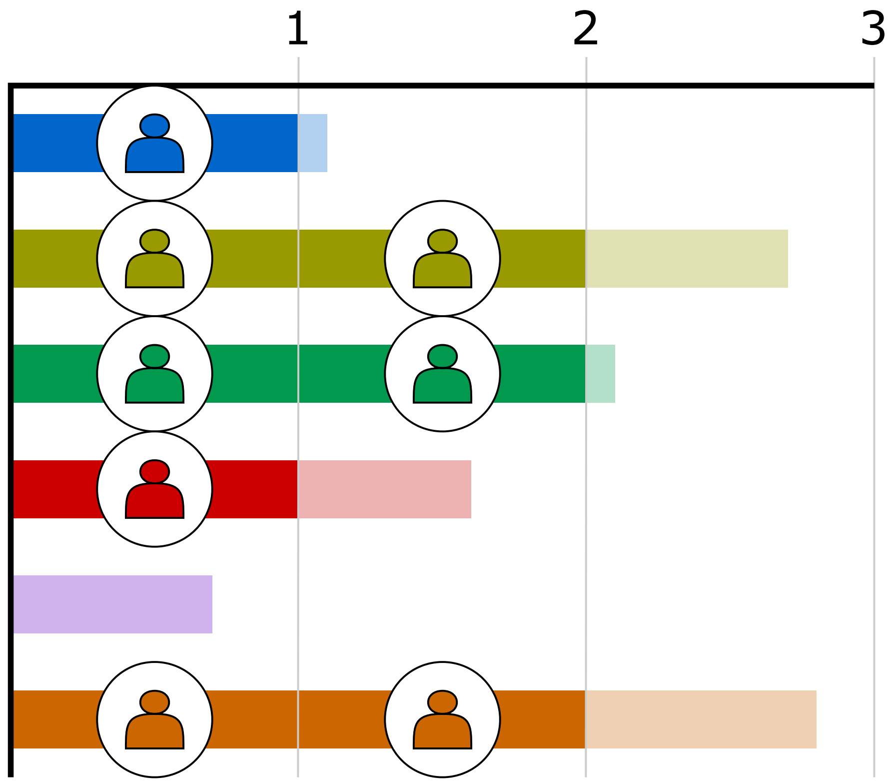

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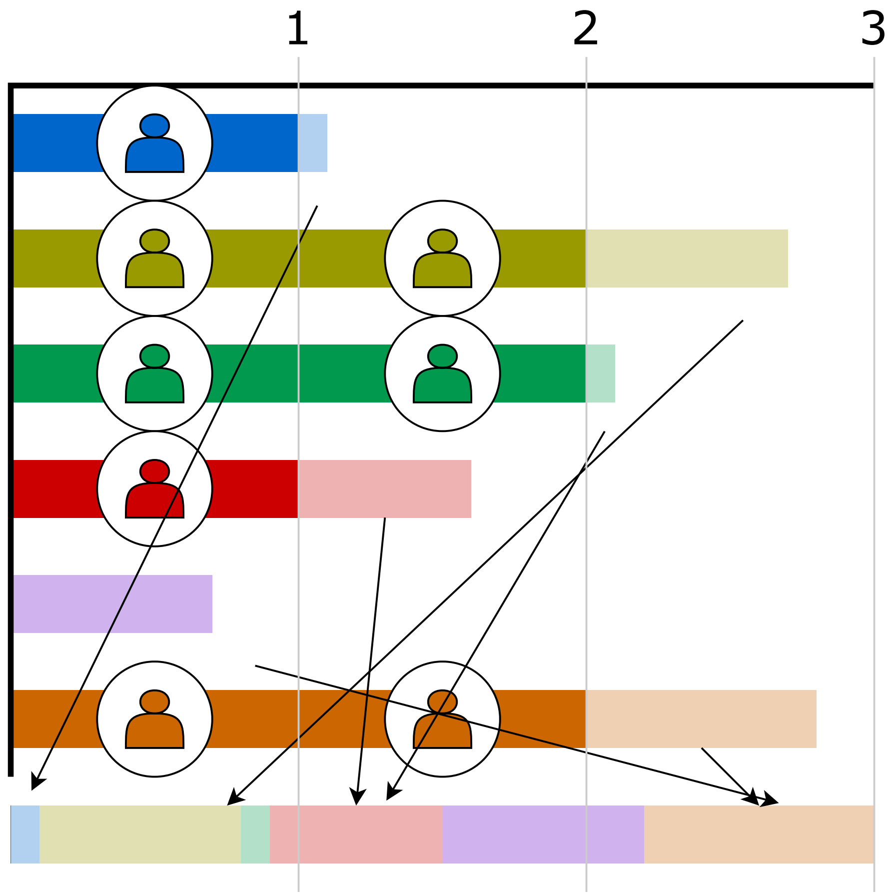

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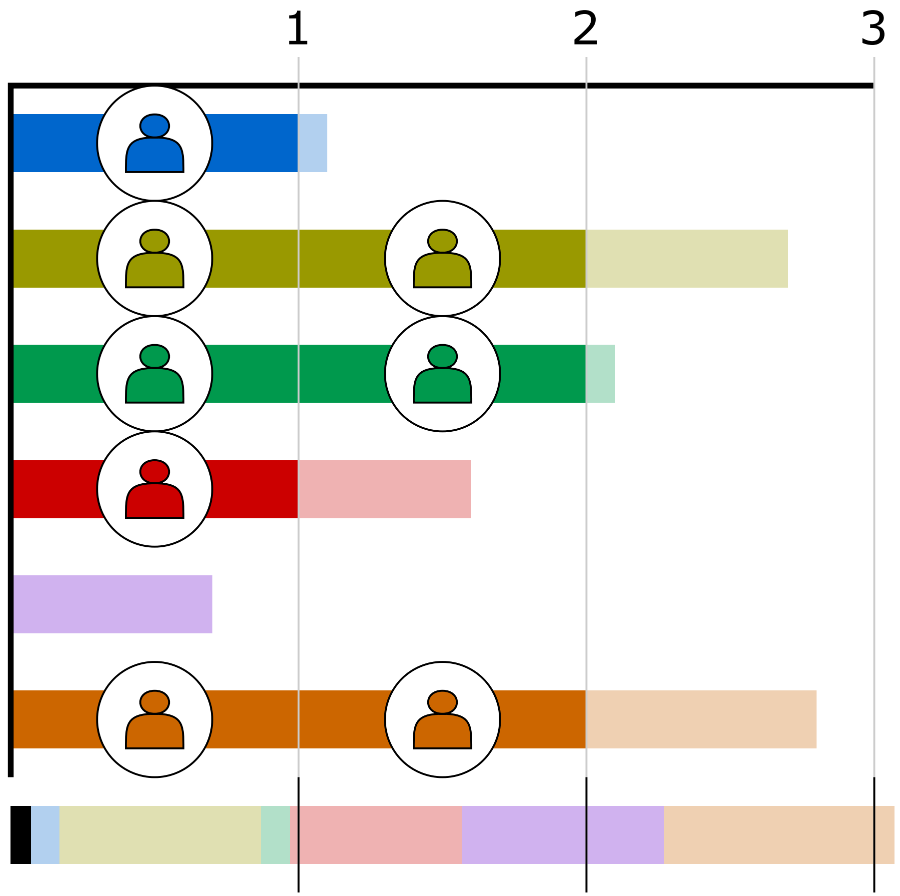

