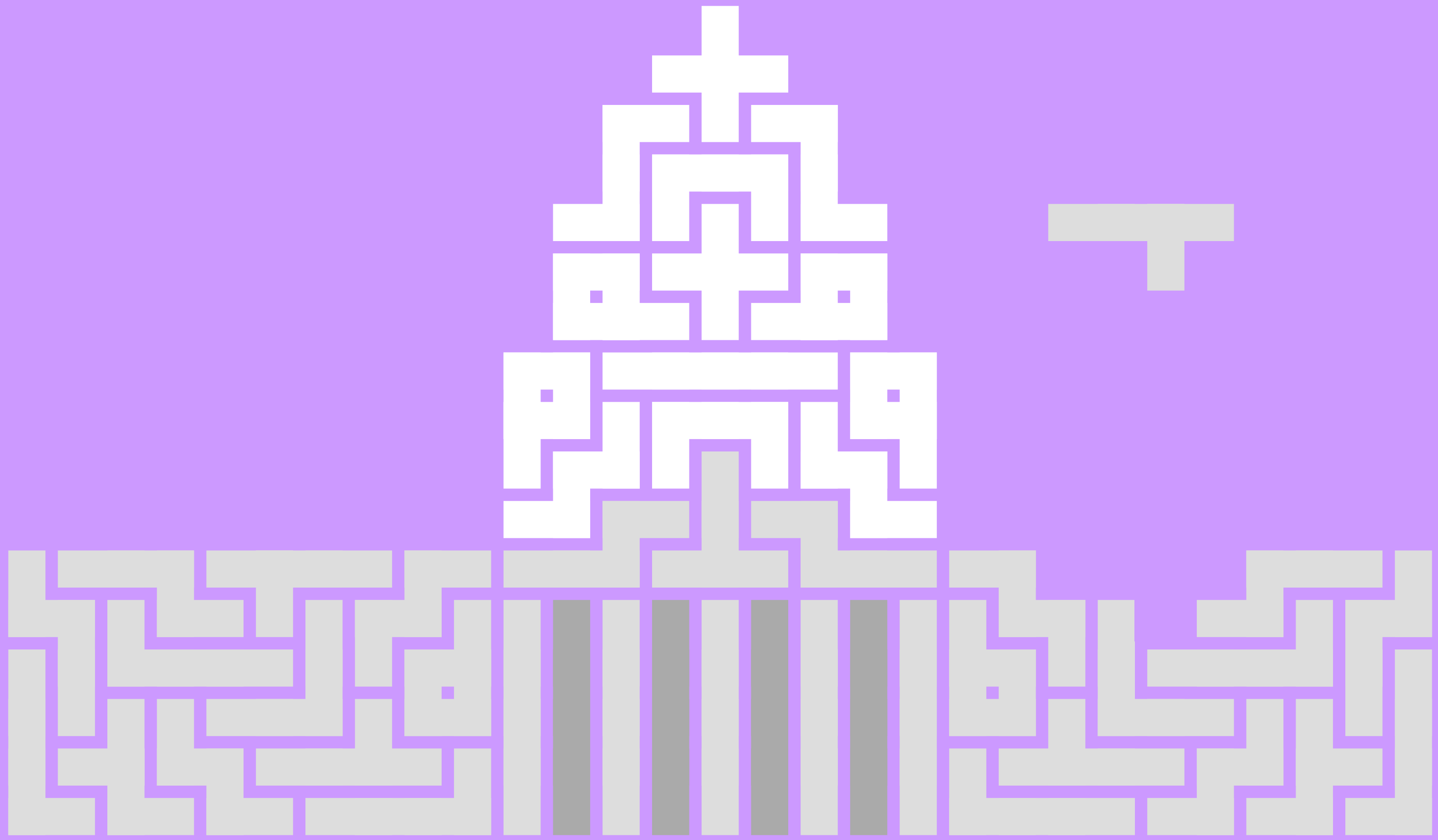

Algorithms For Democratic Decision-Making

Jamie Tucker-Foltz • Yale University • Spring 2026

Lecture 19: **Randomized Apportionment and Weighted Voting**

Announcements

Project updates due tonight! I will start reading them early tomorrow morning.

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Wednesday will be individual meetings with students/groups in my office again, no class.

Grimmett's method - randomize!

Party 1 quota: 1.1

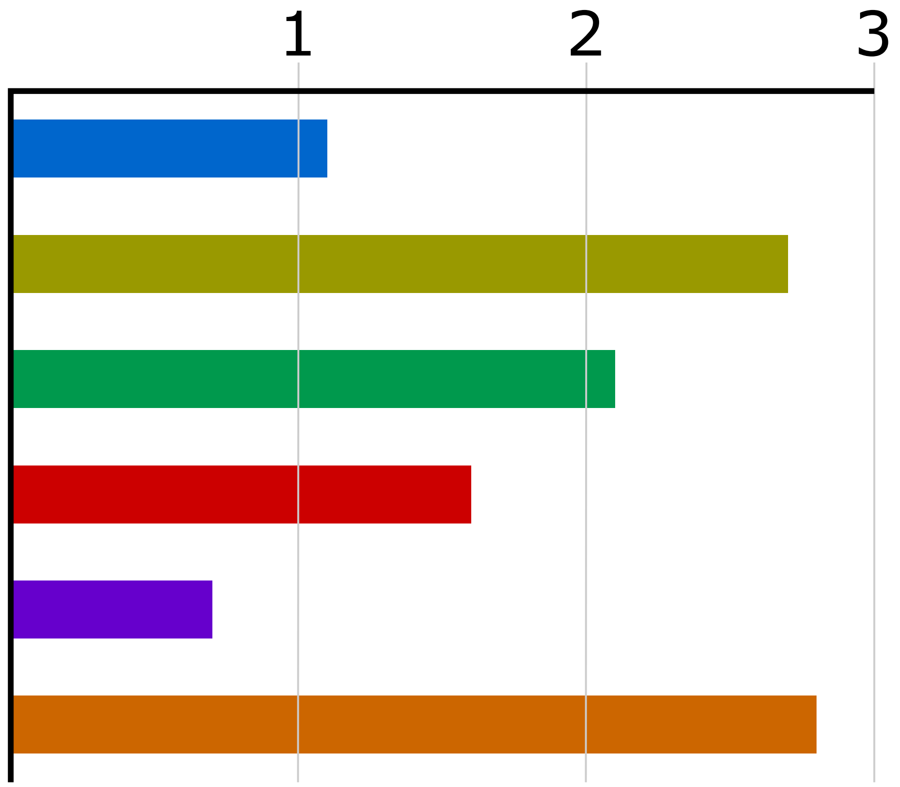
Party 2 quota: 2.7

Party 3 quota: 2.1

Party 4 quota: 1.6

Party 5 quota: 0.7

Party 6 quota: 2.8



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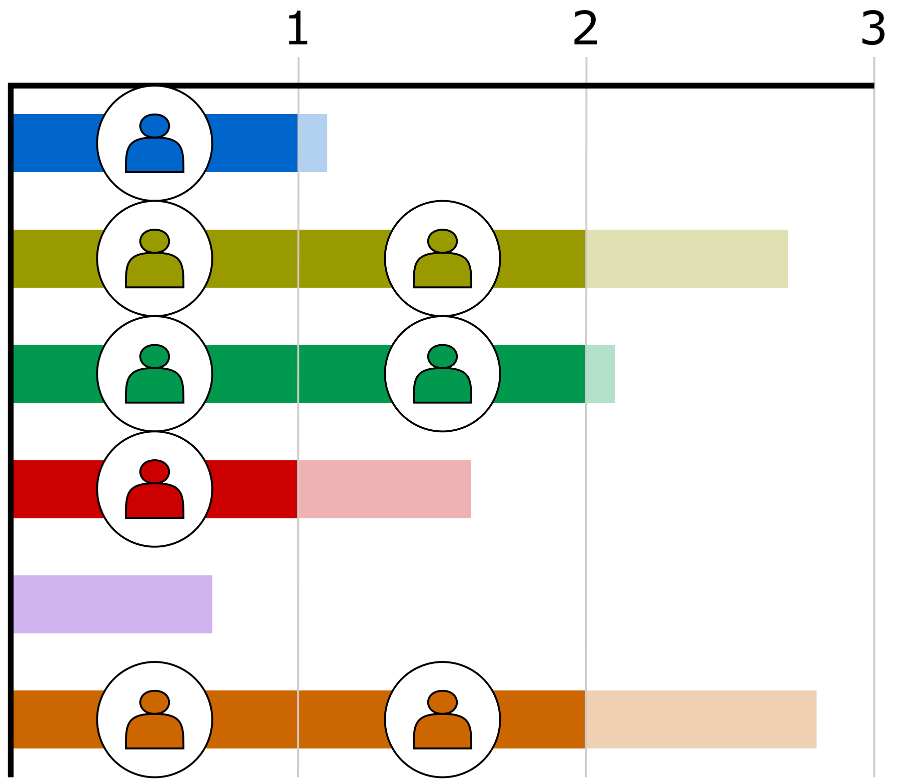

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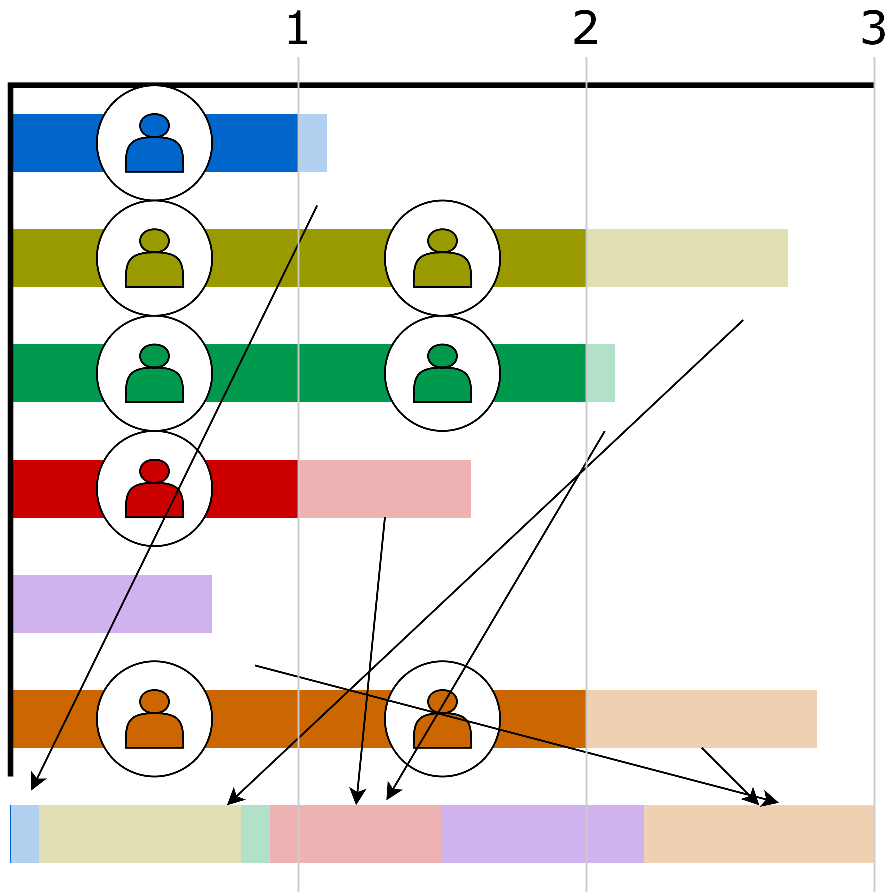

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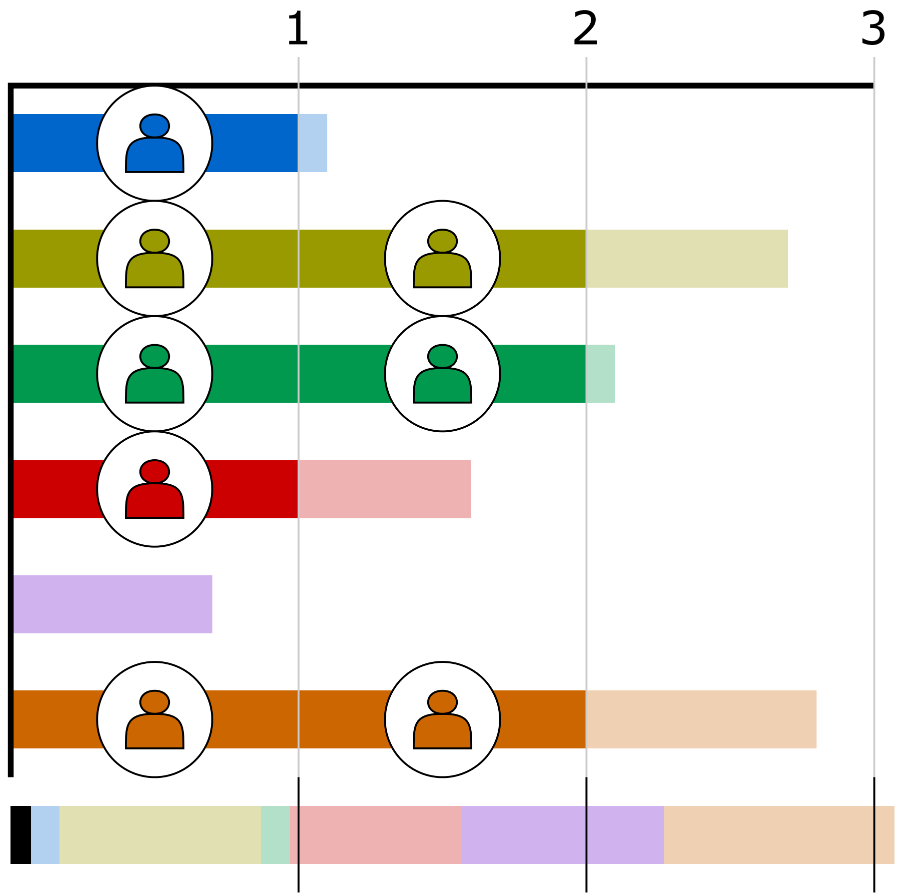

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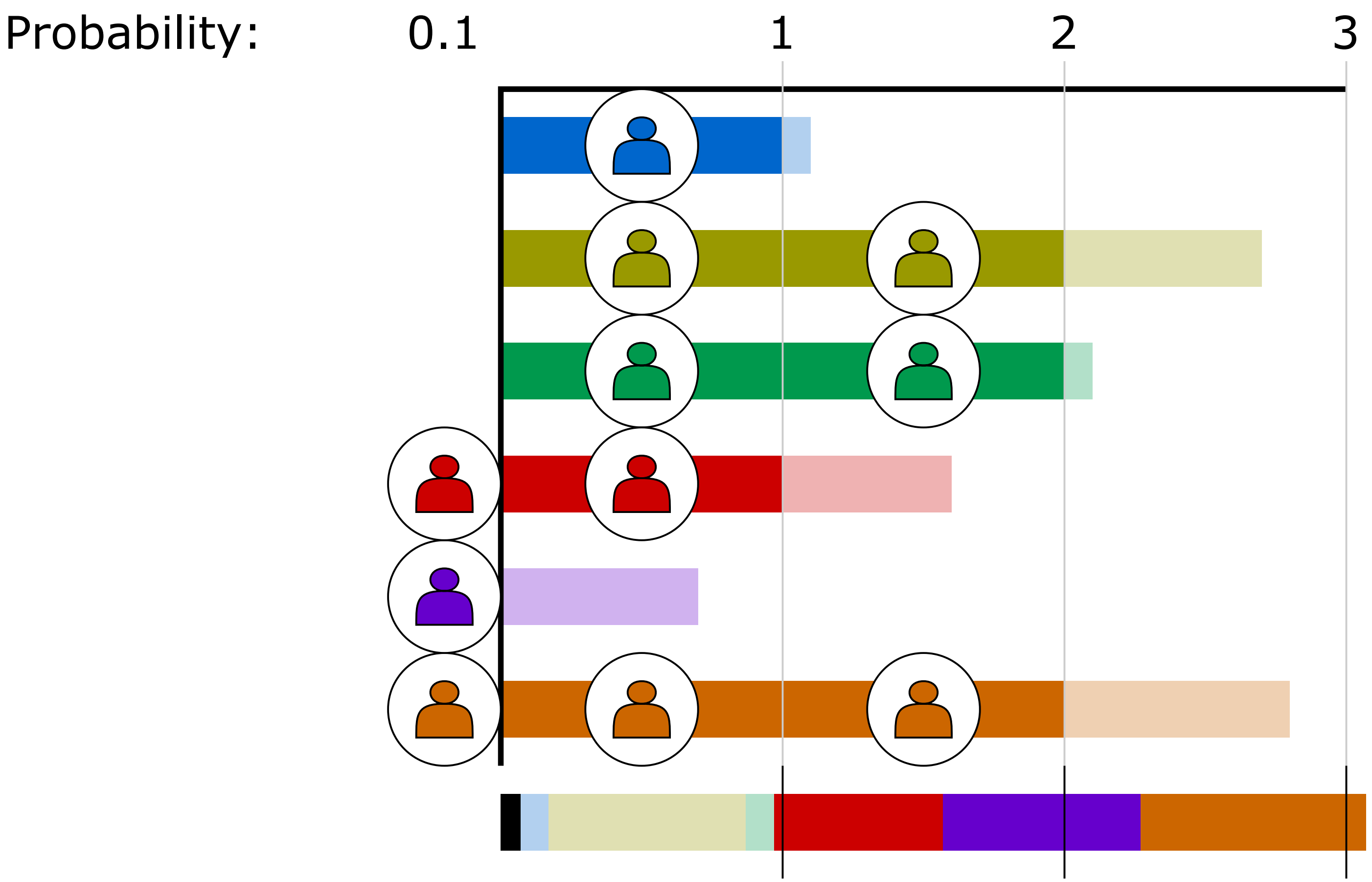