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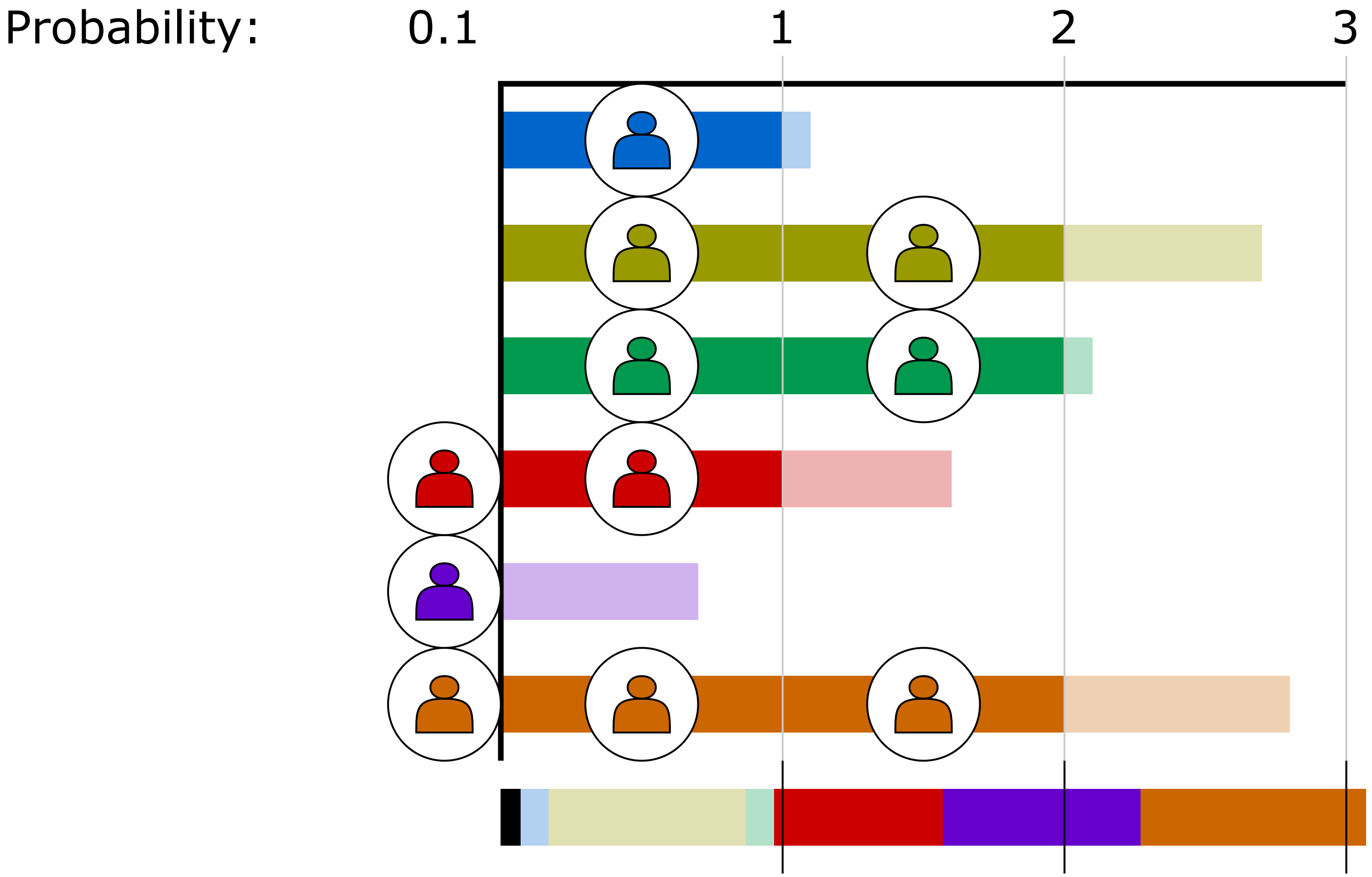

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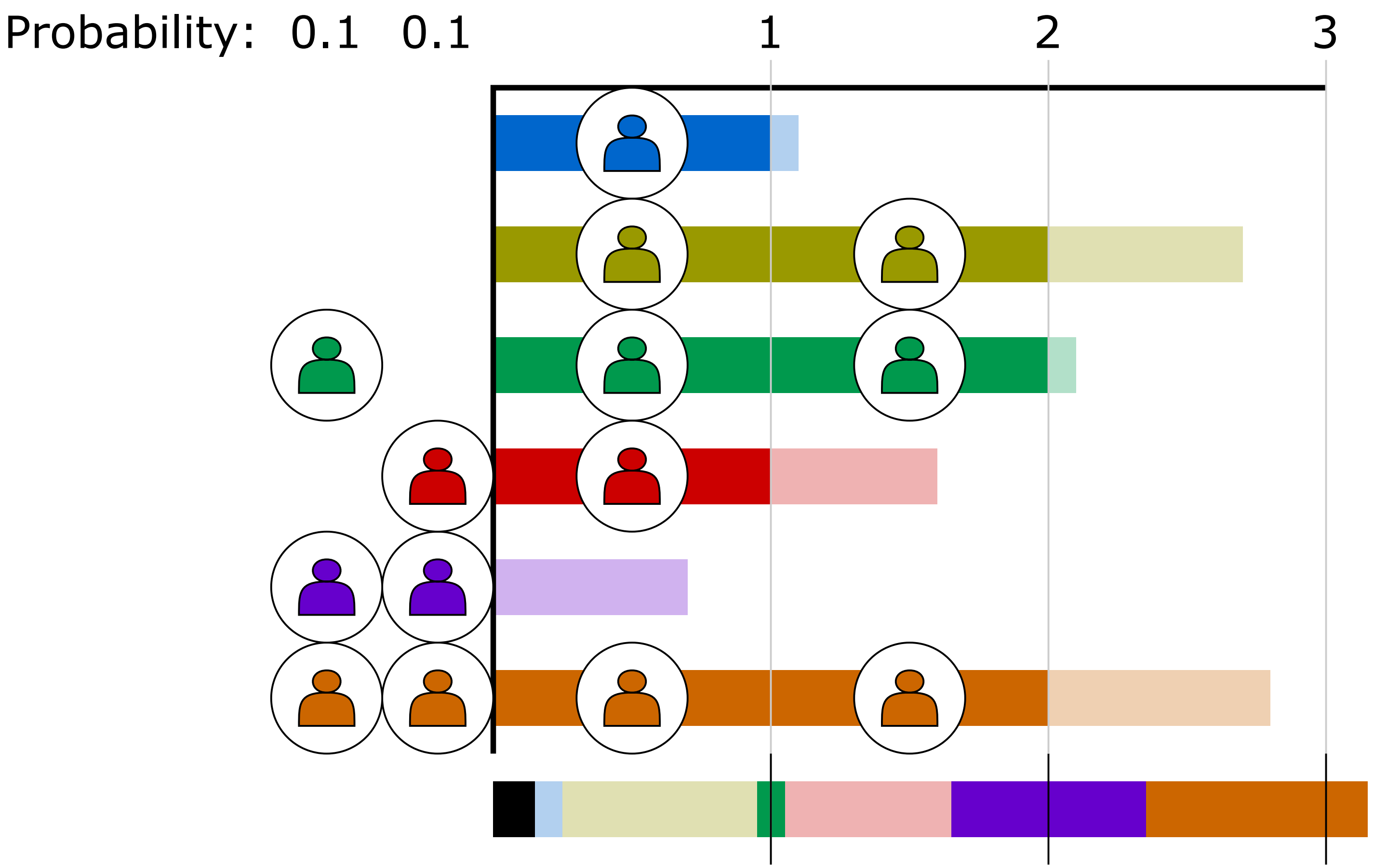

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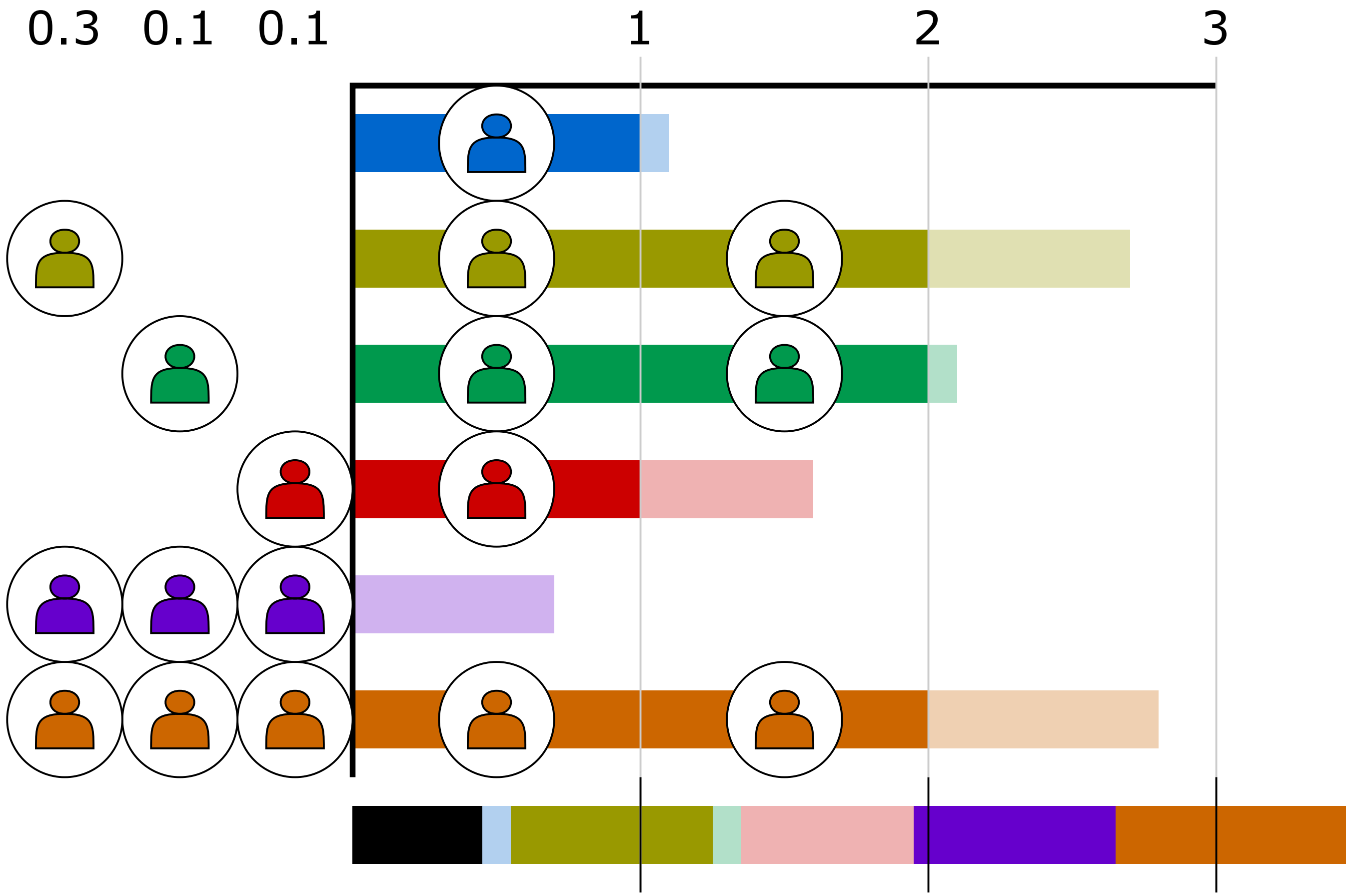

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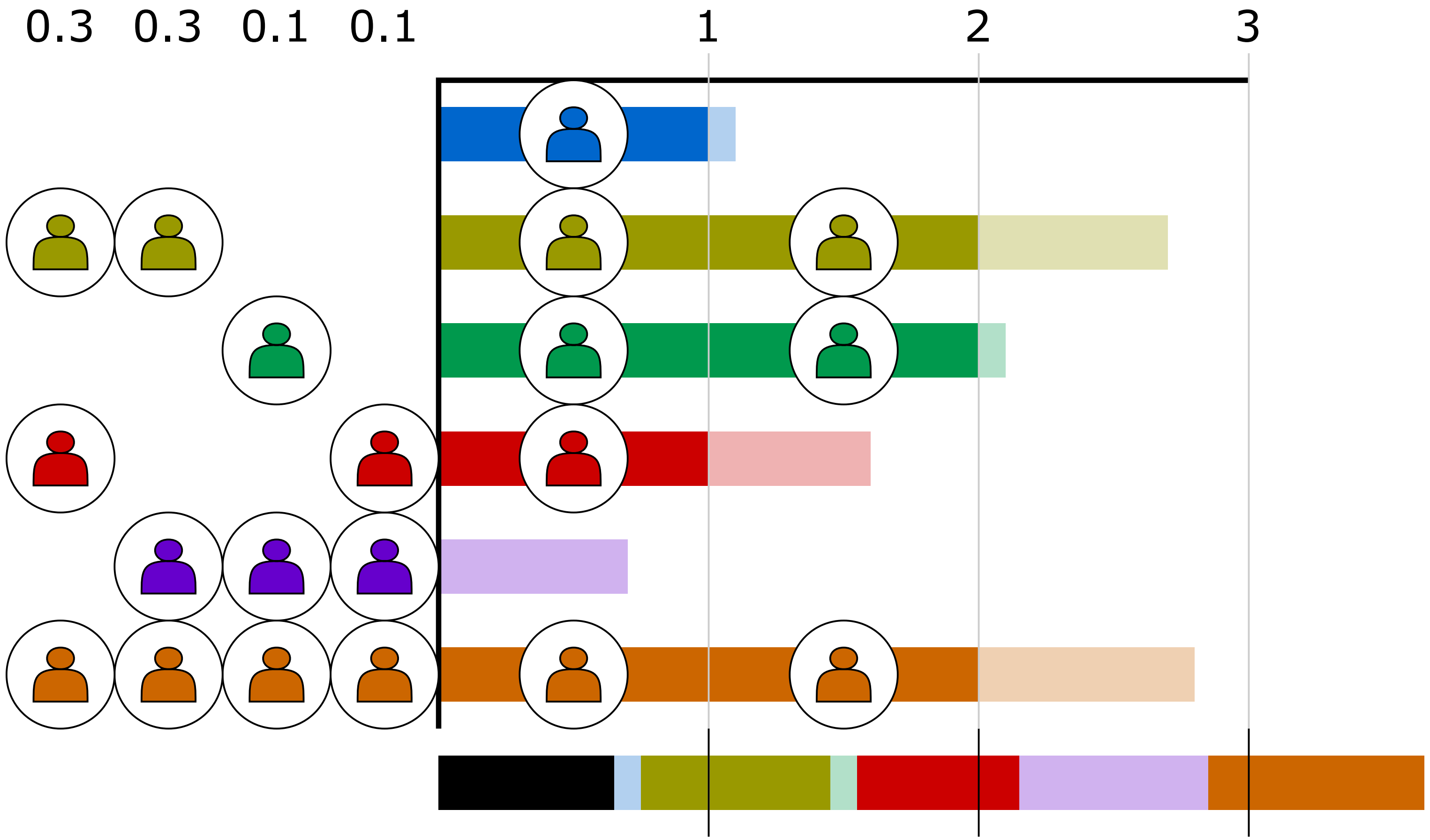

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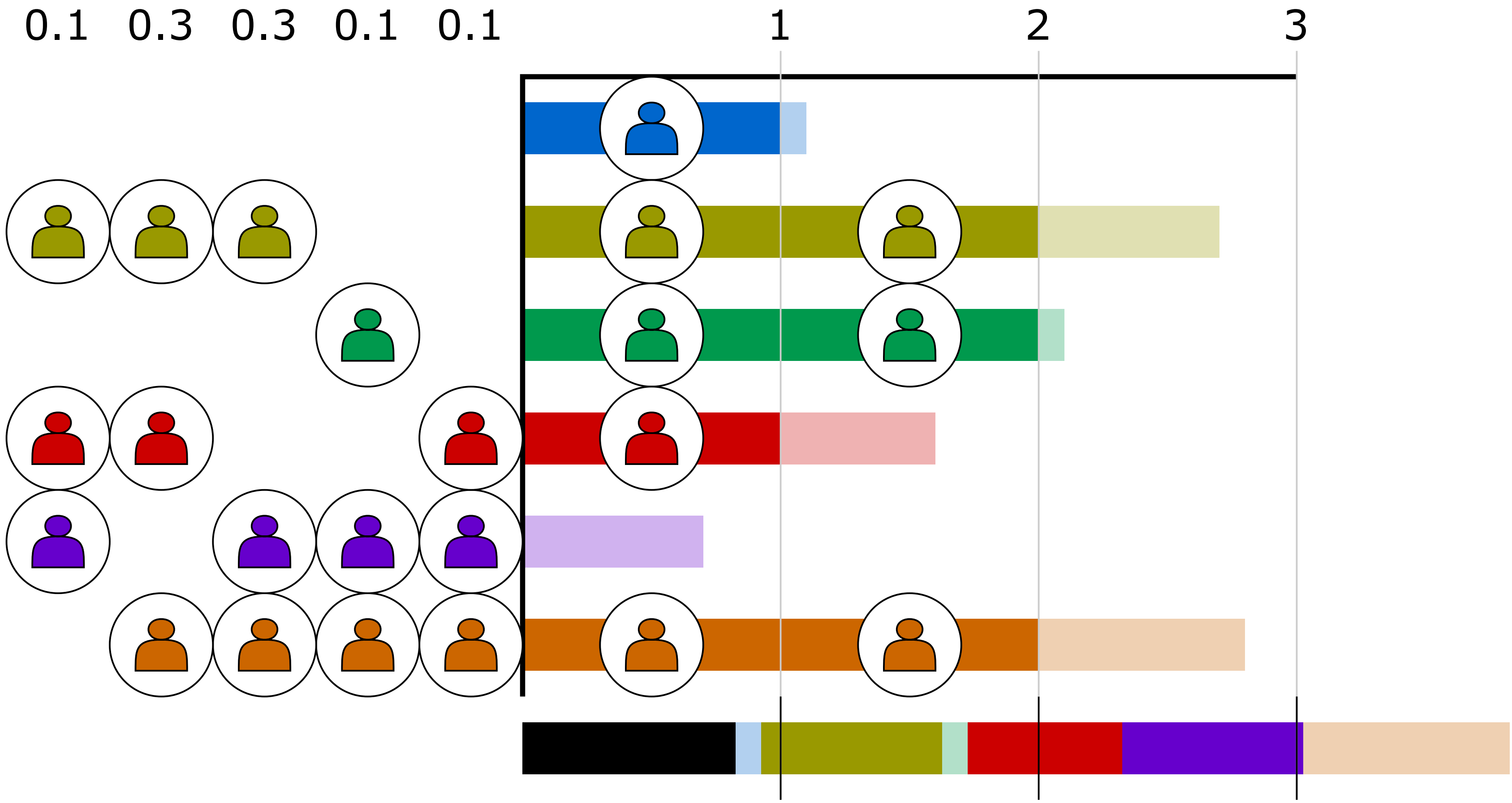