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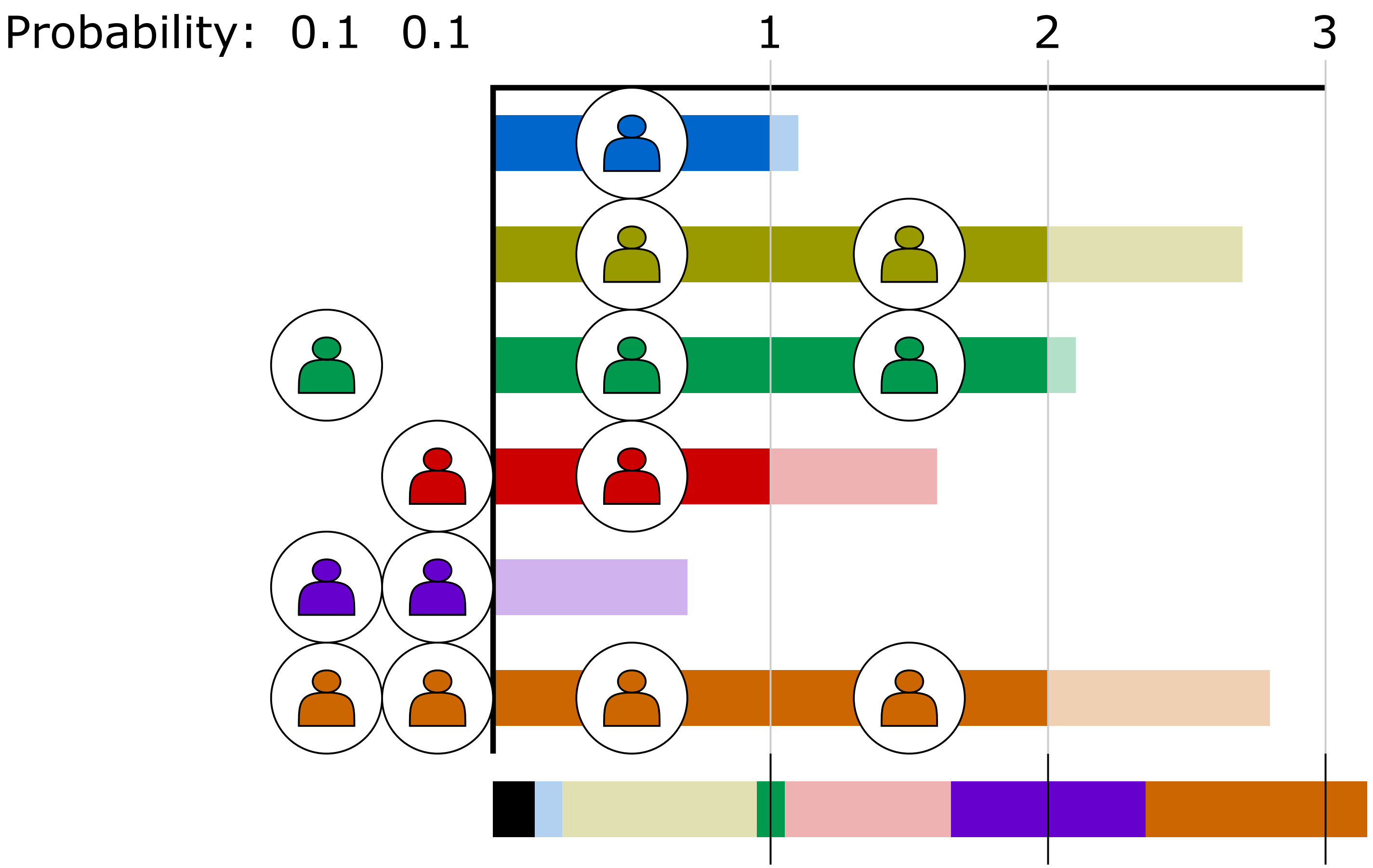

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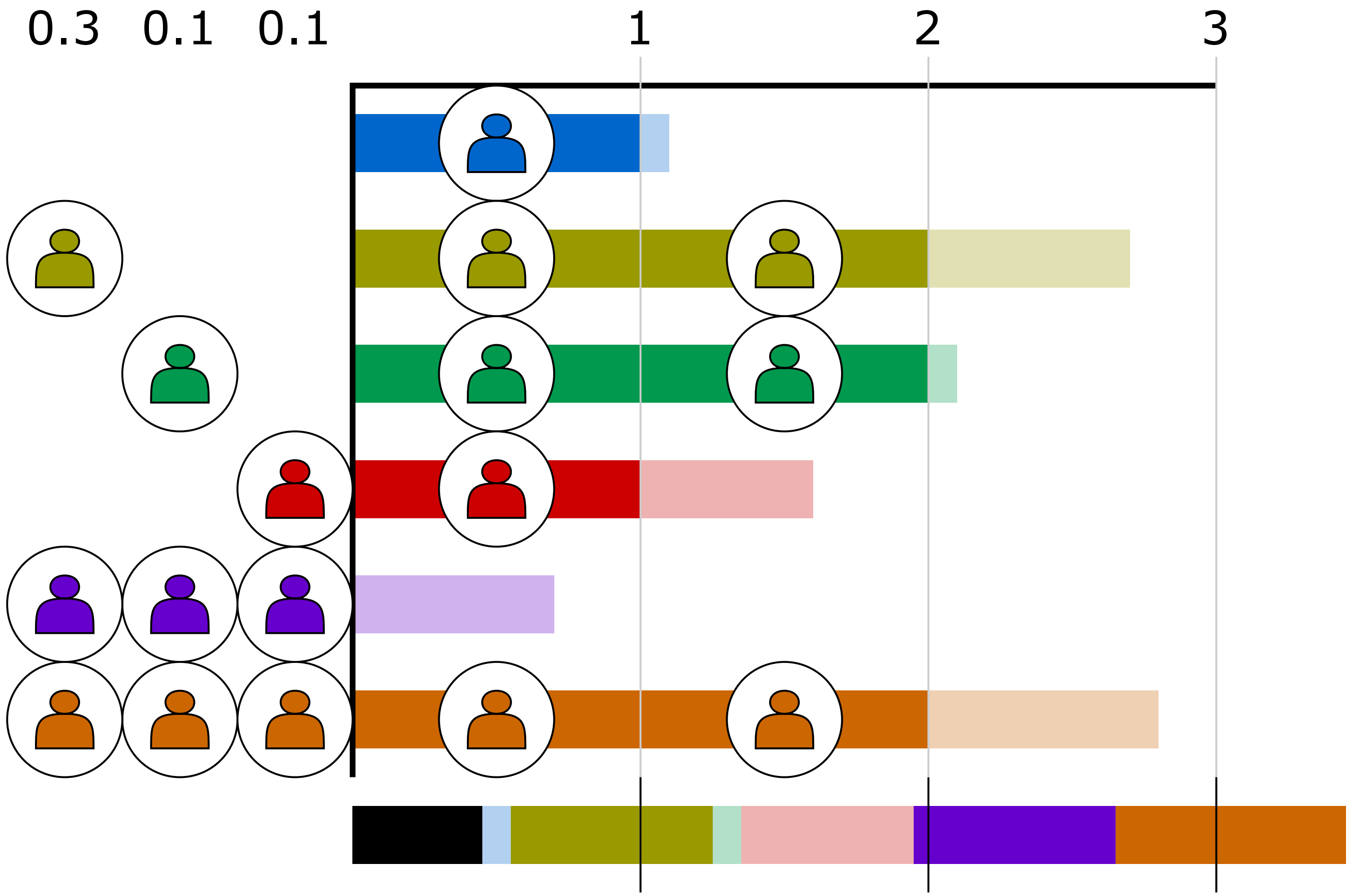

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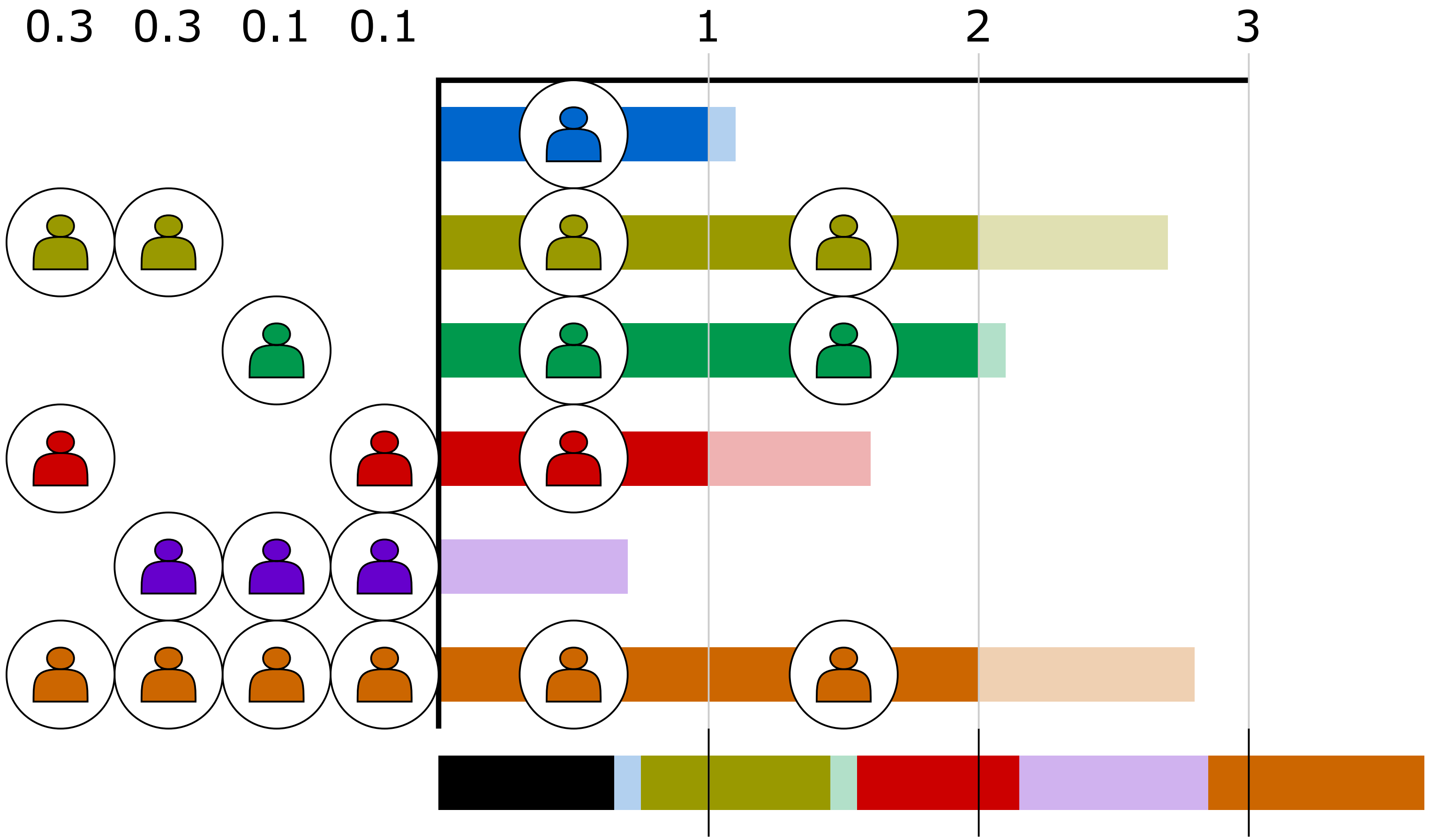

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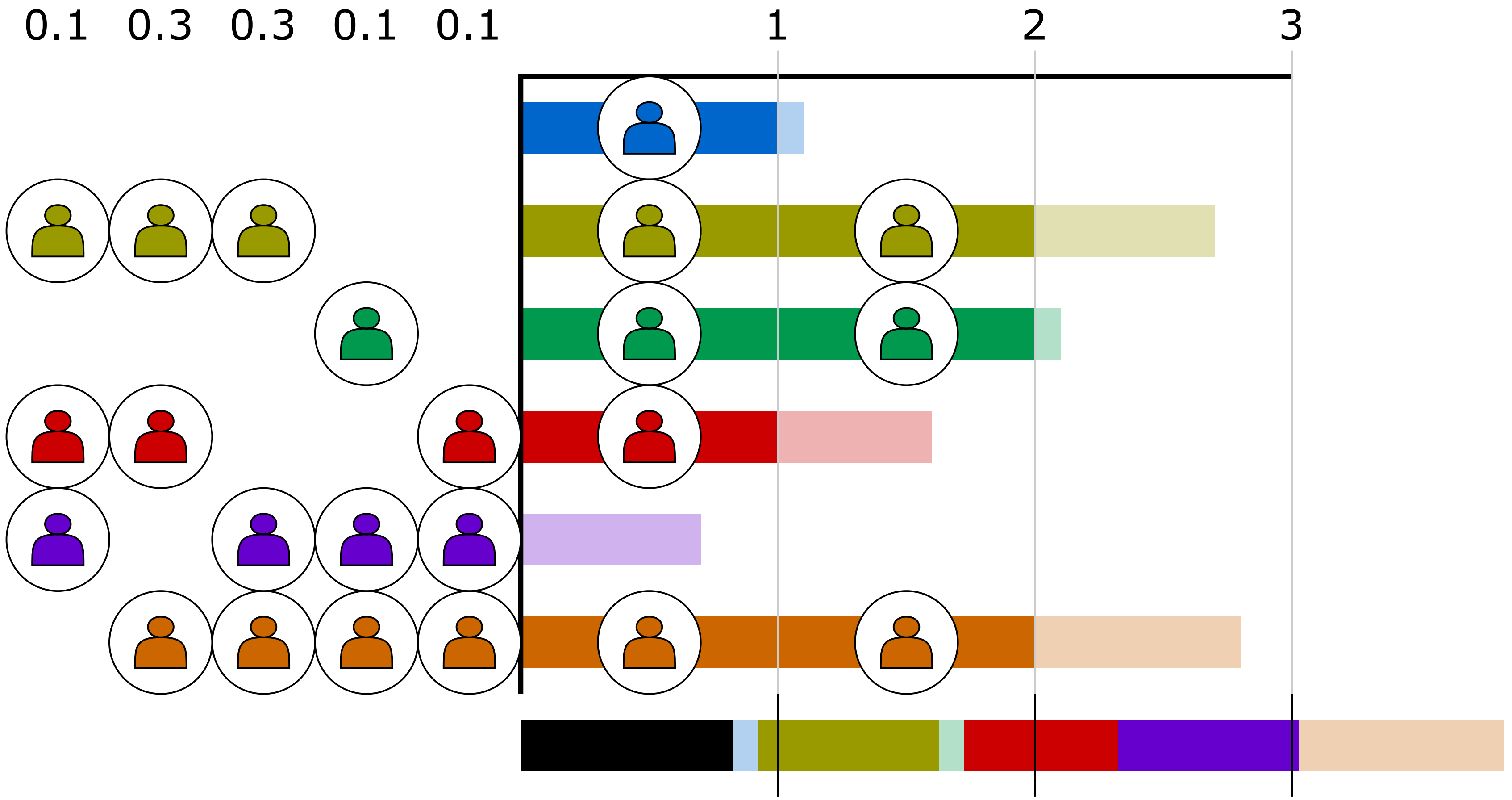

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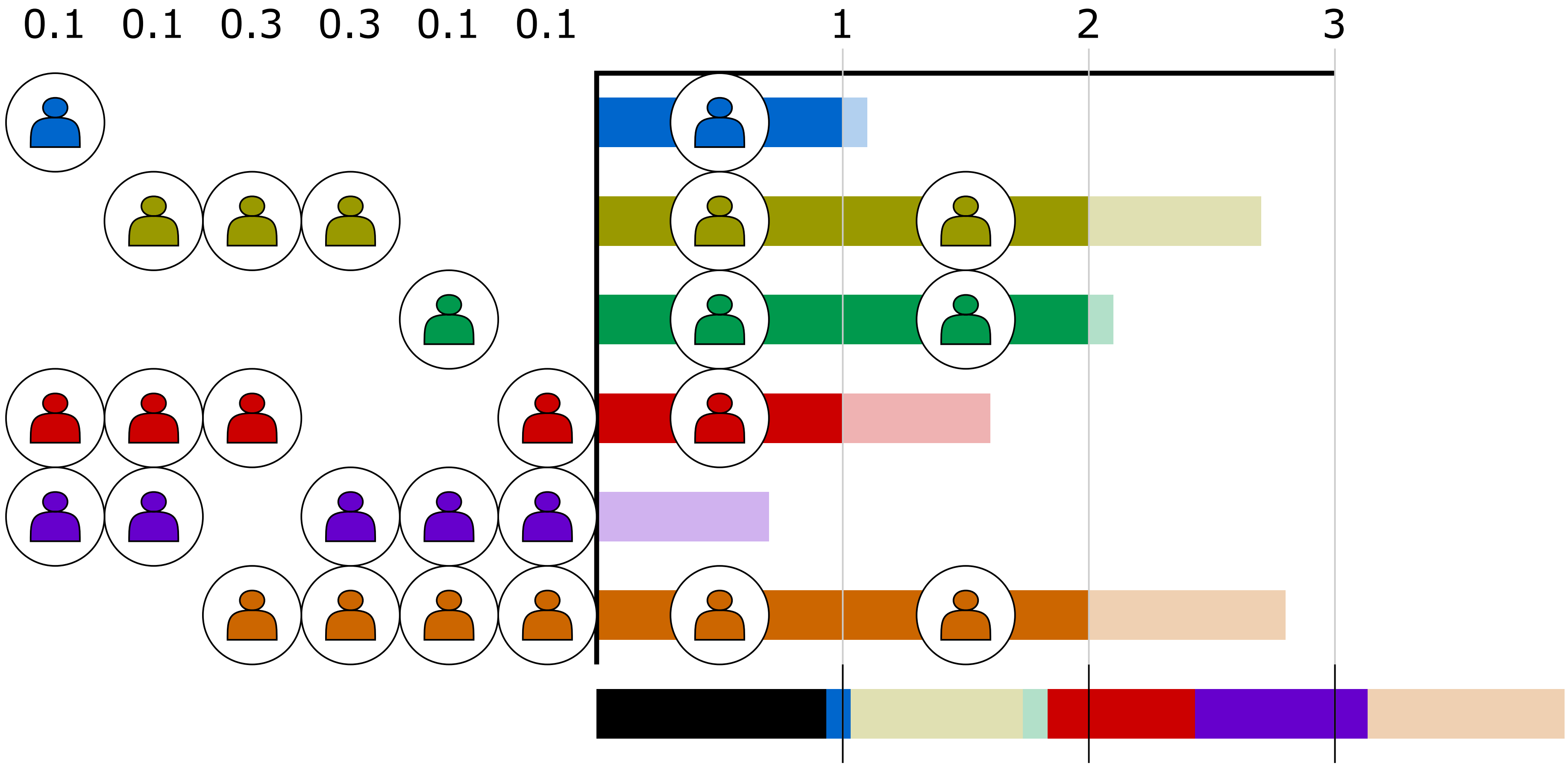