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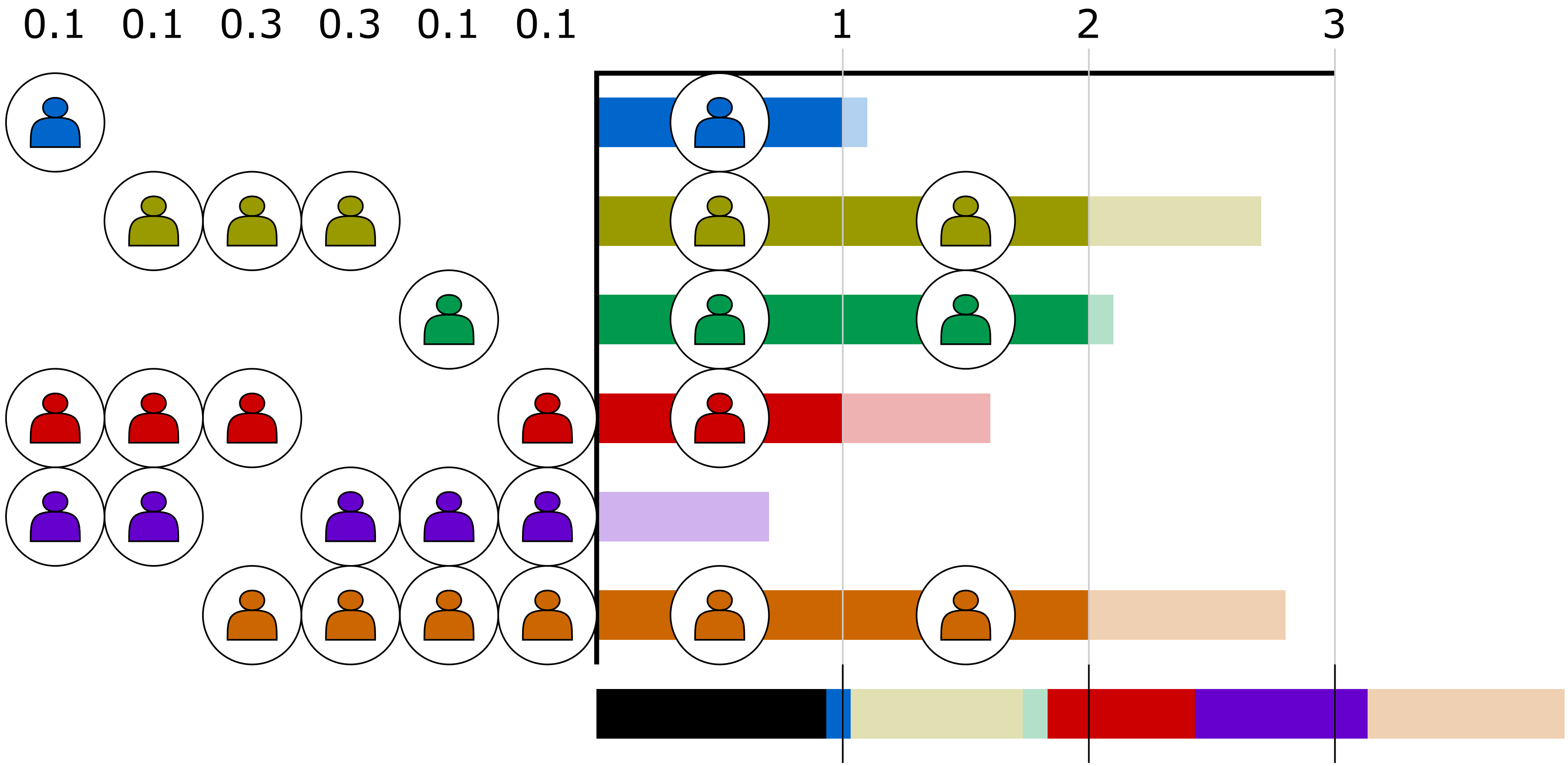

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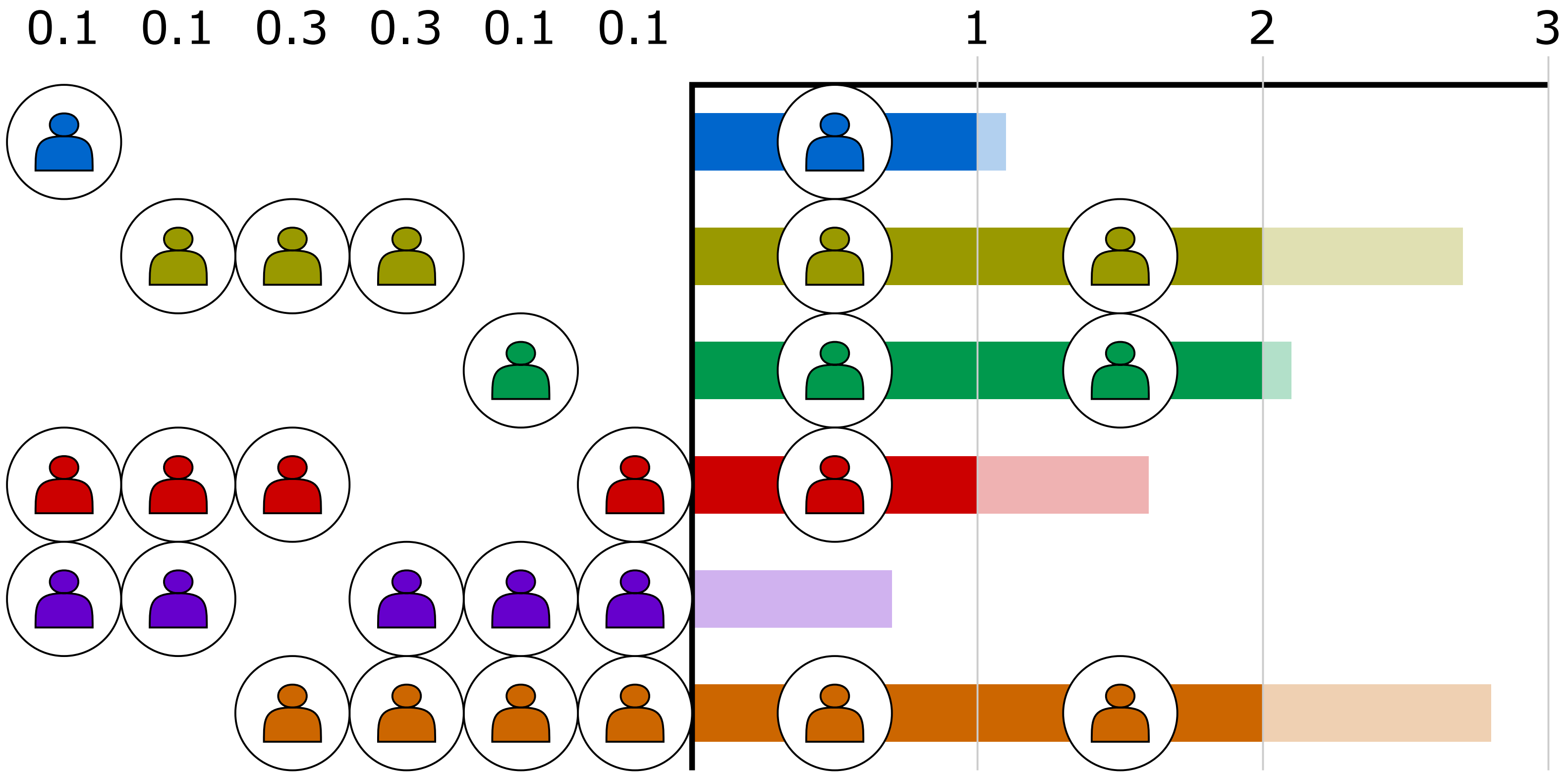