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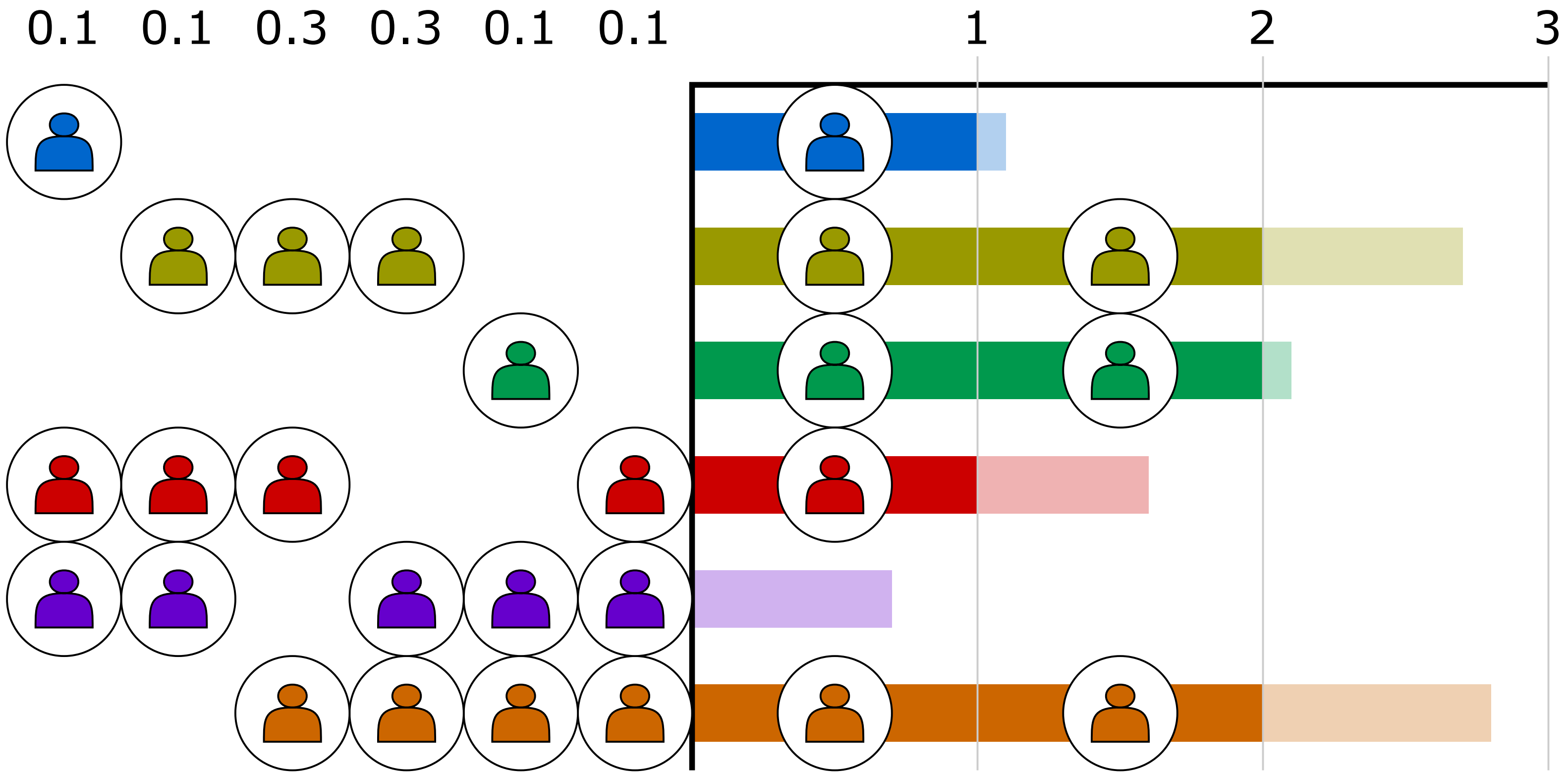

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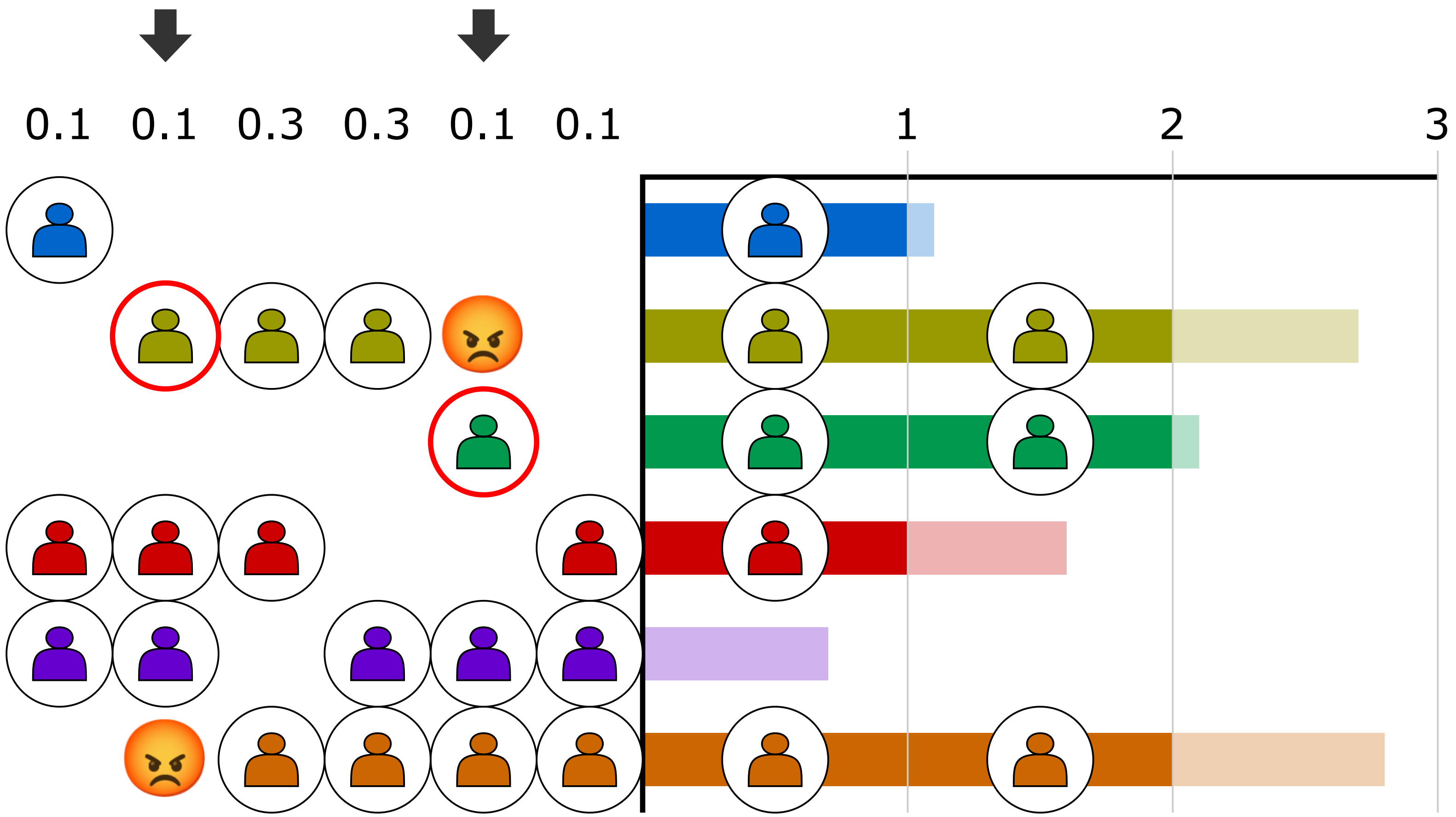

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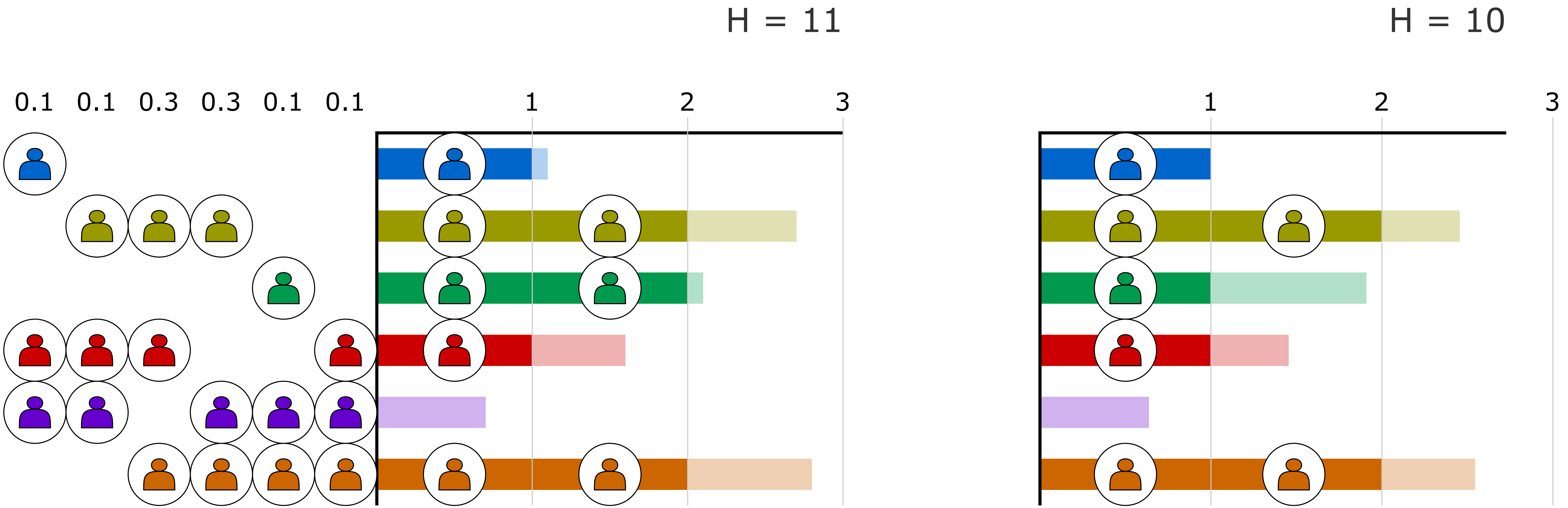

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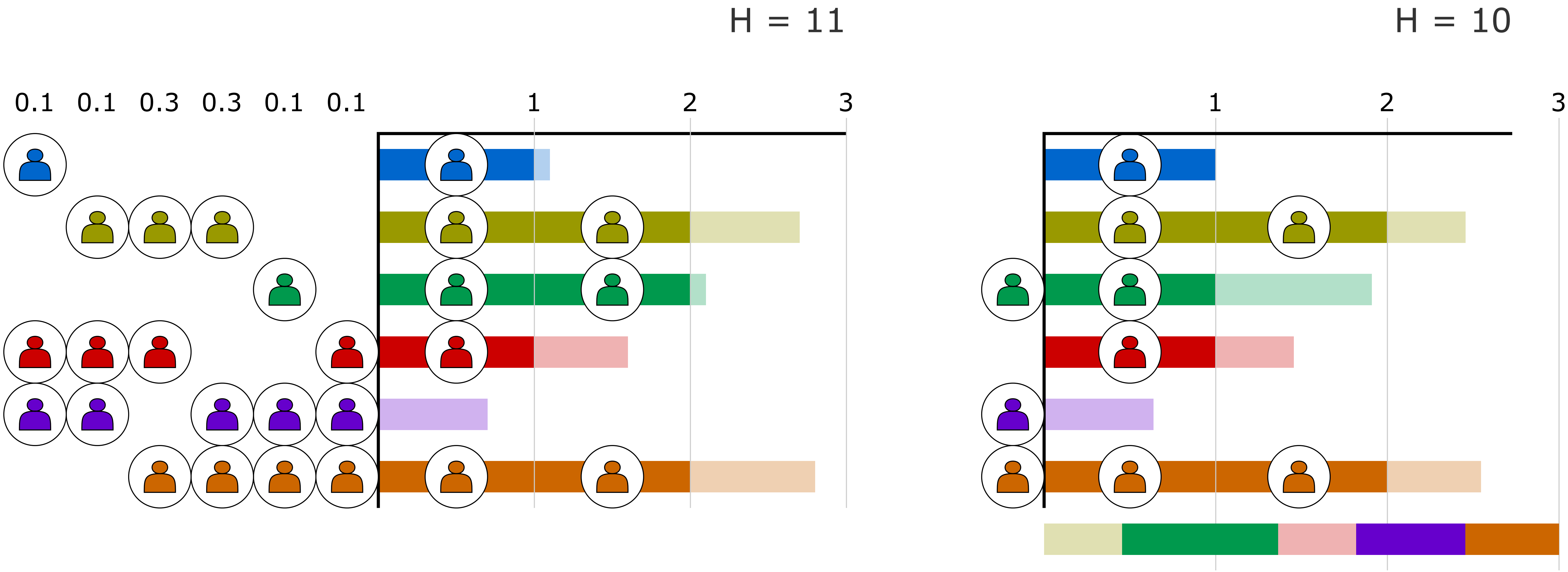

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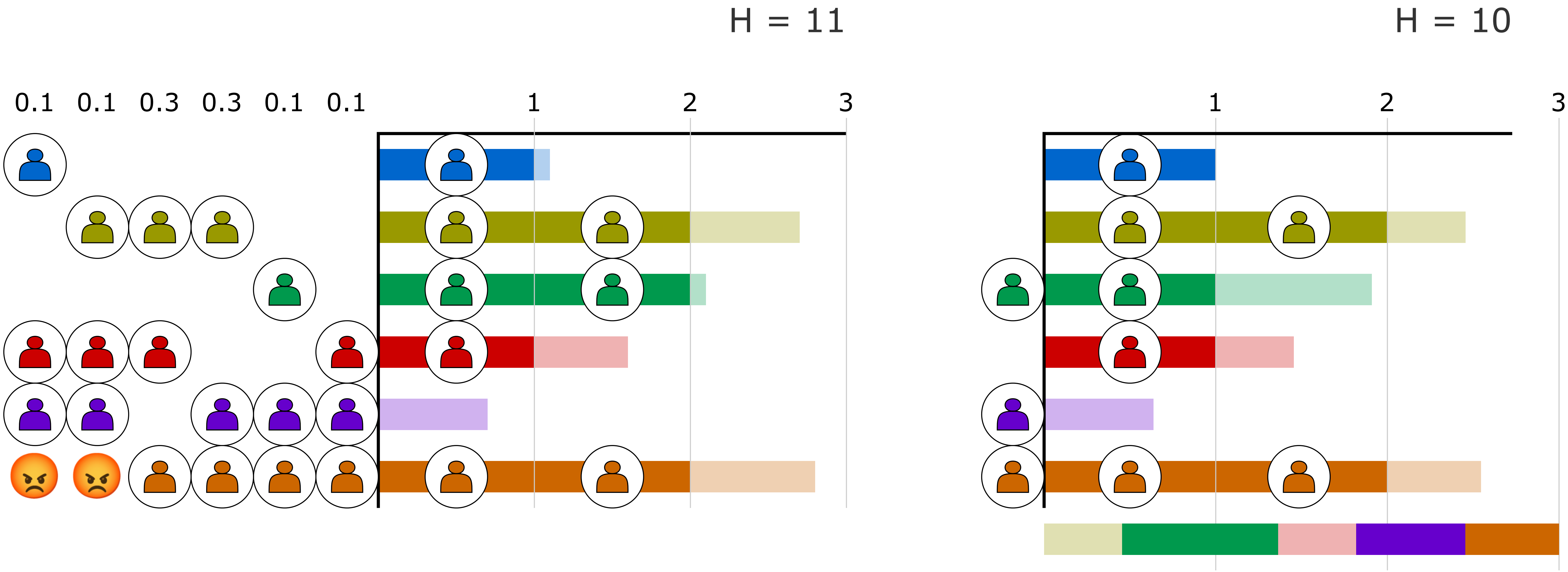

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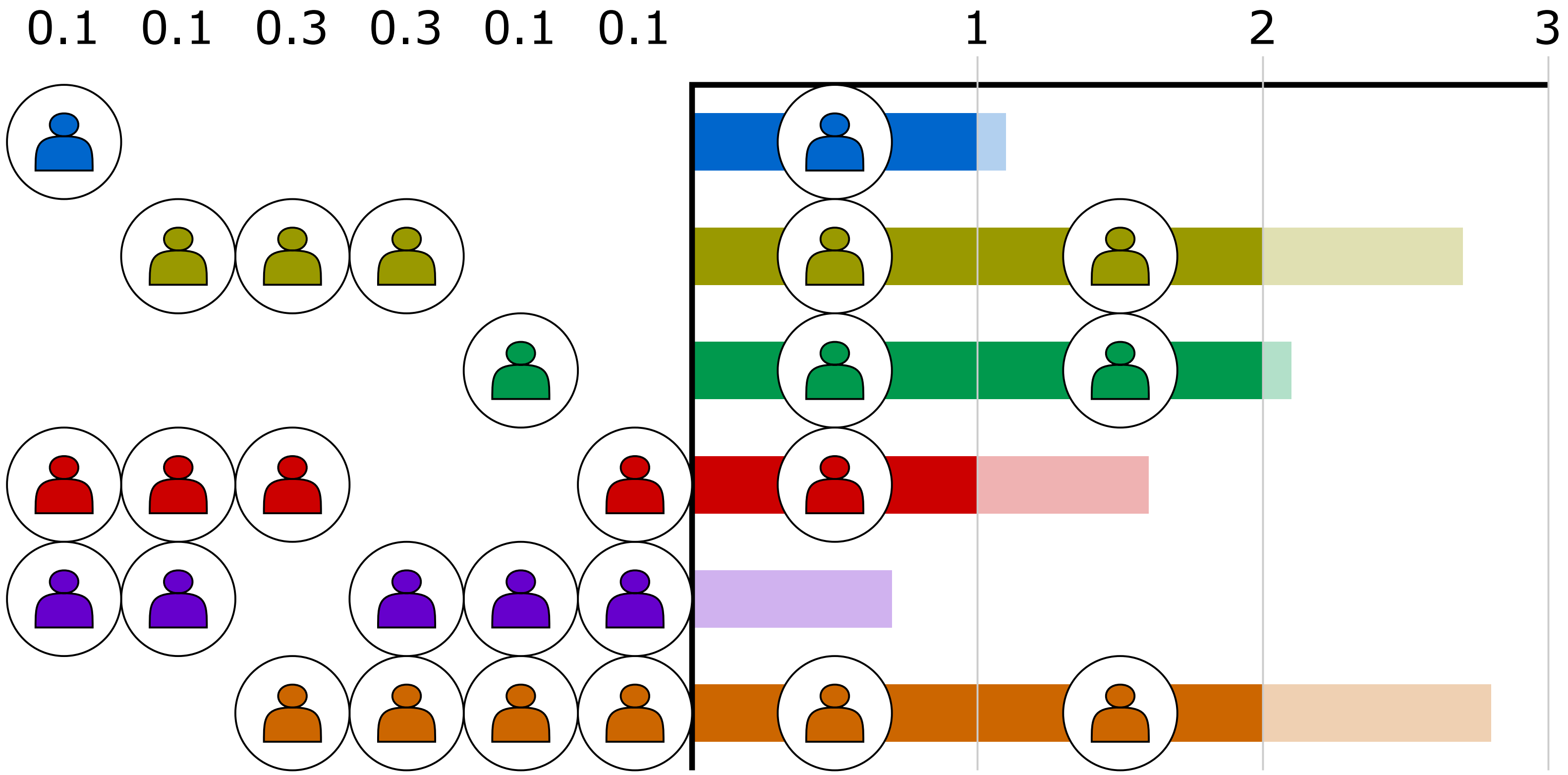

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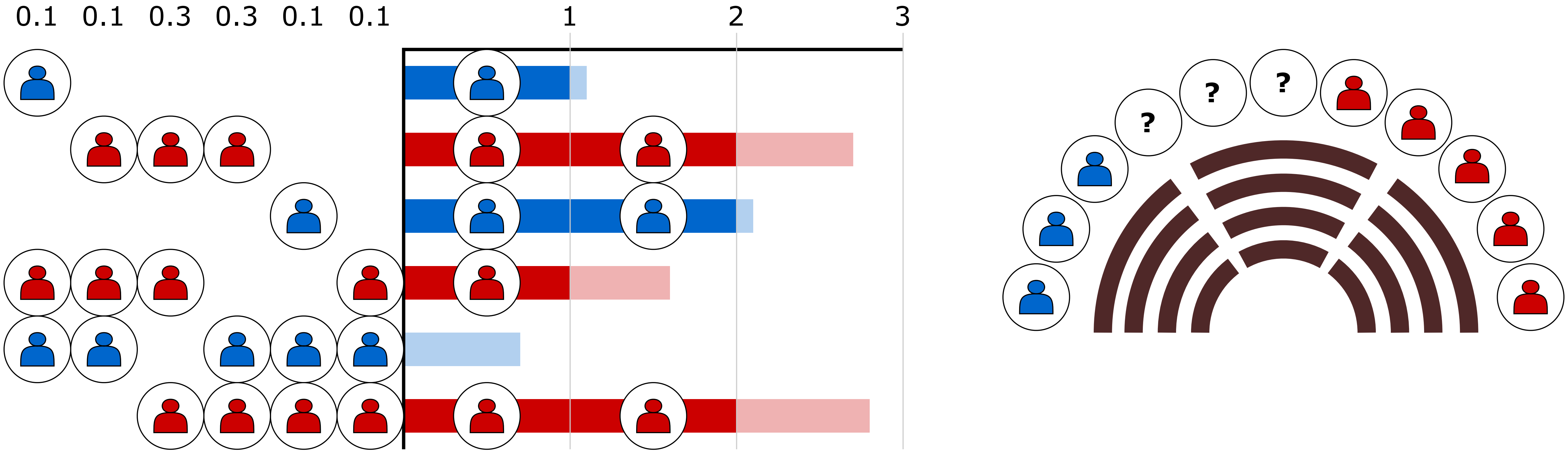

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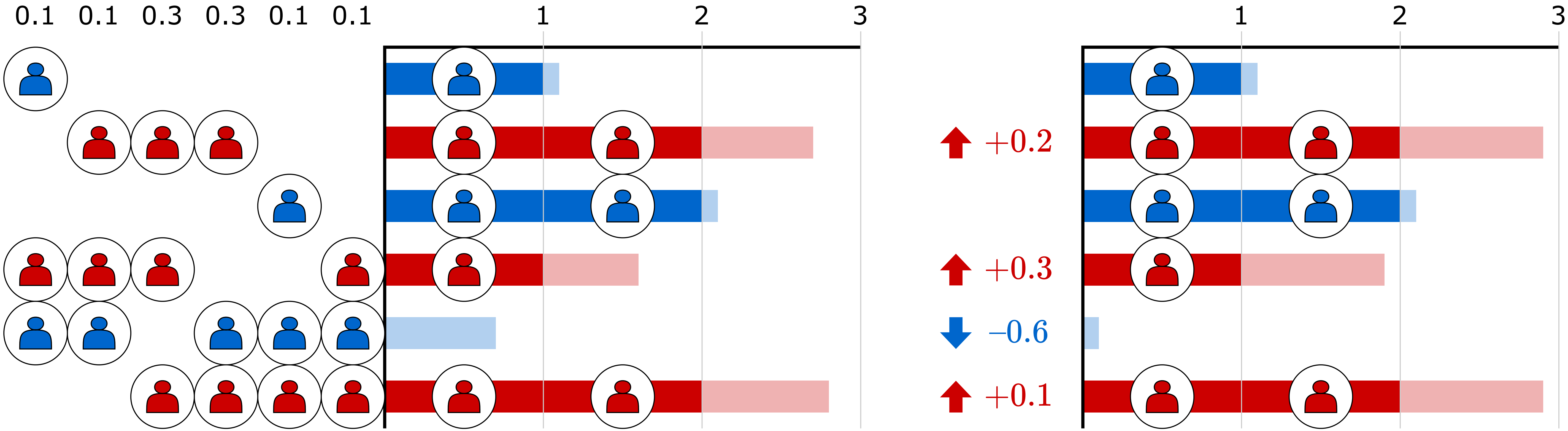

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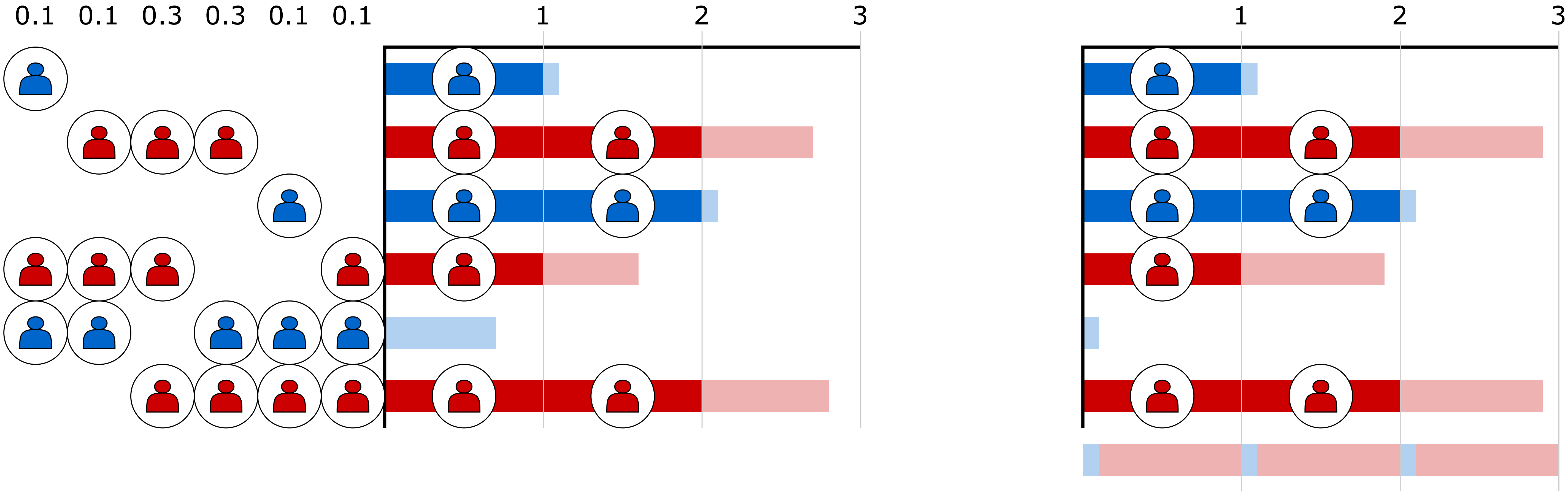

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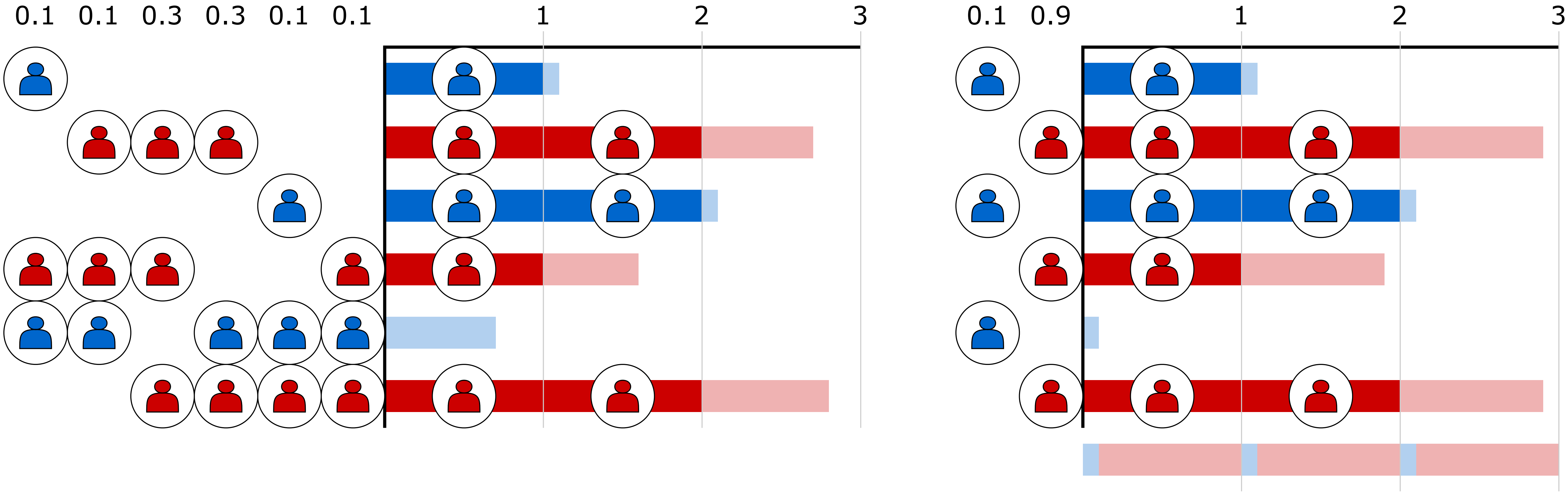

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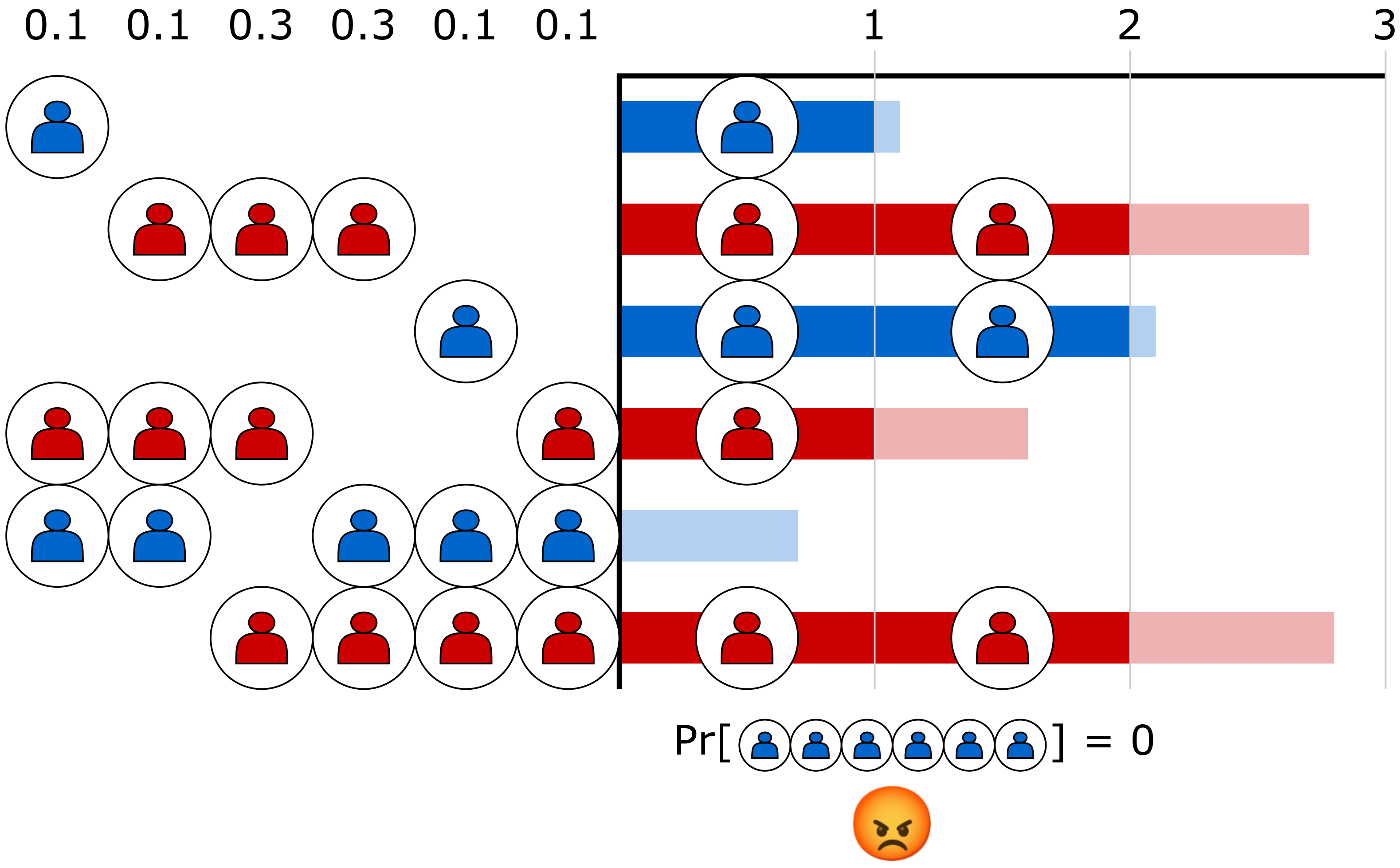

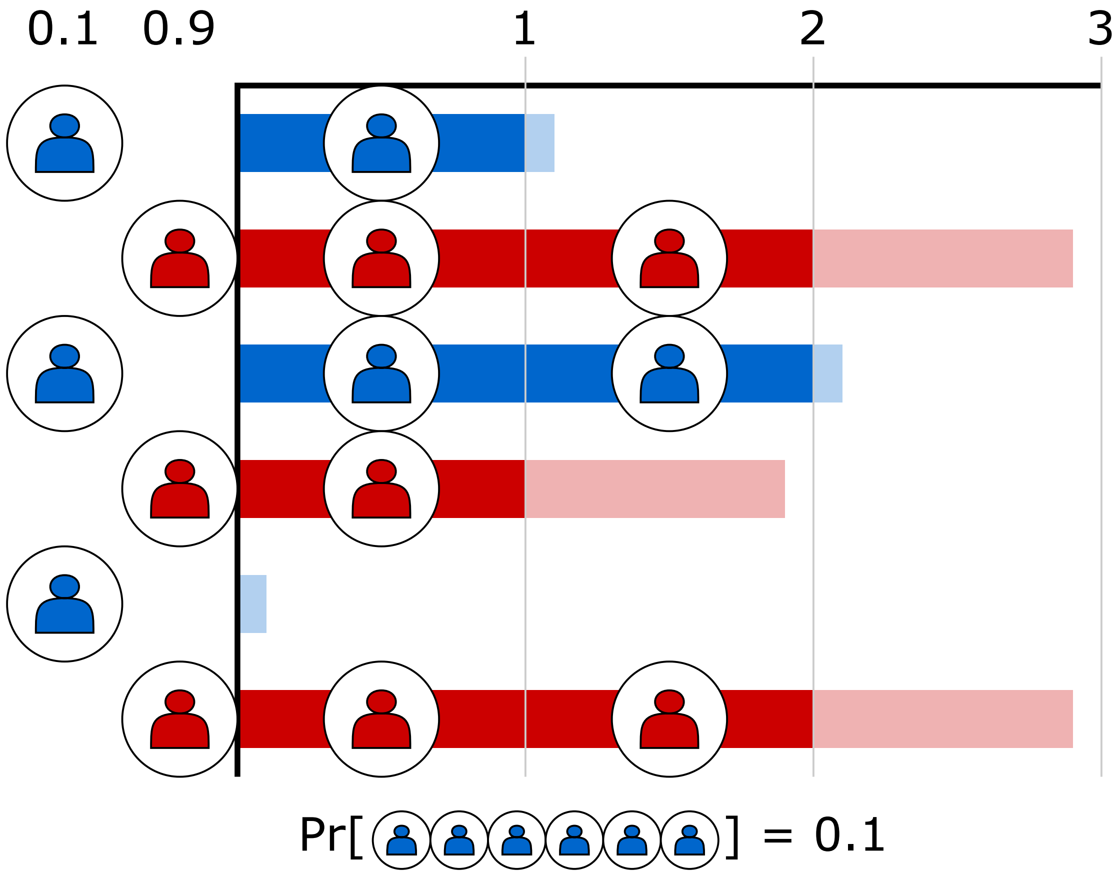

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   > Actually, it's possible to compute a single lottery over rules that are each house-monotone ex-post.
3. Correlations may cause non-monotonicities between joint probabilities of coalition members being rounded up.
   > An old method called Sampford rounding satisfies a weak monotonicity axiom, with impossibilities and conjectures for stronger variants.

Notation

Setup:

- n parties with standard quotas $q_1, q_2, \ldots, q_n \in \mathbb{R}_{\geq 0}$
- House size $H = \sum_{i=1}^{n} q_i \in \mathbb{Z}_{\geq 0}$
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Two kinds of apportionment rules, where the latter is a subclass of the former:

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$$(q_1, q_2, \ldots, q_n) \mapsto (x_1, x_2, \ldots, x_n)$$

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Note: Today, upper/lower quotas and ex ante proportionality are non-negotiable axioms!

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Problem 1: Envy (La Commare, T-F, working paper)

The *envy* of an apportionment $(x_1, x_2, \ldots, x_n)$ is the number of pairs $(i, j) \in [n]^2$ such that $q_i > q_j$ and $x_i < x_j$.

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Theorem

If all lower quotas are equal and residues are distinct, then the expected envy under any ex ante proportional apportionment method is exactly

$$\frac{1}{2}k(n-k) + \frac{1}{2}\sum_{t=1}^{n}(n+1-2t)r_t.$$

Proof of envy equivalence theorem

Proof. Order residues $r_1 < r_2 < \ldots, r_n$. For $i > j$, let $e_{i,j} := \Pr[X_i = 0 \wedge X_j = 1]$. Then the total expected envy is $\sum_{i>j} e_{i,j}$.

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Suppose there are $n = 4$ states, with $H = 10$ and quotas (1.4, 1.8, 3.2, 3.6). Is there an envy-free apportionment?