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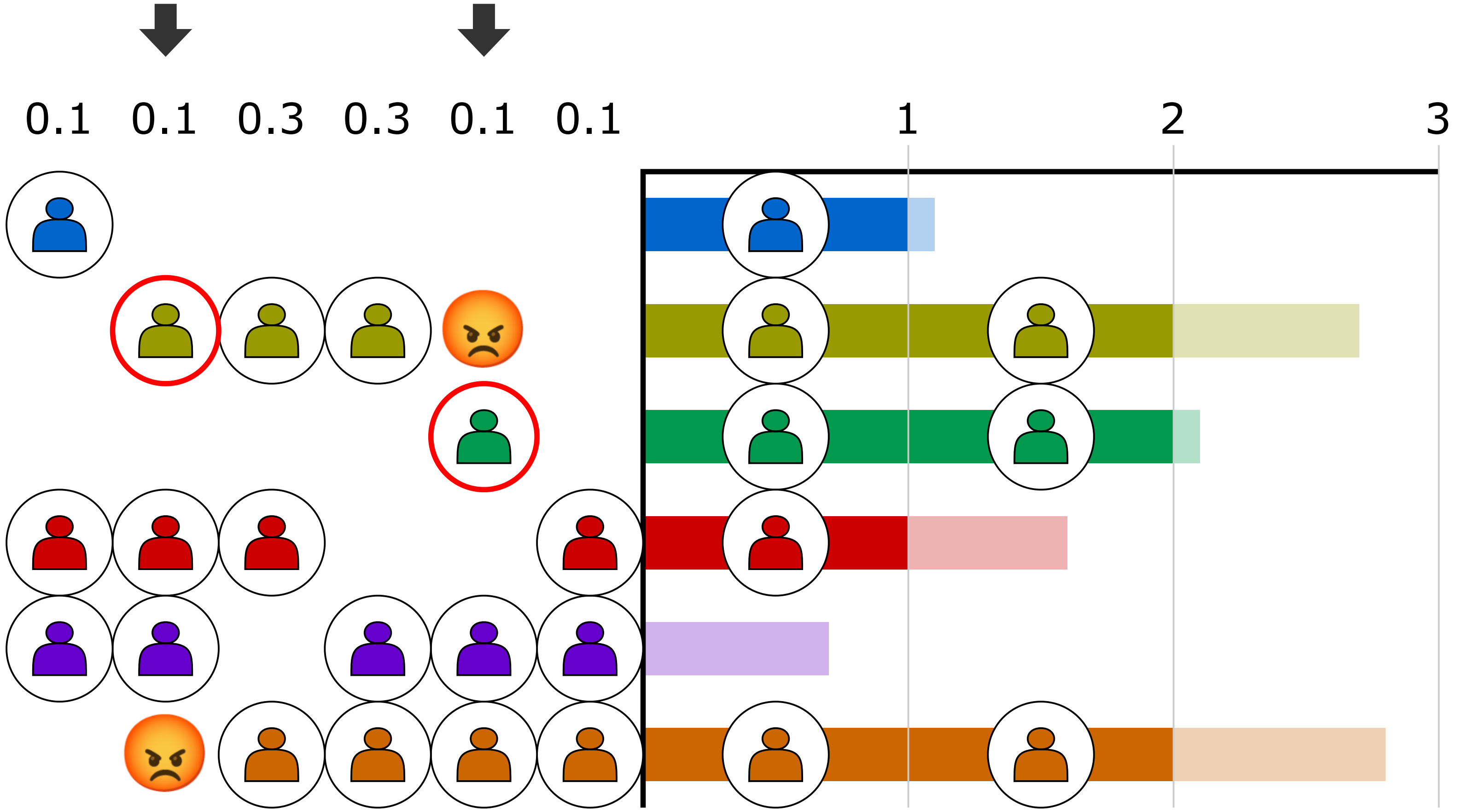

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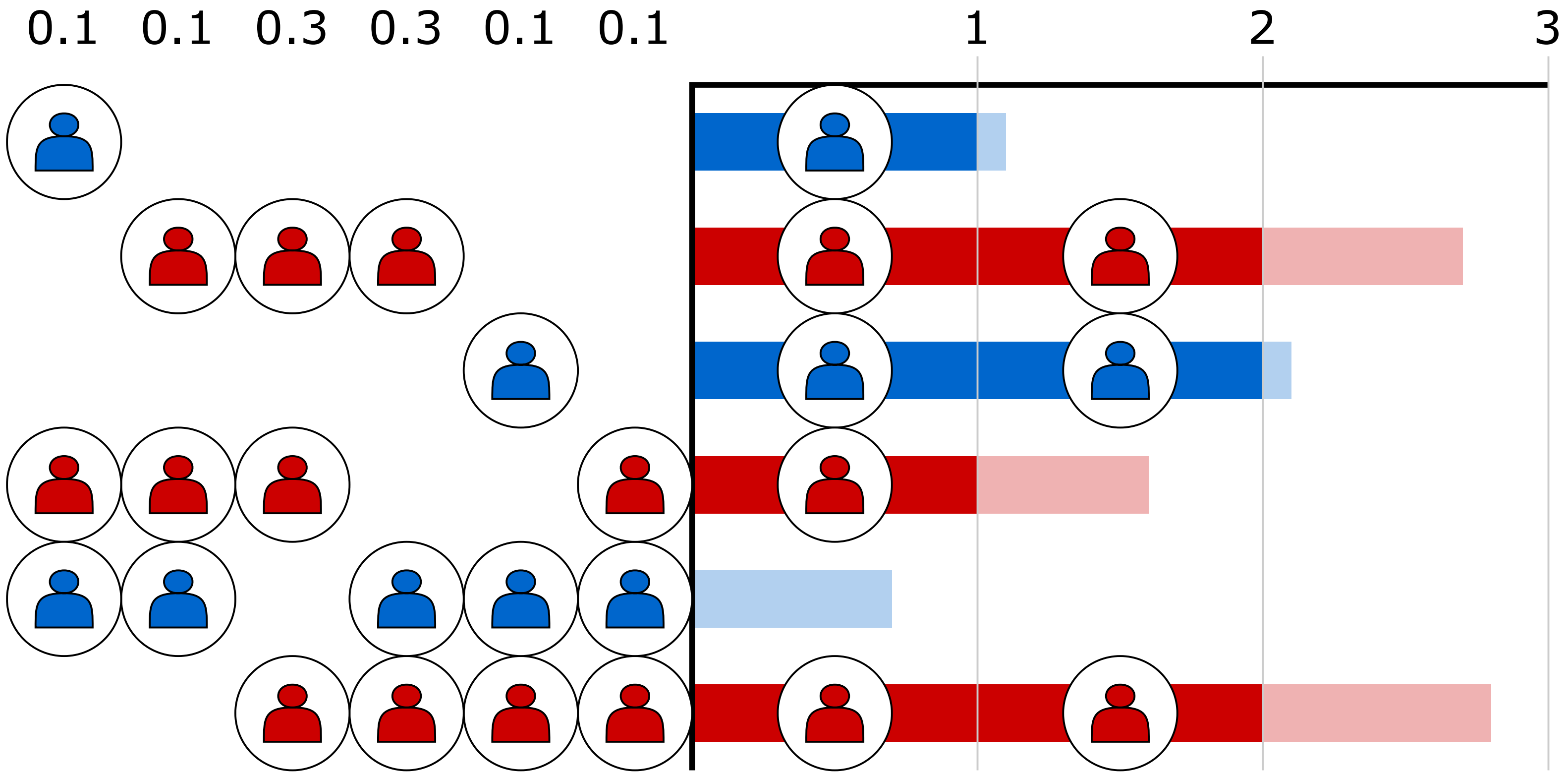