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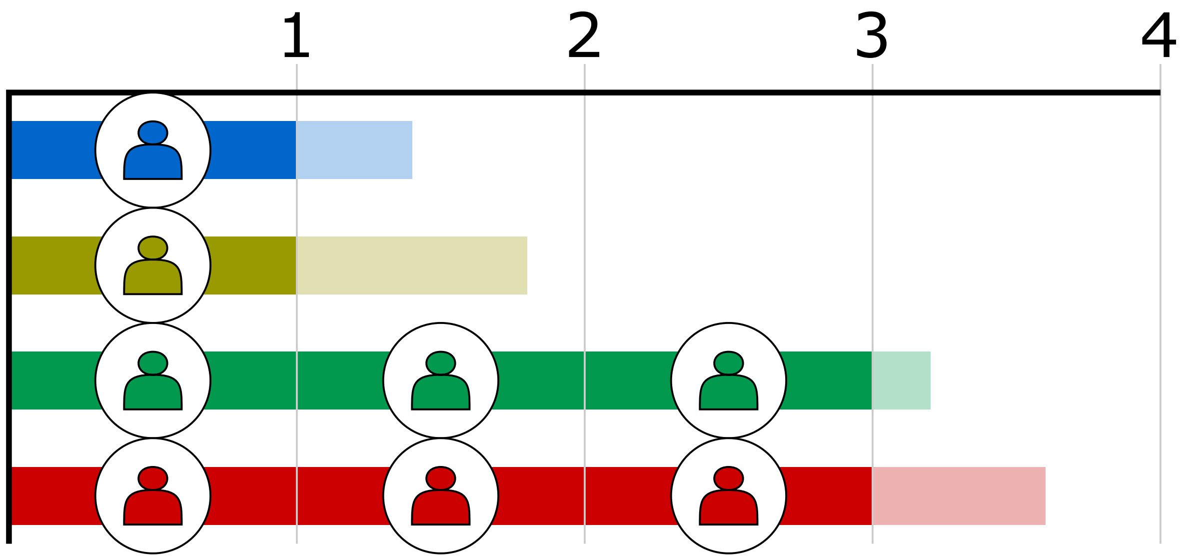

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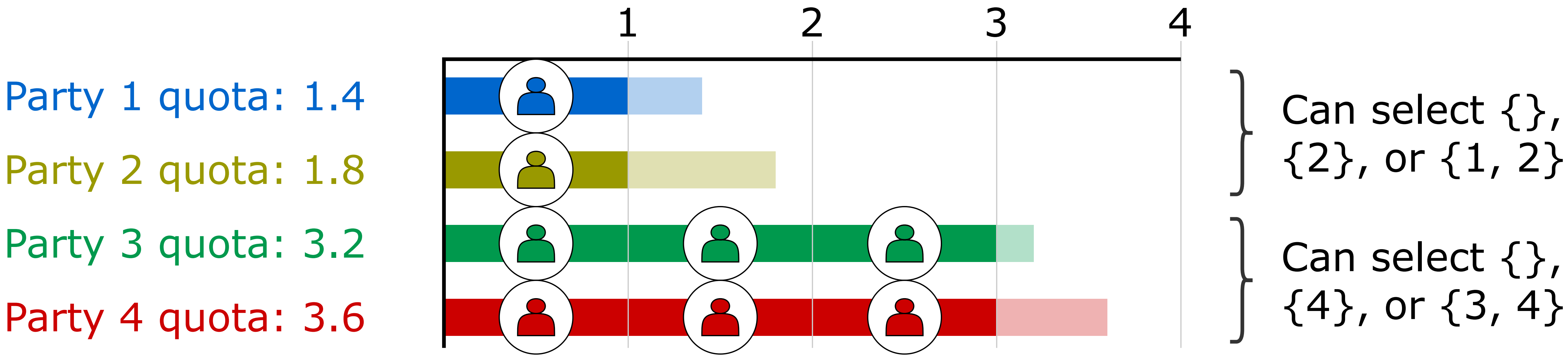

When lower quotas are different

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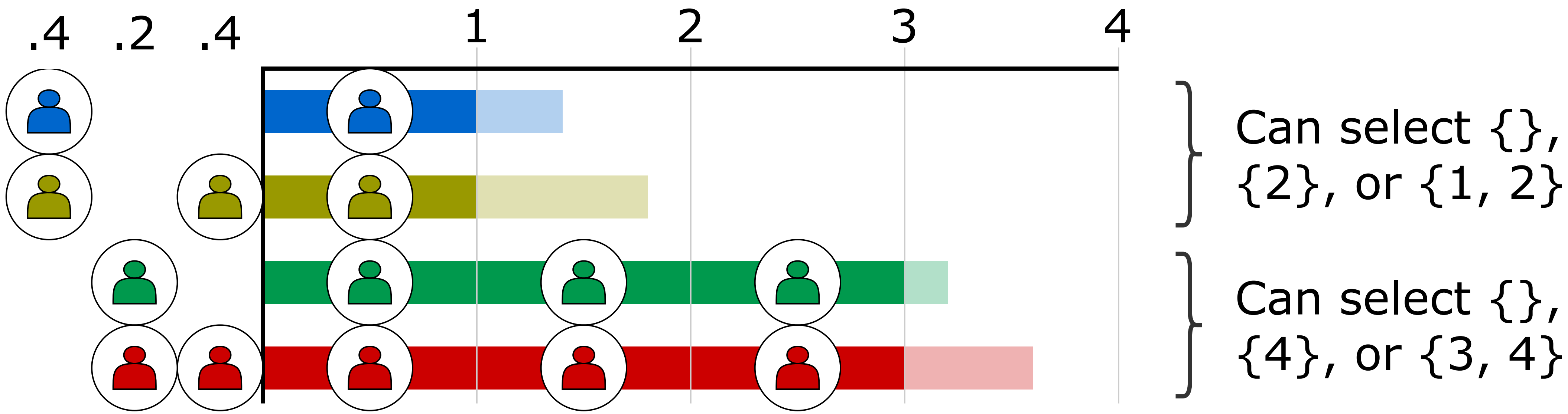

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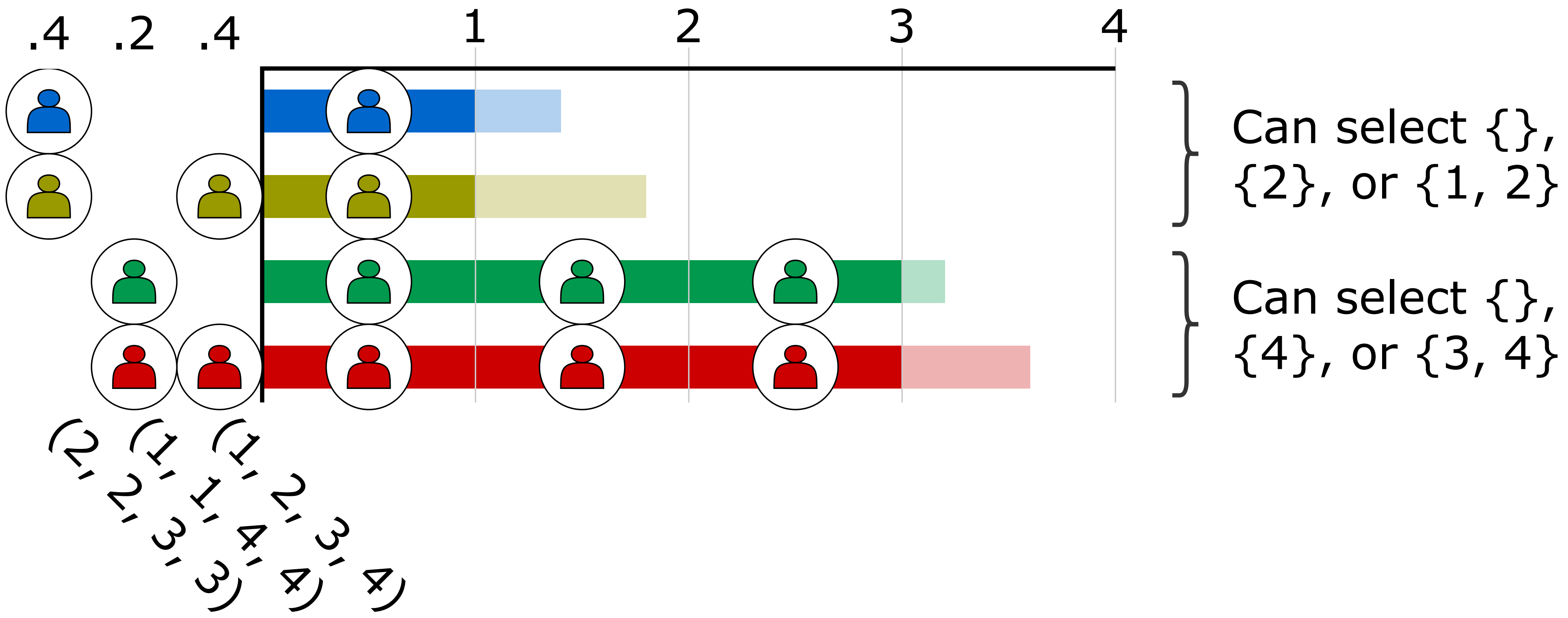

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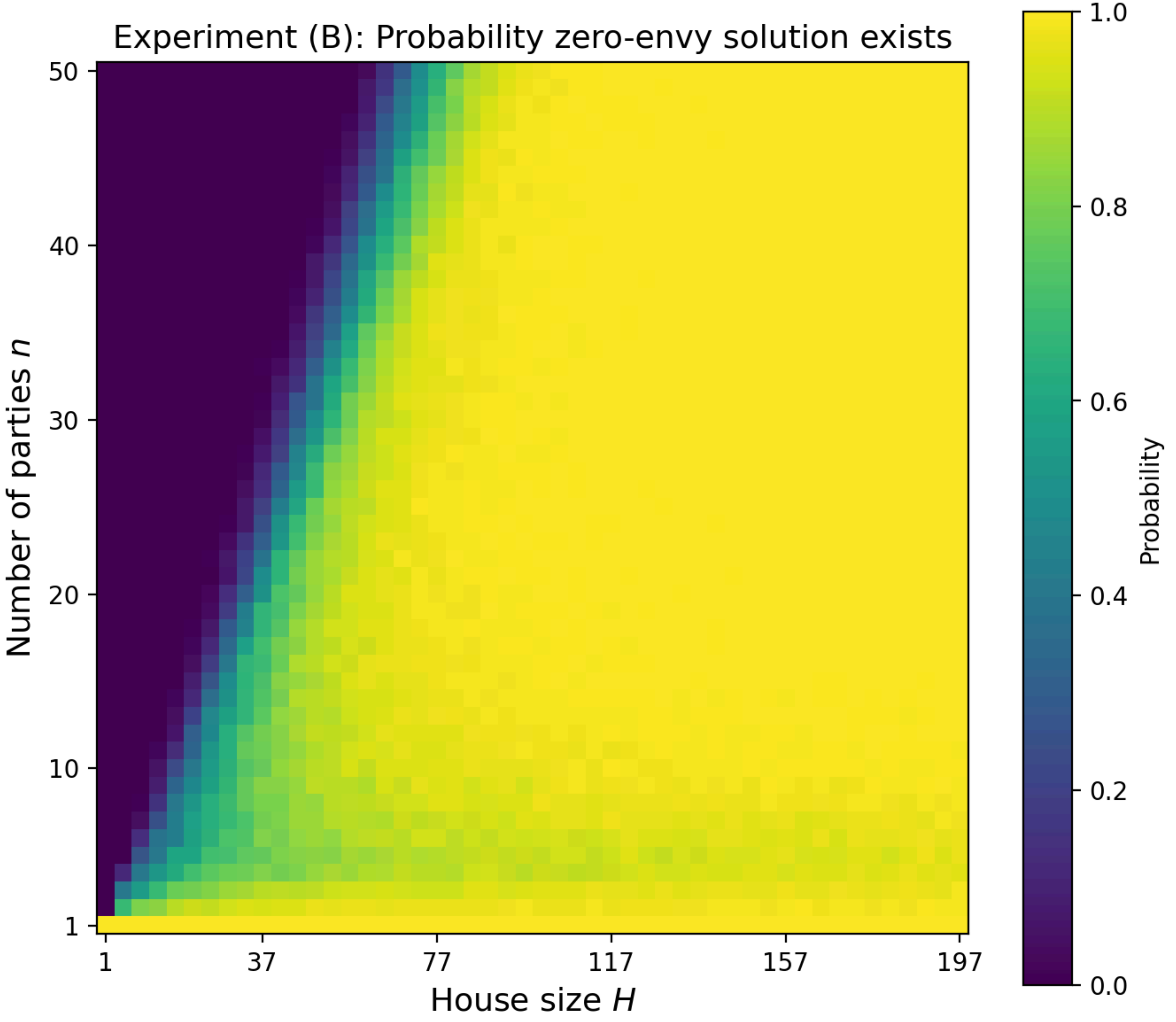
Theorem

In every decennial census in the USA from 1790 to 2020, an envy-free randomized apportionment exists.

Experiments with uniformly random populations



Problem 2: Ex post house monotonicity (Gölz, Peters, Procaccia, 2025)

H = 2

Party 1 population: 100

Party 2 population: 200

Party 3 population: 100

Party 4 population: 200

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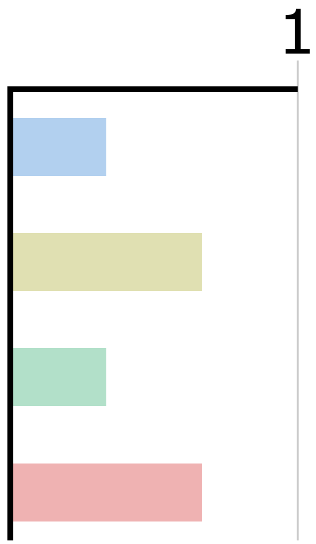

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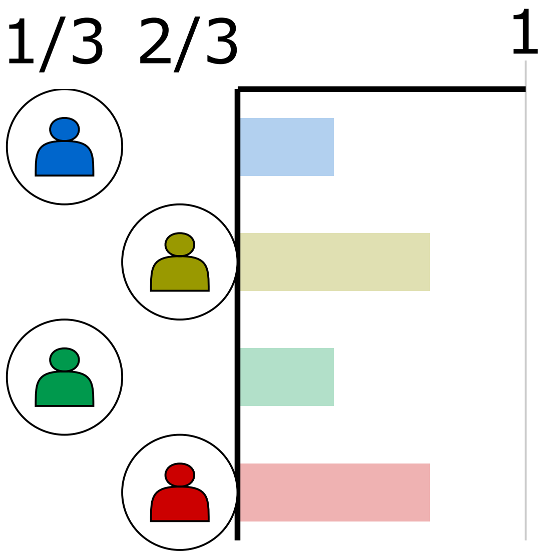

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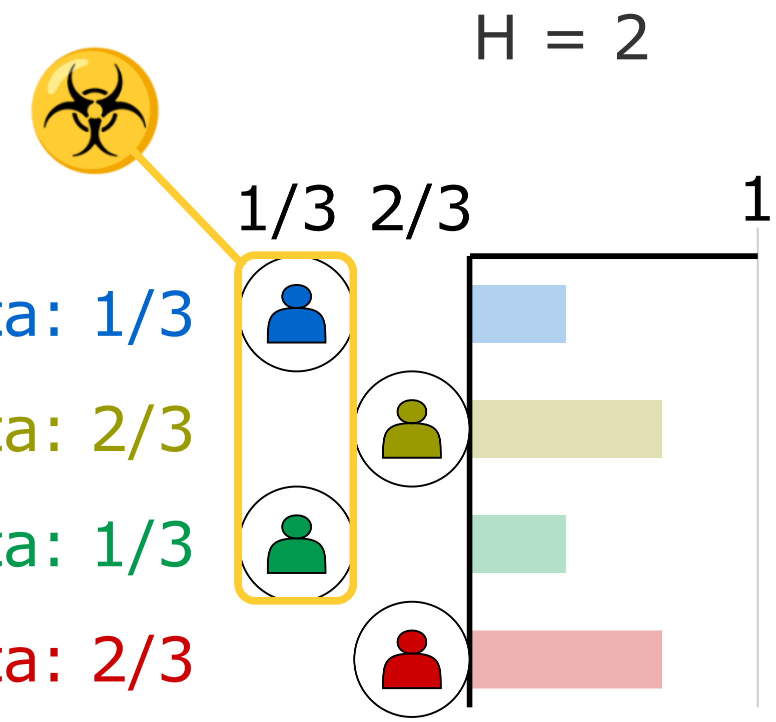

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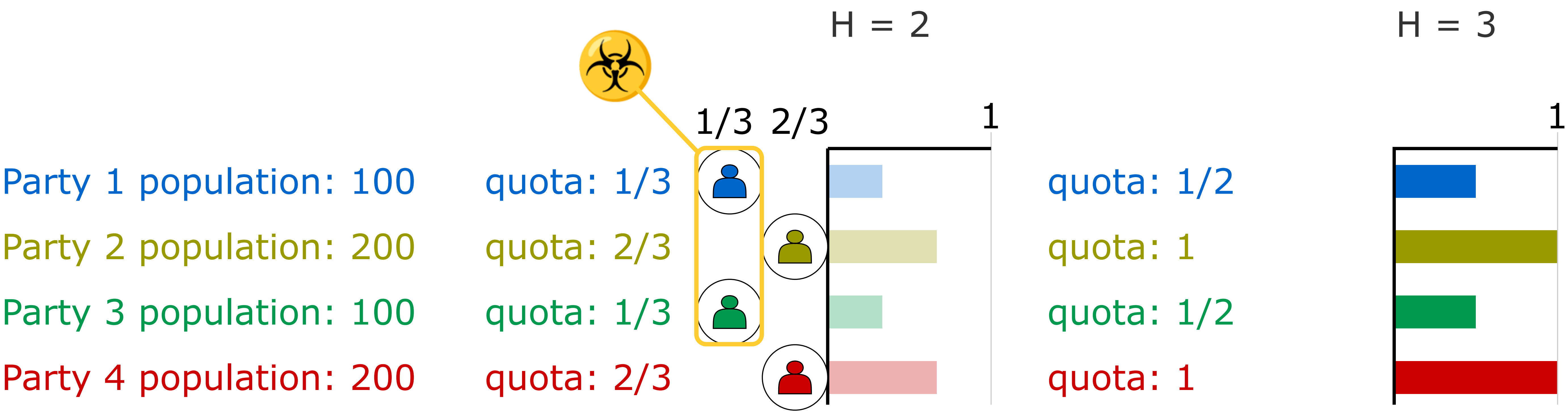

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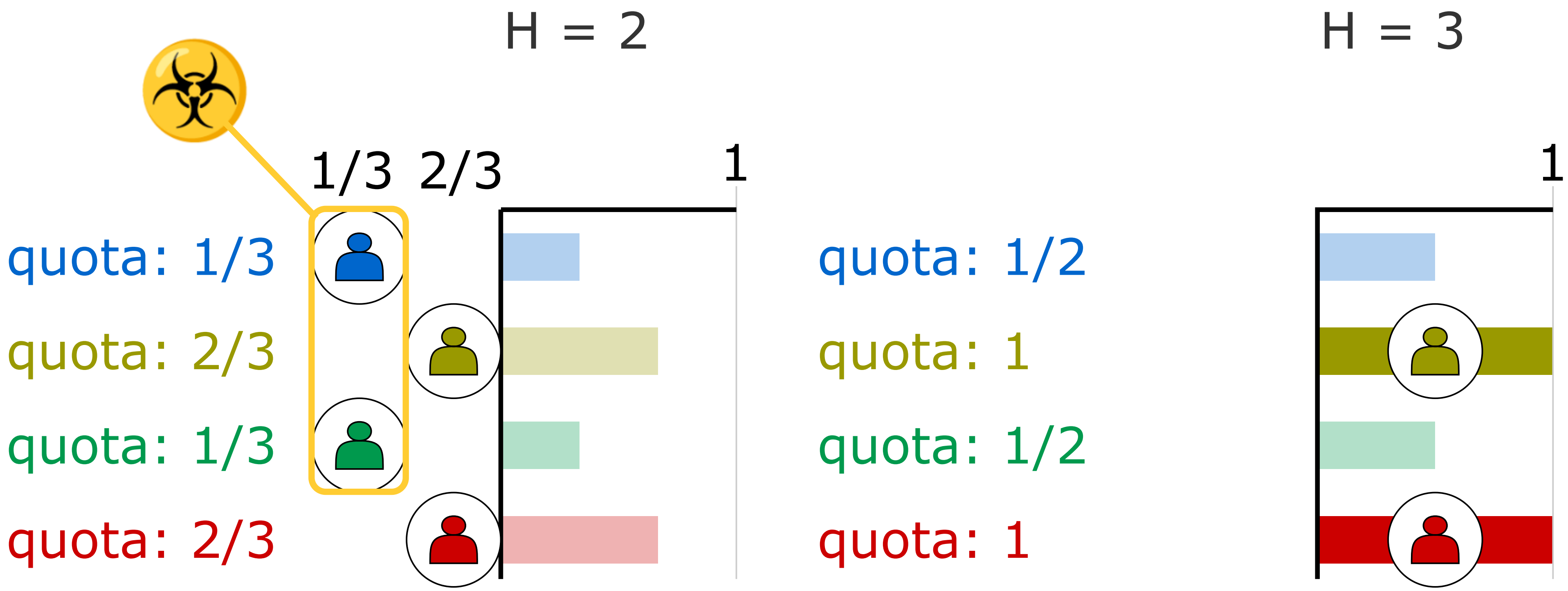