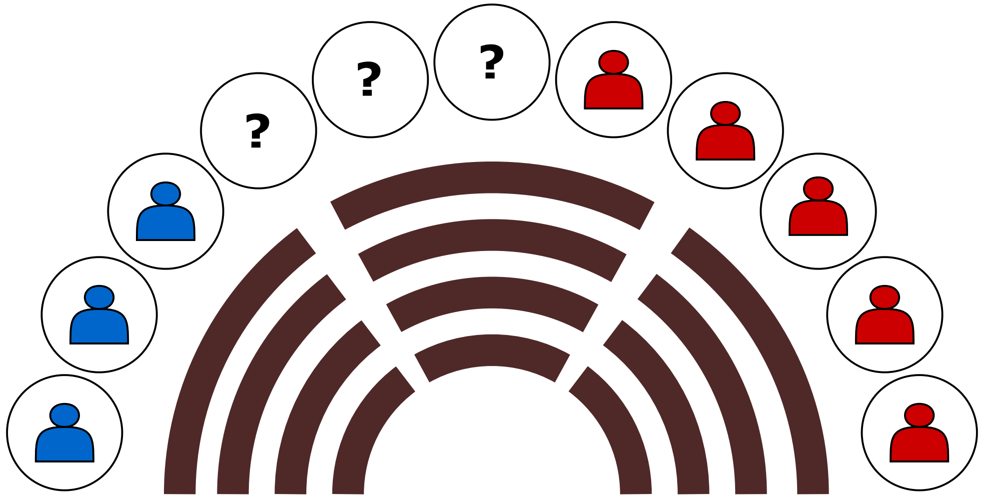

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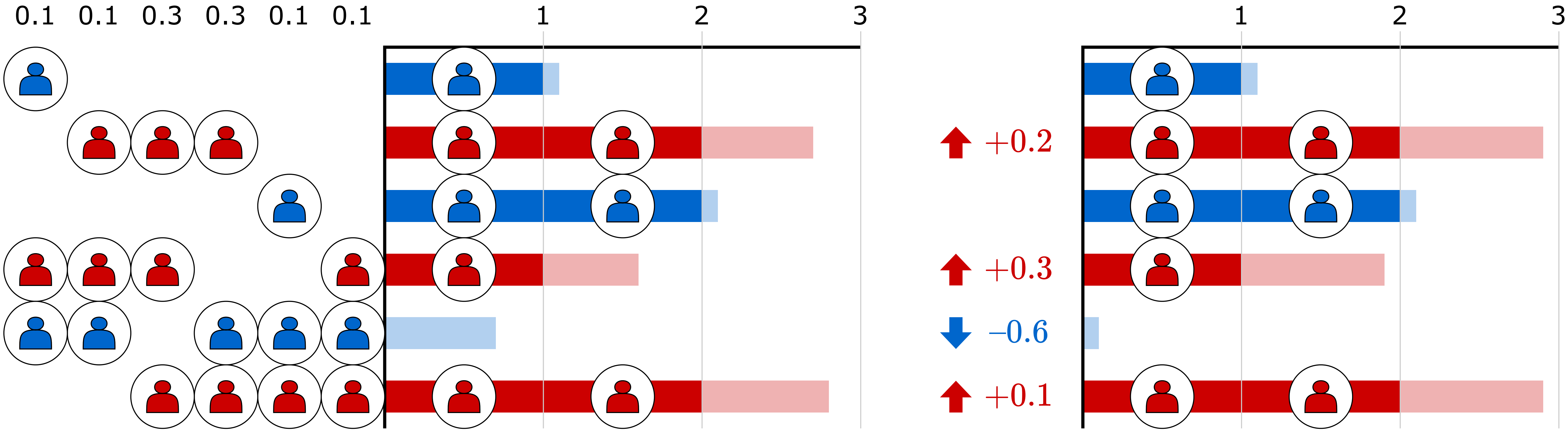

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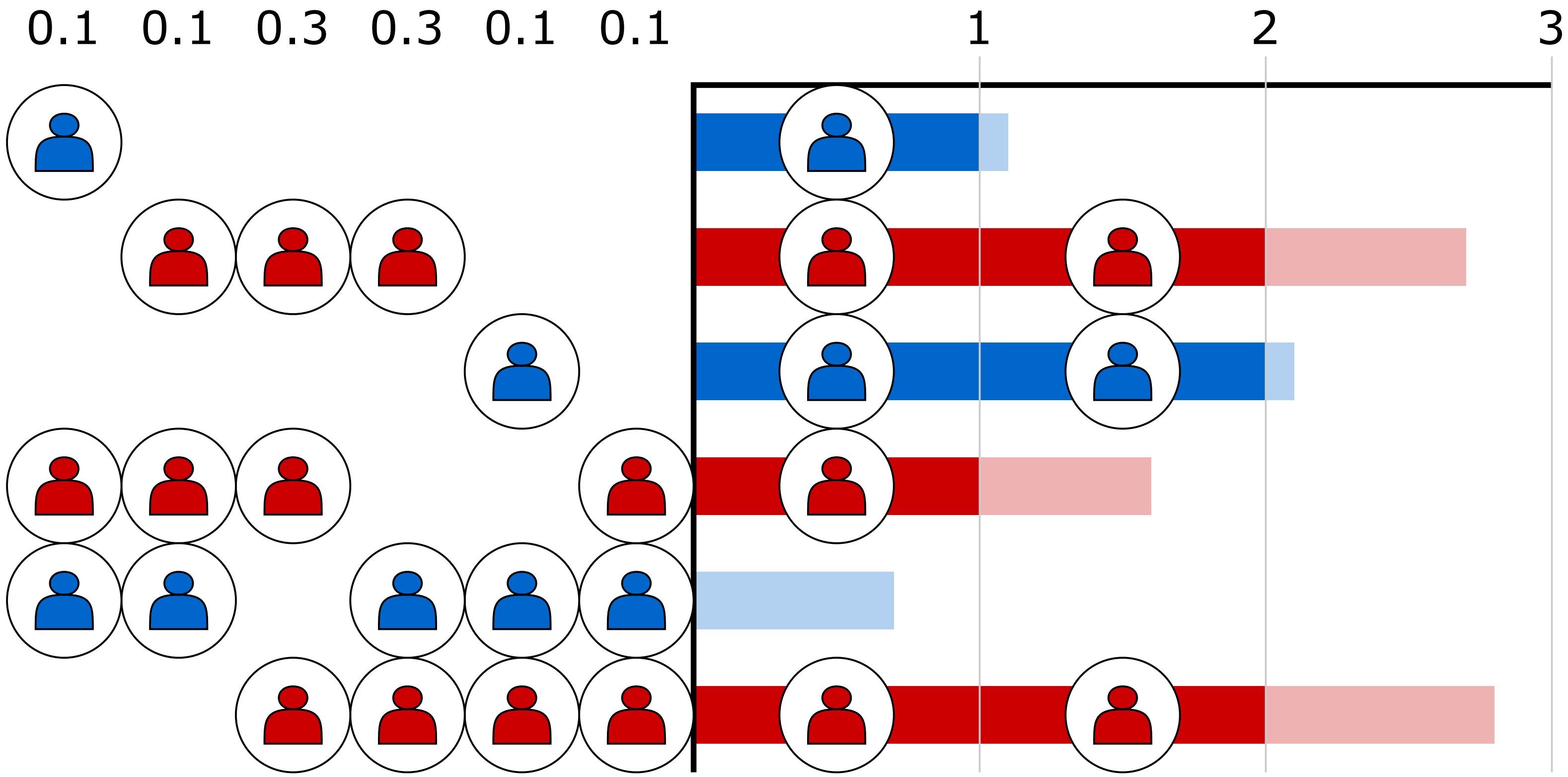