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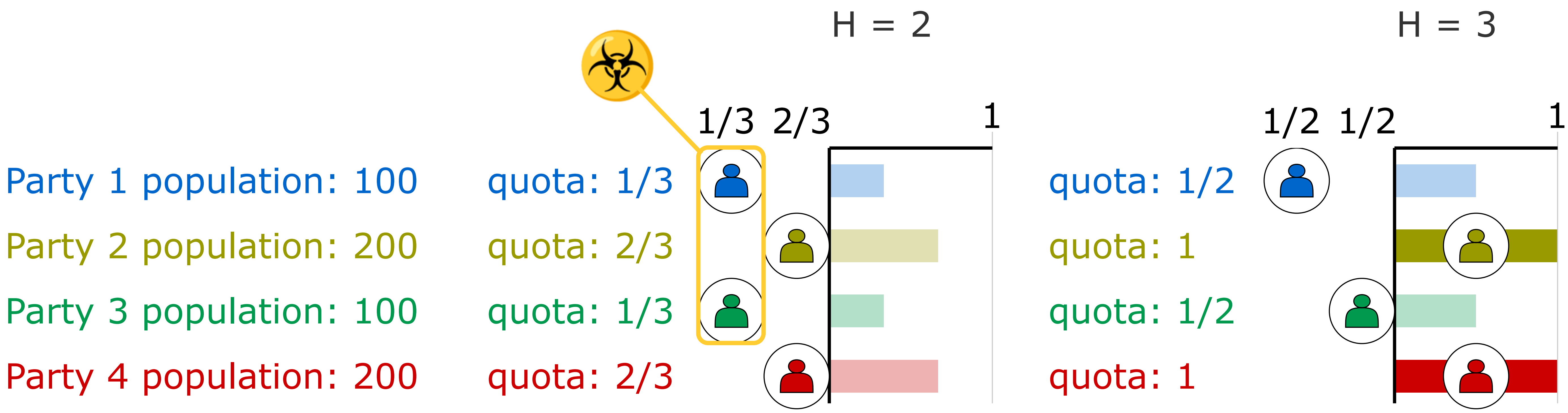

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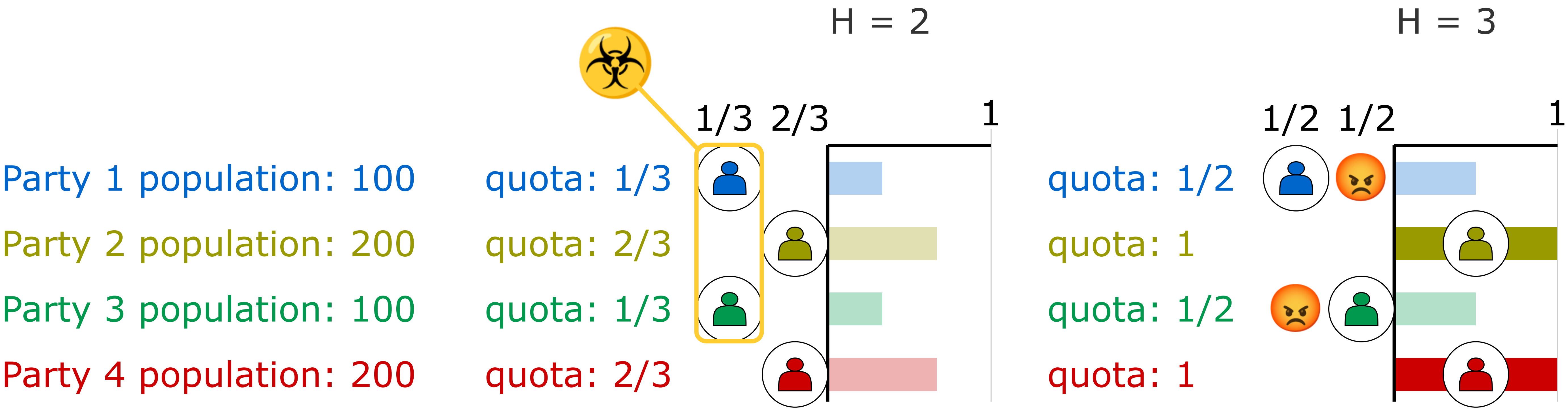

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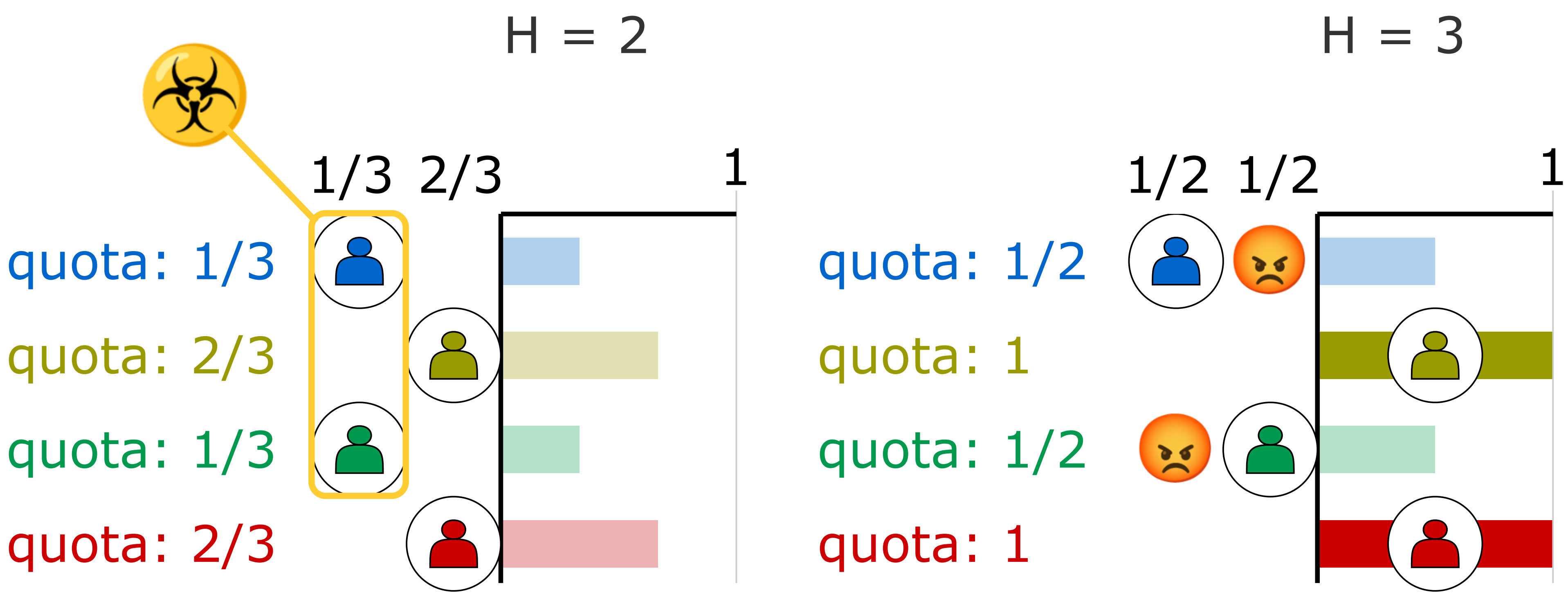

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Theorem

There exists an ex ante proportional lottery over quota-compliant, house-monotone deterministic apportionment rules, which can be computed in polynomial-time for each house size H.

Problem 3: Coalitions (Correa, Gölz, Schmidt-Kraepelin, T-F, Verdugo, 2026)

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We say that f satisfies *selection monotonicity* if, for any pair of residue profiles r and r', and any $T \subseteq [n]$, if $r'_i \geq r_i$ for all $i \in T$ and $r'_i \leq r_i$ for all $i \notin T$, then

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Theorem

Grimmett's method, the two new algorithms we just talked about, and a whole bunch of previously studied methods from the literature on dependent rounding all fail selection monotonicity.

Our hero (?): Conditional Poisson rounding

Find values $\pi_1, \pi_2, \ldots, \pi_n$ and then select each set S of size k with probability $\prod_{i \in S} \pi_i$.

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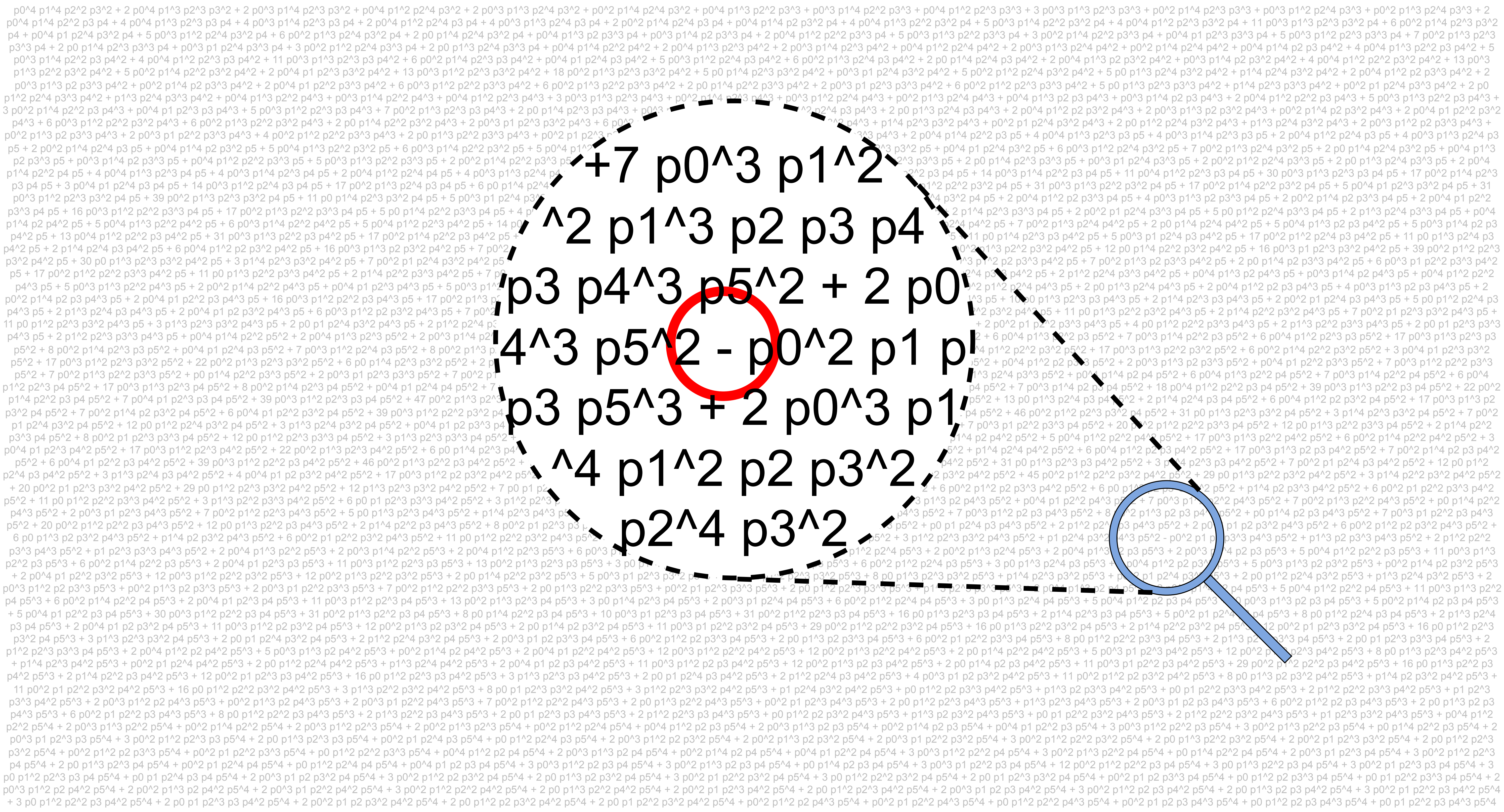
For $n = 6$, $k = 3$, Mathematica told me that selection monotonicity holds if the following polynomial is always positive...

p0^4 p1^4 p2^2 p3^2 + 2 p0^4 p1^3 p2^3 p3^2 + 2 p0^3 p1^4 p2^3 p3^2 + p0^4 p1^2 p2^4 p3^2 + 2 p0^3 p1^3 p2^4 p3^2 + p0^2 p1^4 p2^4 p3^2 + p0^4 p1^3 p2^2 p3^3 + p0^3 p1^4 p2^2 p3^3 + p0^4 p1^2 p2^3 p3^3 + 3 p0^3 p1^3 p2^3 p3^3 + p0^2 p1^4 p2^3 p3^3 + p0^3 p1^2 p2^4 p3^3 + p0^2 p1^3 p2^4 p3^3 + 2
p0^4 p1^4 p2^2 p3 p4 + 4 p0^4 p1^3 p2^3 p3 p4 + 4 p0^3 p1^4 p2^3 p3 p4 + 2 p0^4 p1^2 p2^4 p3 p4 + 4 p0^3 p1^3 p2^4 p3 p4 + 2 p0^2 p1^4 p2^4 p3 p4 + p0^4 p1^4 p2 p3^2 p4 + 4 p0^4 p1^3 p2^2 p3^2 p4 + 5 p0^3 p1^4 p2^2 p3^2 p4 + 4 p0^4 p1^2 p2^3 p3^2 p4 + 11 p0^3 p1^3 p2^3 p3^2 p4 + 6 p0^2 p1^4 p2^3 p3^2
p4 + p0^4 p1 p2^4 p3^2 p4 + 5 p0^3 p1^2 p2^4 p3^2 p4 + 6 p0^2 p1^3 p2^4 p3^2 p4 + 2 p0 p1^4 p2^4 p3^2 p4 + p0^4 p1^3 p2 p3^3 p4 + p0^3 p1^4 p2 p3^3 p4 + 2 p0^4 p1^2 p2^2 p3^3 p4 + 5 p0^3 p1^3 p2^2 p3^3 p4 + 3 p0^2 p1^4 p2^2 p3^3 p4 + p0^4 p1 p2^3 p3^3 p4 + 5 p0^3 p1^2 p2^3 p3^3 p4 + 7 p0^2 p1^3 p2^3
p3^3 p4 + 2 p0 p1^4 p2^3 p3^3 p4 + p0^3 p1 p2^4 p3^3 p4 + 3 p0^2 p1^2 p2^4 p3^3 p4 + 2 p0 p1^3 p2^4 p3^3 p4 + p0^4 p1^4 p2^2 p4^2 + 2 p0^4 p1^3 p2^3 p4^2 + 2 p0^3 p1^4 p2^3 p4^2 + p0^4 p1^2 p2^4 p4^2 + 2 p0^3 p1^3 p2^4 p4^2 + p0^2 p1^4 p2^4 p4^2 + p0^4 p1^4 p2 p3 p4^2 + 4 p0^4 p1^3 p2^2 p3 p4^2 + 5
p0^3 p1^4 p2^2 p3 p4^2 + 4 p0^4 p1^2 p2^3 p3 p4^2 + 11 p0^3 p1^3 p2^3 p3 p4^2 + 6 p0^2 p1^4 p2^3 p3 p4^2 + p0^4 p1 p2^4 p3 p4^2 + 5 p0^3 p1^2 p2^4 p3 p4^2 + 6 p0^2 p1^3 p2^4 p3 p4^2 + 2 p0 p1^4 p2^4 p3 p4^2 + 2 p0^4 p1^3 p2 p3^2 p4^2 + p0^3 p1^4 p2 p3^2 p4^2 + 4 p0^4 p1^2 p2^2 p3^2 p4^2 + 13 p0^3
p1^3 p2^2 p3^2 p4^2 + 5 p0^2 p1^4 p2^2 p3^2 p4^2 + 2 p0^4 p1 p2^3 p3^2 p4^2 + 13 p0^3 p1^2 p2^3 p3^2 p4^2 + 18 p0^2 p1^3 p2^3 p3^2 p4^2 + 5 p0 p1^4 p2^3 p3^2 p4^2 + p0^3 p1 p2^4 p3^2 p4^2 + 5 p0^2 p1^2 p2^4 p3^2 p4^2 + 5 p0 p1^3 p2^4 p3^2 p4^2 + p1^4 p2^4 p3^2 p4^2 + 2 p0^4 p1^2 p2 p3^3 p4^2 + 2
p0^3 p1^3 p2 p3^3 p4^2 + p0^2 p1^4 p2 p3^3 p4^2 + 2 p0^4 p1 p2^2 p3^3 p4^2 + 6 p0^3 p1^2 p2^2 p3^3 p4^2 + 6 p0^2 p1^3 p2^2 p3^3 p4^2 + 2 p0 p1^4 p2^2 p3^3 p4^2 + 2 p0^3 p1 p2^3 p3^3 p4^2 + 6 p0^2 p1^2 p2^3 p3^3 p4^2 + 5 p0 p1^3 p2^3 p3^3 p4^2 + p1^4 p2^3 p3^3 p4^2 + p0^2 p1 p2^4 p3^3 p4^2 + 2 p0
p1^2 p2^4 p3^3 p4^2 + p1^3 p2^4 p3^3 p4^2 + p0^4 p1^3 p2^2 p4^3 + p0^3 p1^4 p2^2 p4^3 + p0^4 p1^2 p2^3 p4^3 + 3 p0^3 p1^3 p2^3 p4^3 + p0^2 p1^4 p2^3 p4^3 + p0^3 p1^2 p2^4 p4^3 + p0^2 p1^3 p2^4 p4^3 + p0^4 p1^3 p2 p3 p4^3 + p0^3 p1^4 p2 p3 p4^3 + 2 p0^4 p1^2 p2^2 p3 p4^3 + 5 p0^3 p1^3 p2^2 p3 p4^3 +
3 p0^2 p1^4 p2^2 p3 p4^3 + p0^4 p1 p2^3 p3 p4^3 + 5 p0^3 p1^2 p2^3 p3 p4^3 + 7 p0^2 p1^3 p2^3 p3 p4^3 + 2 p0 p1^4 p2^3 p3 p4^3 + p0^3 p1 p2^4 p3 p4^3 + 3 p0^2 p1^2 p2^4 p3 p4^3 + 2 p0 p1^3 p2^4 p3 p4^3 + 2 p0^4 p1^2 p2 p3^2 p4^3 + 2 p0^3 p1^3 p2 p3^2 p4^3 + p0^2 p1^4 p2 p3^2 p4^3 + 2 p0^4 p1 p2^2 p3^2
p4^3 + 6 p0^3 p1^2 p2^2 p3^2 p4^3 + 6 p0^2 p1^3 p2^2 p3^2 p4^3 + 2 p0 p1^4 p2^2 p3^2 p4^3 + 2 p0^3 p1 p2^3 p3^2 p4^3 + 6 p0^2 p1^2 p2^3 p3^2 p4^3 + 5 p0 p1^3 p2^3 p3^2 p4^3 + p1^4 p2^3 p3^2 p4^3 + p0^2 p1 p2^4 p3^2 p4^3 + 2 p0 p1^2 p2^4 p3^2 p4^3 + p1^3 p2^4 p3^2 p4^3 + 2 p0^3 p1^2 p2 p3^3 p4^3 +
p0^2 p1^3 p2 p3^3 p4^3 + 2 p0^3 p1 p2^2 p3^3 p4^3 + 4 p0^2 p1^2 p2^2 p3^3 p4^3 + 2 p0 p1^3 p2^2 p3^3 p4^3 + p0^2 p1 p2^3 p3^3 p4^3 + 2 p0 p1^2 p2^3 p3^3 p4^3 + p1^3 p2^3 p3^3 p4^3 + 2 p0^4 p1^4 p2^2 p3 p5 + 4 p0^4 p1^3 p2^3 p3 p5 + 4 p0^3 p1^4 p2^3 p3 p5 + 2 p0^4 p1^2 p2^4 p3 p5 + 4 p0^3 p1^3 p2^4 p3
p5 + 2 p0^2 p1^4 p2^4 p3 p5 + p0^4 p1^4 p2 p3^2 p5 + 5 p0^4 p1^3 p2^2 p3^2 p5 + 6 p0^3 p1^4 p2^2 p3^2 p5 + 5 p0^4 p1^2 p2^3 p3^2 p5 + 14 p0^3 p1^3 p2^3 p3^2 p5 + 7 p0^2 p1^4 p2^3 p3^2 p5 + p0^4 p1 p2^4 p3^2 p5 + 6 p0^3 p1^2 p2^4 p3^2 p5 + 7 p0^2 p1^3 p2^4 p3^2 p5 + 2 p0 p1^4 p2^4 p3^2 p5 + p0^4 p1^3
p2 p3^3 p5 + p0^3 p1^4 p2 p3^3 p5 + p0^4 p1^2 p2^2 p3^3 p5 + 5 p0^3 p1^3 p2^2 p3^3 p5 + 2 p0^2 p1^4 p2^2 p3^3 p5 + p0^4 p1 p2^3 p3^3 p5 + 5 p0^3 p1^2 p2^3 p3^3 p5 + 7 p0^2 p1^3 p2^3 p3^3 p5 + 2 p0 p1^4 p2^3 p3^3 p5 + p0^3 p1 p2^4 p3^3 p5 + 2 p0^2 p1^2 p2^4 p3^3 p5 + 2 p0 p1^3 p2^4 p3^3 p5 + 2 p0^4
p1^4 p2^2 p4 p5 + 4 p0^4 p1^3 p2^3 p4 p5 + 4 p0^3 p1^4 p2^3 p4 p5 + 2 p0^4 p1^2 p2^4 p4 p5 + 4 p0^3 p1^3 p2^4 p4 p5 + 2 p0^2 p1^4 p2^4 p4 p5 + 3 p0^4 p1^4 p2 p3 p4 p5 + 11 p0^4 p1^3 p2^2 p3 p4 p5 + 14 p0^3 p1^4 p2^2 p3 p4 p5 + 11 p0^4 p1^2 p2^3 p3 p4 p5 + 30 p0^3 p1^3 p2^3 p3 p4 p5 + 17 p0^2 p1^4 p2^3
p3 p4 p5 + 3 p0^4 p1 p2^4 p3 p4 p5 + 14 p0^3 p1^2 p2^4 p3 p4 p5 + 17 p0^2 p1^3 p2^4 p3 p4 p5 + 6 p0 p1^4 p2^4 p3 p4 p5 + 5 p0^4 p1^3 p2 p3^2 p4 p5 + 5 p0^3 p1^4 p2 p3^2 p4 p5 + 13 p0^4 p1^2 p2^2 p3^2 p4 p5 + 31 p0^3 p1^3 p2^2 p3^2 p4 p5 + 17 p0^2 p1^4 p2^2 p3^2 p4 p5 + 5 p0^4 p1 p2^3 p3^2 p4 p5 + 31
p0^3 p1^2 p2^3 p3^2 p4 p5 + 39 p0^2 p1^3 p2^3 p3^2 p4 p5 + 11 p0 p1^4 p2^3 p3^2 p4 p5 + 5 p0^3 p1 p2^4 p3^2 p4 p5 + 17 p0^2 p1^2 p2^4 p3^2 p4 p5 + 11 p0 p1^3 p2^4 p3^2 p4 p5 + 2 p1^4 p2^4 p3^2 p4 p5 + 2 p0^4 p1^2 p2 p3^3 p4 p5 + 4 p0^3 p1^3 p2 p3^3 p4 p5 + 2 p0^2 p1^4 p2 p3^3 p4 p5 + 2 p0^4 p1 p2^2
p3^3 p4 p5 + 16 p0^3 p1^2 p2^2 p3^3 p4 p5 + 17 p0^2 p1^3 p2^2 p3^3 p4 p5 + 5 p0 p1^4 p2^2 p3^3 p4 p5 + 4 p0^3 p1 p2^3 p3^3 p4 p5 + 17 p0^2 p1^2 p2^3 p3^3 p4 p5 + 10 p0 p1^3 p2^3 p3^3 p4 p5 + 2 p1^4 p2^3 p3^3 p4 p5 + 2 p0^2 p1 p2^4 p3^3 p4 p5 + 5 p0 p1^2 p2^4 p3^3 p4 p5 + 2 p1^3 p2^4 p3^3 p4 p5 + p0^4
p1^4 p2 p4^2 p5 + 5 p0^4 p1^3 p2^2 p4^2 p5 + 6 p0^3 p1^4 p2^2 p4^2 p5 + 5 p0^4 p1^2 p2^3 p4^2 p5 + 14 p0^3 p1^3 p2^3 p4^2 p5 + 7 p0^2 p1^4 p2^3 p4^2 p5 + p0^4 p1 p2^4 p4^2 p5 + 6 p0^3 p1^2 p2^4 p4^2 p5 + 7 p0^2 p1^3 p2^4 p4^2 p5 + 2 p0 p1^4 p2^4 p4^2 p5 + 5 p0^4 p1^3 p2 p3 p4^2 p5 + 5 p0^3 p1^4 p2 p3
p4^2 p5 + 13 p0^4 p1^2 p2^2 p3 p4^2 p5 + 31 p0^3 p1^3 p2^2 p3 p4^2 p5 + 17 p0^2 p1^4 p2^2 p3 p4^2 p5 + 5 p0^4 p1 p2^3 p3 p4^2 p5 + 31 p0^3 p1^2 p2^3 p3 p4^2 p5 + 39 p0^2 p1^3 p2^3 p3 p4^2 p5 + 11 p0 p1^4 p2^3 p3 p4^2 p5 + 5 p0^3 p1 p2^4 p3 p4^2 p5 + 17 p0^2 p1^2 p2^4 p3 p4^2 p5 + 11 p0 p1^3 p2^4 p3
p4^2 p5 + 2 p1^4 p2^4 p3 p4^2 p5 + 6 p0^4 p1^2 p2 p3^2 p4^2 p5 + 16 p0^3 p1^3 p2 p3^2 p4^2 p5 + 7 p0^2 p1^4 p2 p3^2 p4^2 p5 + 6 p0^4 p1 p2^2 p3^2 p4^2 p5 + 32 p0^3 p1^2 p2^2 p3^2 p4^2 p5 + 39 p0^2 p1^3 p2^2 p3^2 p4^2 p5 + 12 p0 p1^4 p2^2 p3^2 p4^2 p5 + 16 p0^3 p1 p2^3 p3^2 p4^2 p5 + 39 p0^2 p1^2 p2^3
p3^2 p4^2 p5 + 30 p0 p1^3 p2^3 p3^2 p4^2 p5 + 3 p1^4 p2^3 p3^2 p4^2 p5 + 7 p0^2 p1 p2^4 p3^2 p4^2 p5 + 12 p0 p1^2 p2^4 p3^2 p4^2 p5 + 3 p1^3 p2^4 p3^2 p4^2 p5 + 2 p0^4 p1 p2 p3^3 p4^2 p5 + 6 p0^3 p1^2 p2 p3^3 p4^2 p5 + 7 p0^2 p1^3 p2 p3^3 p4^2 p5 + 2 p0 p1^4 p2 p3^3 p4^2 p5 + 6 p0^3 p1 p2^2 p3^3 p4^2
p5 + 17 p0^2 p1^2 p2^2 p3^3 p4^2 p5 + 11 p0 p1^3 p2^2 p3^3 p4^2 p5 + 2 p1^4 p2^2 p3^3 p4^2 p5 + 7 p0^2 p1 p2^3 p3^3 p4^2 p5 + 11 p0 p1^2 p2^3 p3^3 p4^2 p5 + 3 p1^3 p2^3 p3^3 p4^2 p5 + 2 p0 p1 p2^4 p3^3 p4^2 p5 + 2 p1^2 p2^4 p3^3 p4^2 p5 + p0^4 p1^3 p2 p4^3 p5 + p0^3 p1^4 p2 p4^3 p5 + p0^4 p1^2 p2^2
p4^3 p5 + 5 p0^3 p1^3 p2^2 p4^3 p5 + 2 p0^2 p1^4 p2^2 p4^3 p5 + p0^4 p1 p2^3 p4^3 p5 + 5 p0^3 p1^2 p2^3 p4^3 p5 + 7 p0^2 p1^3 p2^3 p4^3 p5 + 2 p0 p1^4 p2^3 p4^3 p5 + p0^3 p1 p2^4 p4^3 p5 + 2 p0^2 p1^2 p2^4 p4^3 p5 + 2 p0 p1^3 p2^4 p4^3 p5 + 2 p0^4 p1^2 p2 p3 p4^3 p5 + 4 p0^3 p1^3 p2 p3 p4^3 p5 + 2
p0^2 p1^4 p2 p3 p4^3 p5 + 2 p0^4 p1 p2^2 p3 p4^3 p5 + 16 p0^3 p1^2 p2^2 p3 p4^3 p5 + 17 p0^2 p1^3 p2^2 p3 p4^3 p5 + 5 p0 p1^4 p2^2 p3 p4^3 p5 + 4 p0^3 p1 p2^3 p3 p4^3 p5 + 17 p0^2 p1^2 p2^3 p3 p4^3 p5 + 10 p0 p1^3 p2^3 p3 p4^3 p5 + 2 p1^4 p2^3 p3 p4^3 p5 + 2 p0^2 p1 p2^4 p3 p4^3 p5 + 5 p0 p1^2 p2^4 p3
p4^3 p5 + 2 p1^3 p2^4 p3 p4^3 p5 + 2 p0^4 p1 p2 p3^2 p4^3 p5 + 6 p0^3 p1^2 p2 p3^2 p4^3 p5 + 7 p0^2 p1^3 p2 p3^2 p4^3 p5 + 2 p0 p1^4 p2 p3^2 p4^3 p5 + 6 p0^3 p1 p2^2 p3^2 p4^3 p5 + 17 p0^2 p1^2 p2^2 p3^2 p4^3 p5 + 11 p0 p1^3 p2^2 p3^2 p4^3 p5 + 2 p1^4 p2^2 p3^2 p4^3 p5 + 7 p0^2 p1 p2^3 p3^2 p4^3 p5 +
11 p0 p1^2 p2^3 p3^2 p4^3 p5 + 3 p1^3 p2^3 p3^2 p4^3 p5 + 2 p0 p1 p2^4 p3^2 p4^3 p5 + 2 p1^2 p2^4 p3^2 p4^3 p5 + 2 p0^3 p1 p2 p3^3 p4^3 p5 + 2 p0^2 p1^2 p2 p3^3 p4^3 p5 + 2 p0 p1^3 p2 p3^3 p4^3 p5 + 2 p0^2 p1 p2^2 p3^3 p4^3 p5 + 4 p0 p1^2 p2^2 p3^3 p4^3 p5 + 2 p1^3 p2^2 p3^3 p4^3 p5 + 2 p0 p1 p2^3 p3^3
p4^3 p5 + 2 p1^2 p2^3 p3^3 p4^3 p5 + p0^4 p1^4 p2^2 p5^2 + 2 p0^4 p1^3 p2^3 p5^2 + 2 p0^3 p1^4 p2^3 p5^2 + p0^4 p1^2 p2^4 p5^2 + 2 p0^3 p1^3 p2^4 p5^2 + p0^2 p1^4 p2^4 p5^2 + p0^4 p1^4 p2 p3 p5^2 + 6 p0^4 p1^3 p2^2 p3 p5^2 + 7 p0^3 p1^4 p2^2 p3 p5^2 + 6 p0^4 p1^2 p2^3 p3 p5^2 + 17 p0^3 p1^3 p2^3 p3
p5^2 + 8 p0^2 p1^4 p2^3 p3 p5^2 + p0^4 p1 p2^4 p3 p5^2 + 7 p0^3 p1^2 p2^4 p3 p5^2 + 8 p0^2 p1^3 p2^4 p3 p5^2 + 2 p0 p1^4 p2^4 p3 p5^2 + 3 p0^4 p1^3 p2 p3^2 p5^2 + 2 p0^3 p1^4 p2 p3^2 p5^2 + 5 p0^4 p1^2 p2^2 p3^2 p5^2 + 17 p0^3 p1^3 p2^2 p3^2 p5^2 + 6 p0^2 p1^4 p2^2 p3^2 p5^2 + 3 p0^4 p1 p2^3 p3^2
p5^2 + 17 p0^3 p1^2 p2^3 p3^2 p5^2 + 22 p0^2 p1^3 p2^3 p3^2 p5^2 + 6 p0 p1^4 p2^3 p3^2 p5^2 + 2 p0^3 p1 p2^4 p3^2 p5^2 + 6 p0^2 p1^2 p2^4 p3^2 p5^2 + 6 p0 p1^3 p2^4 p3^2 p5^2 + p1^4 p2^4 p3^2 p5^2 + p0^4 p1^2 p2 p3^3 p5^2 + 2 p0^3 p1^3 p2 p3^3 p5^2 + p0^4 p1 p2^2 p3^3 p5^2 + 7 p0^3 p1^2 p2^2 p3^3
p5^2 + 7 p0^2 p1^3 p2^2 p3^3 p5^2 + p0 p1^4 p2^2 p3^3 p5^2 + 2 p0^3 p1 p2^3 p3^3 p5^2 + 7 p0^2 p1^2 p2^3 p3^3 p5^2 + 5 p0 p1^3 p2^3 p3^3 p5^2 + p1^4 p2^3 p3^3 p5^2 + p0 p1^2 p2^4 p3^3 p5^2 + p1^3 p2^4 p3^3 p5^2 + p0^4 p1^4 p2 p4 p5^2 + 6 p0^4 p1^3 p2^2 p4 p5^2 + 7 p0^3 p1^4 p2^2 p4 p5^2 + 6 p0^4
p1^2 p2^3 p4 p5^2 + 17 p0^3 p1^3 p2^3 p4 p5^2 + 8 p0^2 p1^4 p2^3 p4 p5^2 + p0^4 p1 p2^4 p4 p5^2 + 7 p0^3 p1^2 p2^4 p4 p5^2 + 8 p0^2 p1^3 p2^4 p4 p5^2 + 2 p0 p1^4 p2^4 p4 p5^2 + 7 p0^4 p1^3 p2 p3 p4 p5^2 + 7 p0^3 p1^4 p2 p3 p4 p5^2 + 18 p0^4 p1^2 p2^2 p3 p4 p5^2 + 39 p0^3 p1^3 p2^2 p3 p4 p5^2 + 22 p0^2