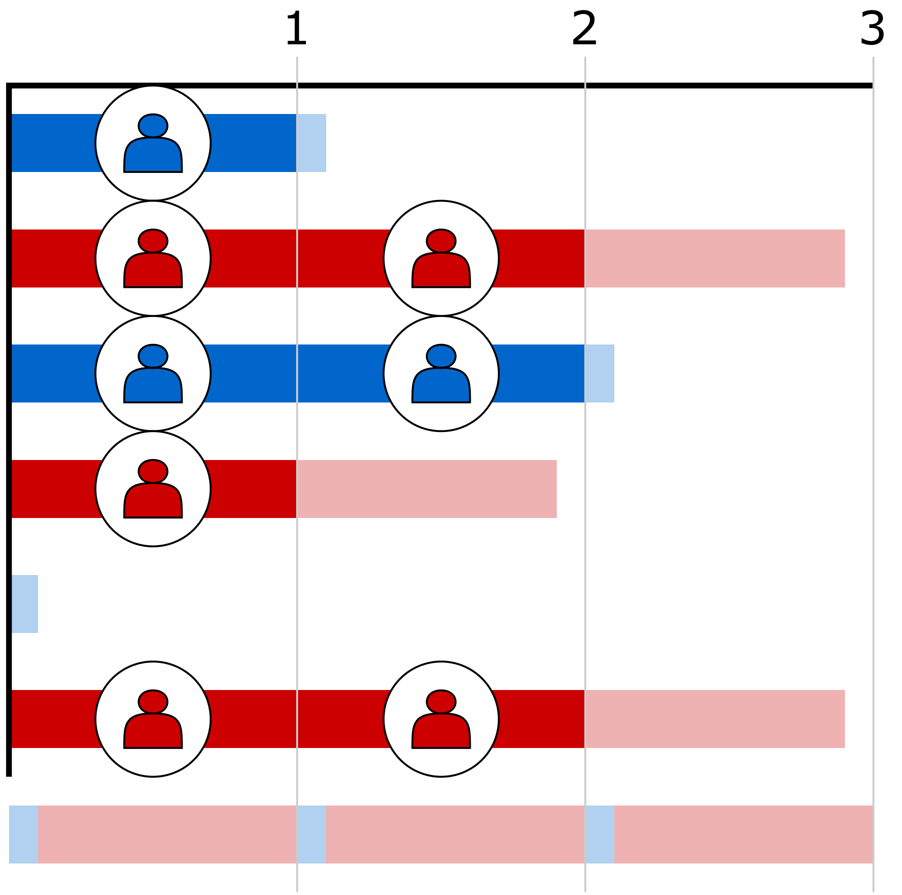

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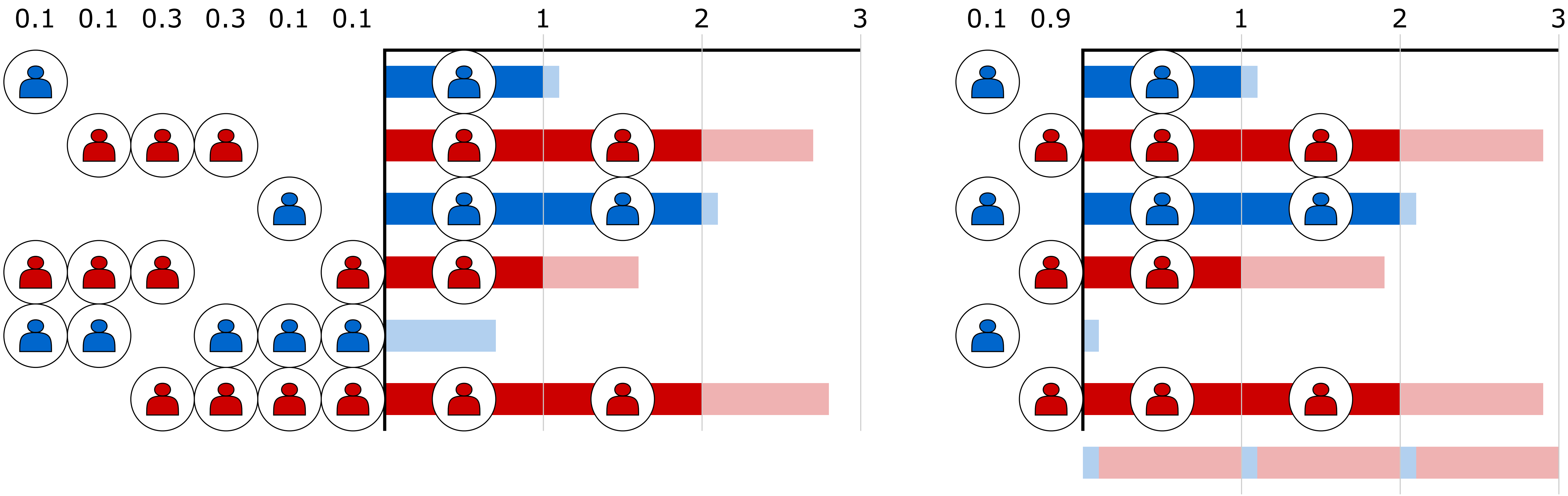

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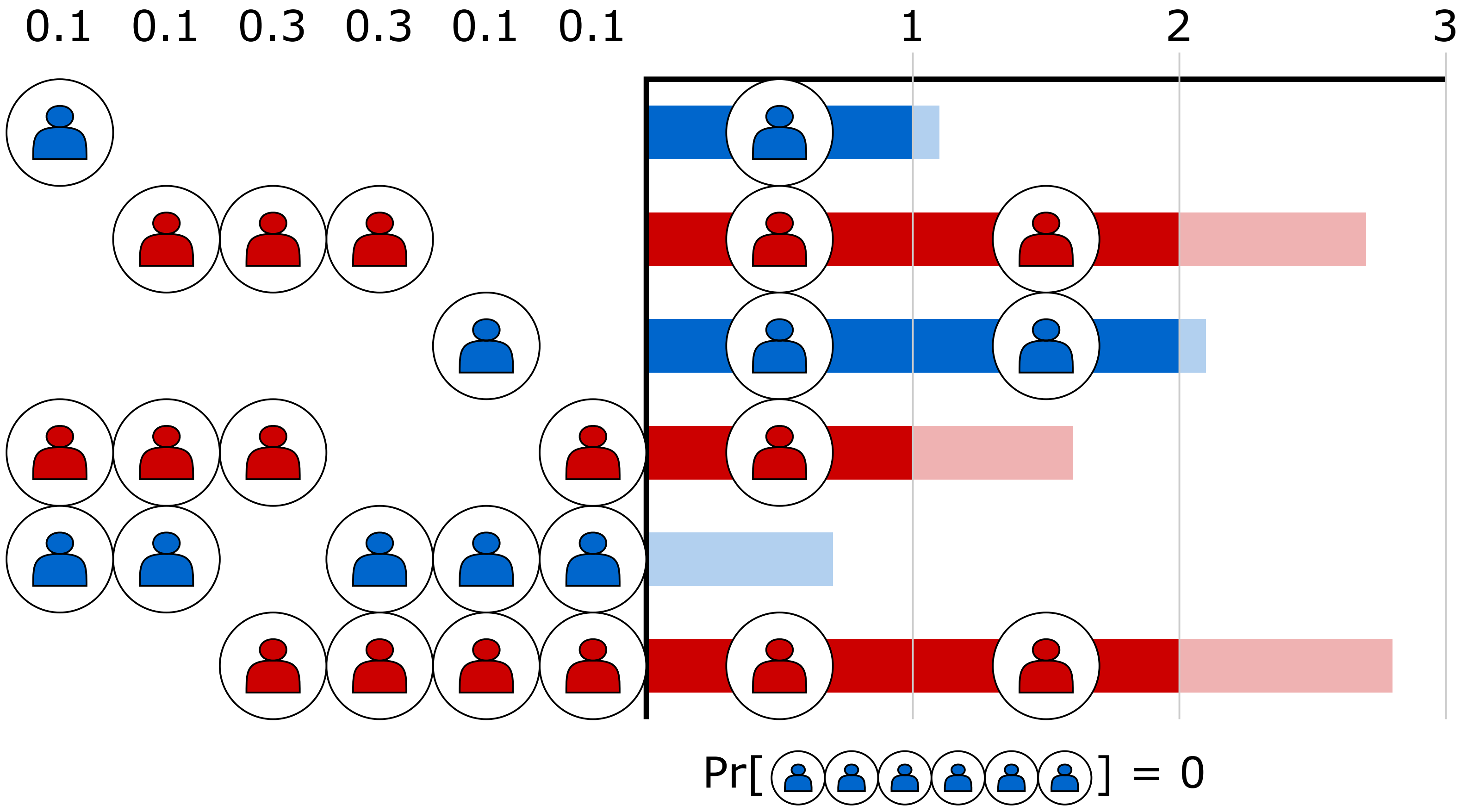

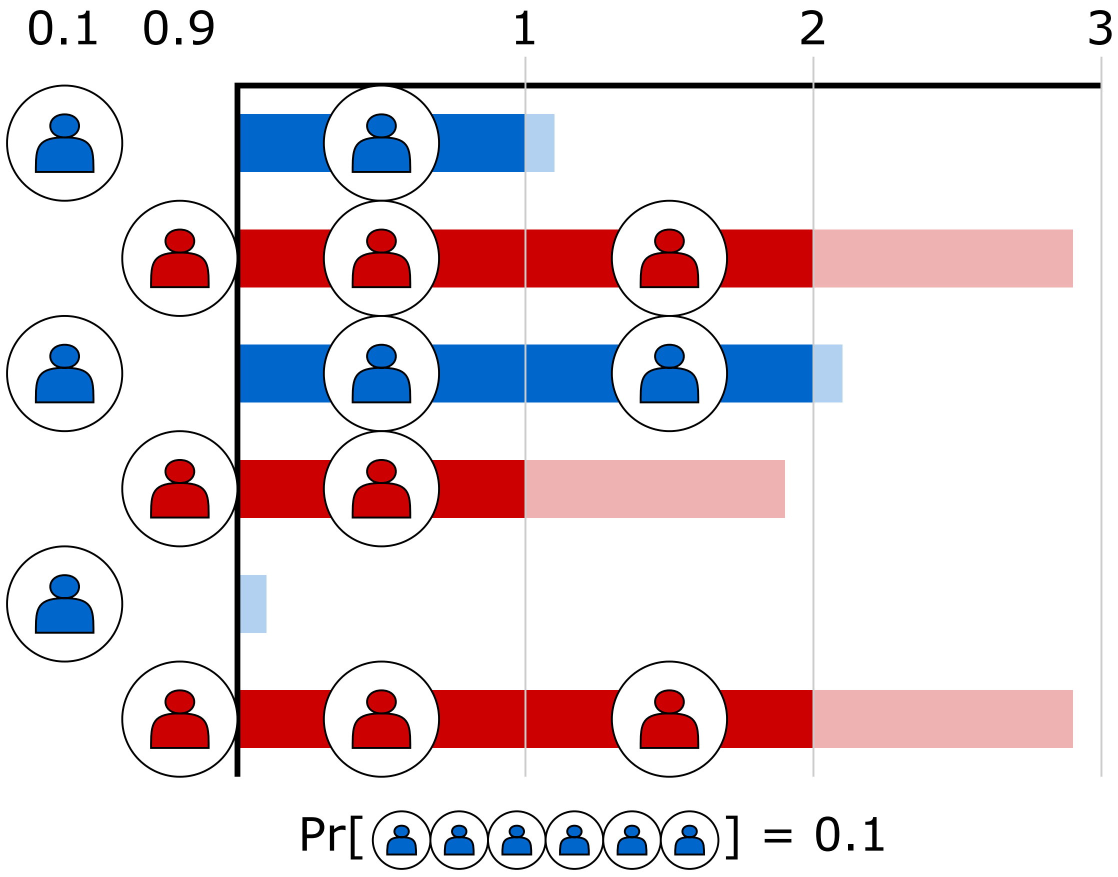

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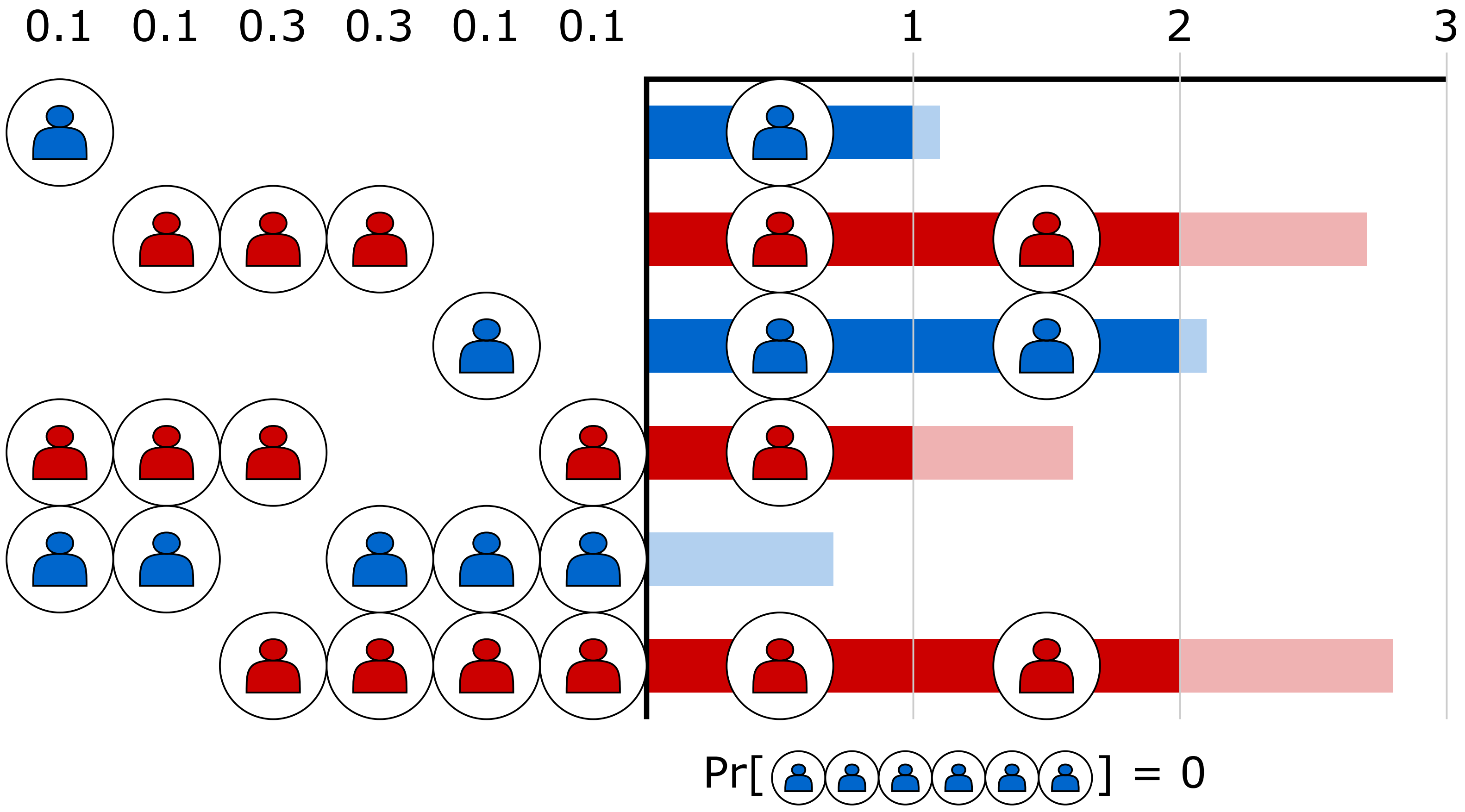

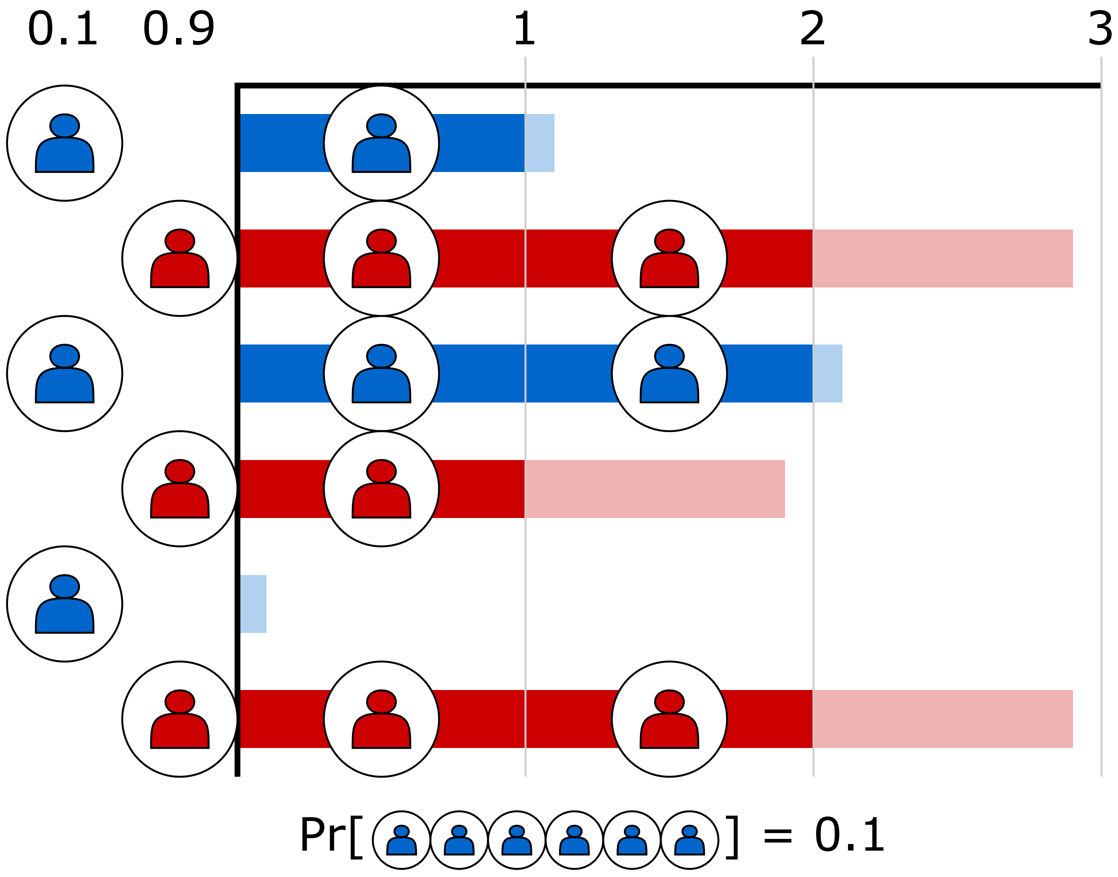

Next class: More clever ways to correlate the lotteries with better axiomatic properties.