p1^4 p2^2 p3 p4 p5^2 + 7 p0^4 p1 p2^3 p3 p4 p5^2 + 39 p0^3 p1^2 p2^3 p3 p4 p5^2 + 47 p0^2 p1^3 p2^3 p3 p4 p5^2 + 13 p0 p1^4 p2^3 p3 p4 p5^2 + 7 p0^3 p1 p2^4 p3 p4 p5^2 + 22 p0^2 p1^2 p2^4 p3 p4 p5^2 + 13 p0 p1^3 p2^4 p3 p4 p5^2 + 2 p1^4 p2^4 p3 p4 p5^2 + 6 p0^4 p1^2 p2 p3^2 p4 p5^2 + 17 p0^3 p1^3 p2
p3^2 p4 p5^2 + 7 p0^2 p1^4 p2 p3^2 p4 p5^2 + 6 p0^4 p1 p2^2 p3^2 p4 p5^2 + 39 p0^3 p1^2 p2^2 p3^2 p4 p5^2 + 46 p0^2 p1^3 p2^2 p3^2 p4 p5^2 + 12 p0 p1^4 p2^2 p3^2 p4 p5^2 + 17 p0^3 p1 p2^3 p3^2 p4 p5^2 + 46 p0^2 p1^2 p2^3 p3^2 p4 p5^2 + 31 p0 p1^3 p2^3 p3^2 p4 p5^2 + 3 p1^4 p2^3 p3^2 p4 p5^2 + 7 p0^2
p1 p2^4 p3^2 p4 p5^2 + 12 p0 p1^2 p2^4 p3^2 p4 p5^2 + 3 p1^3 p2^4 p3^2 p4 p5^2 + p0^4 p1 p2 p3^3 p4 p5^2 + 7 p0^3 p1^2 p2 p3^3 p4 p5^2 + 8 p0^2 p1^3 p2 p3^3 p4 p5^2 + p0 p1^4 p2 p3^3 p4 p5^2 + 7 p0^3 p1 p2^2 p3^3 p4 p5^2 + 20 p0^2 p1^2 p2^2 p3^3 p4 p5^2 + 12 p0 p1^3 p2^2 p3^3 p4 p5^2 + 2 p1^4 p2^2
p3^3 p4 p5^2 + 8 p0^2 p1 p2^3 p3^3 p4 p5^2 + 12 p0 p1^2 p2^3 p3^3 p4 p5^2 + 3 p1^3 p2^3 p3^3 p4 p5^2 + p0 p1 p2^4 p3^3 p4 p5^2 + 2 p1^2 p2^4 p3^3 p4 p5^2 + 3 p0^4 p1^3 p2 p4^2 p5^2 + 2 p0^3 p1^4 p2 p4^2 p5^2 + 5 p0^4 p1^2 p2^2 p4^2 p5^2 + 17 p0^3 p1^3 p2^2 p4^2 p5^2 + 6 p0^2 p1^4 p2^2 p4^2 p5^2 + 3
p0^4 p1 p2^3 p4^2 p5^2 + 17 p0^3 p1^2 p2^3 p4^2 p5^2 + 22 p0^2 p1^3 p2^3 p4^2 p5^2 + 6 p0 p1^4 p2^3 p4^2 p5^2 + 2 p0^3 p1 p2^4 p4^2 p5^2 + 6 p0^2 p1^2 p2^4 p4^2 p5^2 + 6 p0 p1^3 p2^4 p4^2 p5^2 + p1^4 p2^4 p4^2 p5^2 + 6 p0^4 p1^2 p2 p3 p4^2 p5^2 + 17 p0^3 p1^3 p2 p3 p4^2 p5^2 + 7 p0^2 p1^4 p2 p3 p4^2
p5^2 + 6 p0^4 p1 p2^2 p3 p4^2 p5^2 + 39 p0^3 p1^2 p2^2 p3 p4^2 p5^2 + 46 p0^2 p1^3 p2^2 p3 p4^2 p5^2 + 12 p0 p1^4 p2^2 p3 p4^2 p5^2 + 17 p0^3 p1 p2^3 p3 p4^2 p5^2 + 46 p0^2 p1^2 p2^3 p3 p4^2 p5^2 + 31 p0 p1^3 p2^3 p3 p4^2 p5^2 + 3 p1^4 p2^3 p3 p4^2 p5^2 + 7 p0^2 p1 p2^4 p3 p4^2 p5^2 + 12 p0 p1^2
p2^4 p3 p4^2 p5^2 + 3 p1^3 p2^4 p3 p4^2 p5^2 + 4 p0^4 p1 p2 p3^2 p4^2 p5^2 + 17 p0^3 p1^2 p2 p3^2 p4^2 p5^2 + 20 p0^2 p1^3 p2 p3^2 p4^2 p5^2 + 7 p0 p1^4 p2 p3^2 p4^2 p5^2 + 17 p0^3 p1 p2^2 p3^2 p4^2 p5^2 + 45 p0^2 p1^2 p2^2 p3^2 p4^2 p5^2 + 29 p0 p1^3 p2^2 p3^2 p4^2 p5^2 + 3 p1^4 p2^2 p3^2 p4^2 p5^2
+ 20 p0^2 p1 p2^3 p3^2 p4^2 p5^2 + 29 p0 p1^2 p2^3 p3^2 p4^2 p5^2 + 12 p1^3 p2^3 p3^2 p4^2 p5^2 + 7 p0 p1 p2^4 p3^2 p4^2 p5^2 + 3 p1^2 p2^4 p3^2 p4^2 p5^2 + 2 p0^3 p1 p2 p3^3 p4^2 p5^2 + 6 p0^2 p1^2 p2 p3^3 p4^2 p5^2 + 6 p0 p1^3 p2 p3^3 p4^2 p5^2 + p1^4 p2 p3^3 p4^2 p5^2 + 6 p0^2 p1 p2^2 p3^3 p4^2
p5^2 + 11 p0 p1^2 p2^2 p3^3 p4^2 p5^2 + 3 p1^3 p2^2 p3^3 p4^2 p5^2 + 6 p0 p1 p2^3 p3^3 p4^2 p5^2 + 3 p1^2 p2^3 p3^3 p4^2 p5^2 + p1 p2^4 p3^3 p4^2 p5^2 + p0^4 p1^2 p2 p4^3 p5^2 + 2 p0^3 p1^3 p2 p4^3 p5^2 + p0^4 p1 p2^2 p4^3 p5^2 + 7 p0^3 p1^2 p2^2 p4^3 p5^2 + 7 p0^2 p1^3 p2^2 p4^3 p5^2 + p0 p1^4 p2^2
p4^3 p5^2 + 2 p0^3 p1 p2^3 p4^3 p5^2 + 7 p0^2 p1^2 p2^3 p4^3 p5^2 + 5 p0 p1^3 p2^3 p4^3 p5^2 + p1^4 p2^3 p4^3 p5^2 + p0 p1^2 p2^4 p4^3 p5^2 + p1^3 p2^4 p4^3 p5^2 + p0^4 p1 p2 p3 p4^3 p5^2 + 7 p0^3 p1^2 p2 p3 p4^3 p5^2 + 8 p0^2 p1^3 p2 p3 p4^3 p5^2 + p0 p1^4 p2 p3 p4^3 p5^2 + 7 p0^3 p1 p2^2 p3 p4^3
p5^2 + 20 p0^2 p1^2 p2^2 p3 p4^3 p5^2 + 12 p0 p1^3 p2^2 p3 p4^3 p5^2 + 2 p1^4 p2^2 p3 p4^3 p5^2 + 8 p0^2 p1 p2^3 p3 p4^3 p5^2 + 12 p0 p1^2 p2^3 p3 p4^3 p5^2 + 3 p1^3 p2^3 p3 p4^3 p5^2 + p0 p1 p2^4 p3 p4^3 p5^2 + 2 p1^2 p2^4 p3 p4^3 p5^2 + 2 p0^3 p1 p2 p3^2 p4^3 p5^2 + 6 p0^2 p1^2 p2 p3^2 p4^3 p5^2 +
6 p0 p1^3 p2 p3^2 p4^3 p5^2 + p1^4 p2 p3^2 p4^3 p5^2 + 6 p0^2 p1 p2^2 p3^2 p4^3 p5^2 + 11 p0 p1^2 p2^2 p3^2 p4^3 p5^2 + 3 p1^3 p2^2 p3^2 p4^3 p5^2 + 6 p0 p1 p2^3 p3^2 p4^3 p5^2 + 3 p1^2 p2^3 p3^2 p4^3 p5^2 + p1 p2^4 p3^2 p4^3 p5^2 - p0^2 p1 p2 p3^3 p4^3 p5^2 + p1^3 p2 p3^3 p4^3 p5^2 + 2 p1^2 p2^2
p3^3 p4^3 p5^2 + p1 p2^3 p3^3 p4^3 p5^2 + 2 p0^4 p1^3 p2^2 p5^3 + 2 p0^3 p1^4 p2^2 p5^3 + 2 p0^4 p1^2 p2^3 p5^3 + 6 p0^3 p1^3 p2^3 p5^3 + 2 p0^2 p1^4 p2^3 p5^3 + 2 p0^3 p1^2 p2^4 p5^3 + 2 p0^2 p1^3 p2^4 p5^3 + 2 p0^4 p1^3 p2 p3 p5^3 + 2 p0^3 p1^4 p2 p3 p5^3 + 5 p0^4 p1^2 p2^2 p3 p5^3 + 11 p0^3 p1^3
p2^2 p3 p5^3 + 6 p0^2 p1^4 p2^2 p3 p5^3 + 2 p0^4 p1 p2^3 p3 p5^3 + 11 p0^3 p1^2 p2^3 p3 p5^3 + 13 p0^2 p1^3 p2^3 p3 p5^3 + 3 p0 p1^4 p2^3 p3 p5^3 + 2 p0^3 p1 p2^4 p3 p5^3 + 6 p0^2 p1^2 p2^4 p3 p5^3 + 3 p0 p1^3 p2^4 p3 p5^3 + 2 p0^4 p1^2 p2 p3^2 p5^3 + 5 p0^3 p1^3 p2 p3^2 p5^3 + p0^2 p1^4 p2 p3^2 p5^3
+ 2 p0^4 p1 p2^2 p3^2 p5^3 + 12 p0^3 p1^2 p2^2 p3^2 p5^3 + 12 p0^2 p1^3 p2^2 p3^2 p5^3 + 2 p0 p1^4 p2^2 p3^2 p5^3 + 5 p0^3 p1 p2^3 p3^2 p5^3 + 12 p0^2 p1^2 p2^3 p3^2 p5^3 + 8 p0 p1^3 p2^3 p3^2 p5^3 + p1^4 p2^3 p3^2 p5^3 + p0^2 p1 p2^4 p3^2 p5^3 + 2 p0 p1^2 p2^4 p3^2 p5^3 + p1^3 p2^4 p3^2 p5^3 + 2
p0^3 p1^2 p2 p3^3 p5^3 + p0^2 p1^3 p2 p3^3 p5^3 + 2 p0^3 p1 p2^2 p3^3 p5^3 + 7 p0^2 p1^2 p2^2 p3^3 p5^3 + 2 p0 p1^3 p2^2 p3^3 p5^3 + p0^2 p1 p2^3 p3^3 p5^3 + 2 p0 p1^2 p2^3 p3^3 p5^3 + p1^3 p2^3 p3^3 p5^3 + 2 p0^4 p1^3 p2 p4 p5^3 + 2 p0^3 p1^4 p2 p4 p5^3 + 5 p0^4 p1^2 p2^2 p4 p5^3 + 11 p0^3 p1^3 p2^2
p4 p5^3 + 6 p0^2 p1^4 p2^2 p4 p5^3 + 2 p0^4 p1 p2^3 p4 p5^3 + 11 p0^3 p1^2 p2^3 p4 p5^3 + 13 p0^2 p1^3 p2^3 p4 p5^3 + 3 p0 p1^4 p2^3 p4 p5^3 + 2 p0^3 p1 p2^4 p4 p5^3 + 6 p0^2 p1^2 p2^4 p4 p5^3 + 3 p0 p1^3 p2^4 p4 p5^3 + 5 p0^4 p1^2 p2 p3 p4 p5^3 + 10 p0^3 p1^3 p2 p3 p4 p5^3 + 5 p0^2 p1^4 p2 p3 p4 p5^3
+ 5 p0^4 p1 p2^2 p3 p4 p5^3 + 30 p0^3 p1^2 p2^2 p3 p4 p5^3 + 31 p0^2 p1^3 p2^2 p3 p4 p5^3 + 8 p0 p1^4 p2^2 p3 p4 p5^3 + 10 p0^3 p1 p2^3 p3 p4 p5^3 + 31 p0^2 p1^2 p2^3 p3 p4 p5^3 + 16 p0 p1^3 p2^3 p3 p4 p5^3 + 2 p1^4 p2^3 p3 p4 p5^3 + 5 p0^2 p1 p2^4 p3 p4 p5^3 + 8 p0 p1^2 p2^4 p3 p4 p5^3 + 2 p1^3 p2^4
p3 p4 p5^3 + 2 p0^4 p1 p2 p3^2 p4 p5^3 + 11 p0^3 p1^2 p2 p3^2 p4 p5^3 + 12 p0^2 p1^3 p2 p3^2 p4 p5^3 + 2 p0 p1^4 p2 p3^2 p4 p5^3 + 11 p0^3 p1 p2^2 p3^2 p4 p5^3 + 29 p0^2 p1^2 p2^2 p3^2 p4 p5^3 + 16 p0 p1^3 p2^2 p3^2 p4 p5^3 + 2 p1^4 p2^2 p3^2 p4 p5^3 + 12 p0^2 p1 p2^3 p3^2 p4 p5^3 + 16 p0 p1^2 p2^3
p3^2 p4 p5^3 + 3 p1^3 p2^3 p3^2 p4 p5^3 + 2 p0 p1 p2^4 p3^2 p4 p5^3 + 2 p1^2 p2^4 p3^2 p4 p5^3 + 2 p0^3 p1 p2 p3^3 p4 p5^3 + 6 p0^2 p1^2 p2 p3^3 p4 p5^3 + 2 p0 p1^3 p2 p3^3 p4 p5^3 + 6 p0^2 p1 p2^2 p3^3 p4 p5^3 + 8 p0 p1^2 p2^2 p3^3 p4 p5^3 + 2 p1^3 p2^2 p3^3 p4 p5^3 + 2 p0 p1 p2^3 p3^3 p4 p5^3 + 2
p1^2 p2^3 p3^3 p4 p5^3 + 2 p0^4 p1^2 p2 p4^2 p5^3 + 5 p0^3 p1^3 p2 p4^2 p5^3 + p0^2 p1^4 p2 p4^2 p5^3 + 2 p0^4 p1 p2^2 p4^2 p5^3 + 12 p0^3 p1^2 p2^2 p4^2 p5^3 + 12 p0^2 p1^3 p2^2 p4^2 p5^3 + 2 p0 p1^4 p2^2 p4^2 p5^3 + 5 p0^3 p1 p2^3 p4^2 p5^3 + 12 p0^2 p1^2 p2^3 p4^2 p5^3 + 8 p0 p1^3 p2^3 p4^2 p5^3
+ p1^4 p2^3 p4^2 p5^3 + p0^2 p1 p2^4 p4^2 p5^3 + 2 p0 p1^2 p2^4 p4^2 p5^3 + p1^3 p2^4 p4^2 p5^3 + 2 p0^4 p1 p2 p3 p4^2 p5^3 + 11 p0^3 p1^2 p2 p3 p4^2 p5^3 + 12 p0^2 p1^3 p2 p3 p4^2 p5^3 + 2 p0 p1^4 p2 p3 p4^2 p5^3 + 11 p0^3 p1 p2^2 p3 p4^2 p5^3 + 29 p0^2 p1^2 p2^2 p3 p4^2 p5^3 + 16 p0 p1^3 p2^2 p3
p4^2 p5^3 + 2 p1^4 p2^2 p3 p4^2 p5^3 + 12 p0^2 p1 p2^3 p3 p4^2 p5^3 + 16 p0 p1^2 p2^3 p3 p4^2 p5^3 + 3 p1^3 p2^3 p3 p4^2 p5^3 + 2 p0 p1 p2^4 p3 p4^2 p5^3 + 2 p1^2 p2^4 p3 p4^2 p5^3 + 4 p0^3 p1 p2 p3^2 p4^2 p5^3 + 11 p0^2 p1^2 p2 p3^2 p4^2 p5^3 + 8 p0 p1^3 p2 p3^2 p4^2 p5^3 + p1^4 p2 p3^2 p4^2 p5^3 +
11 p0^2 p1 p2^2 p3^2 p4^2 p5^3 + 16 p0 p1^2 p2^2 p3^2 p4^2 p5^3 + 3 p1^3 p2^2 p3^2 p4^2 p5^3 + 8 p0 p1 p2^3 p3^2 p4^2 p5^3 + 3 p1^2 p2^3 p3^2 p4^2 p5^3 + p1 p2^4 p3^2 p4^2 p5^3 + p0 p1^2 p2 p3^3 p4^2 p5^3 + p1^3 p2 p3^3 p4^2 p5^3 + p0 p1 p2^2 p3^3 p4^2 p5^3 + 2 p1^2 p2^2 p3^3 p4^2 p5^3 + p1 p2^3
p3^3 p4^2 p5^3 + 2 p0^3 p1^2 p2 p4^3 p5^3 + p0^2 p1^3 p2 p4^3 p5^3 + 2 p0^3 p1 p2^2 p4^3 p5^3 + 7 p0^2 p1^2 p2^2 p4^3 p5^3 + 2 p0 p1^3 p2^2 p4^3 p5^3 + p0^2 p1 p2^3 p4^3 p5^3 + 2 p0 p1^2 p2^3 p4^3 p5^3 + p1^3 p2^3 p4^3 p5^3 + 2 p0^3 p1 p2 p3 p4^3 p5^3 + 6 p0^2 p1^2 p2 p3 p4^3 p5^3 + 2 p0 p1^3 p2 p3
p4^3 p5^3 + 6 p0^2 p1 p2^2 p3 p4^3 p5^3 + 8 p0 p1^2 p2^2 p3 p4^3 p5^3 + 2 p1^3 p2^2 p3 p4^3 p5^3 + 2 p0 p1 p2^3 p3 p4^3 p5^3 + 2 p1^2 p2^3 p3 p4^3 p5^3 + p0 p1^2 p2 p3^2 p4^3 p5^3 + p1^3 p2 p3^2 p4^3 p5^3 + p0 p1 p2^2 p3^2 p4^3 p5^3 + 2 p1^2 p2^2 p3^2 p4^3 p5^3 + p1 p2^3 p3^2 p4^3 p5^3 + p0^4 p1^2
p2^2 p5^4 + 2 p0^3 p1^3 p2^2 p5^4 + p0^2 p1^4 p2^2 p5^4 + 2 p0^3 p1^2 p2^3 p5^4 + 2 p0^2 p1^3 p2^3 p5^4 + p0^2 p1^2 p2^4 p5^4 + p0^4 p1^2 p2 p3 p5^4 + 2 p0^3 p1^3 p2 p3 p5^4 + p0^2 p1^4 p2 p3 p5^4 + p0^4 p1 p2^2 p3 p5^4 + 3 p0^3 p1^2 p2^2 p3 p5^4 + 3 p0^2 p1^3 p2^2 p3 p5^4 + p0 p1^4 p2^2 p3 p5^4 + 2
p0^3 p1 p2^3 p3 p5^4 + 3 p0^2 p1^2 p2^3 p3 p5^4 + 2 p0 p1^3 p2^3 p3 p5^4 + p0^2 p1 p2^4 p3 p5^4 + p0 p1^2 p2^4 p3 p5^4 + 2 p0^3 p1^2 p2 p3^2 p5^4 + 2 p0^2 p1^3 p2 p3^2 p5^4 + 2 p0^3 p1 p2^2 p3^2 p5^4 + 3 p0^2 p1^2 p2^2 p3^2 p5^4 + 2 p0 p1^3 p2^2 p3^2 p5^4 + 2 p0^2 p1 p2^3 p3^2 p5^4 + 2 p0 p1^2 p2^3
p3^2 p5^4 + p0^2 p1^2 p2 p3^3 p5^4 + p0^2 p1 p2^2 p3^3 p5^4 + p0 p1^2 p2^2 p3^3 p5^4 + p0^4 p1^2 p2 p4 p5^4 + 2 p0^3 p1^3 p2 p4 p5^4 + p0^2 p1^4 p2 p4 p5^4 + p0^4 p1 p2^2 p4 p5^4 + 3 p0^3 p1^2 p2^2 p4 p5^4 + 3 p0^2 p1^3 p2^2 p4 p5^4 + p0 p1^4 p2^2 p4 p5^4 + 2 p0^3 p1 p2^3 p4 p5^4 + 3 p0^2 p1^2 p2^3
p4 p5^4 + 2 p0 p1^3 p2^3 p4 p5^4 + p0^2 p1 p2^4 p4 p5^4 + p0 p1^2 p2^4 p4 p5^4 + p0^4 p1 p2 p3 p4 p5^4 + 3 p0^3 p1^2 p2 p3 p4 p5^4 + 3 p0^2 p1^3 p2 p3 p4 p5^4 + p0 p1^4 p2 p3 p4 p5^4 + 3 p0^3 p1 p2^2 p3 p4 p5^4 + 12 p0^2 p1^2 p2^2 p3 p4 p5^4 + 3 p0 p1^3 p2^2 p3 p4 p5^4 + 3 p0^2 p1 p2^3 p3 p4 p5^4 + 3
p0 p1^2 p2^3 p3 p4 p5^4 + p0 p1 p2^4 p3 p4 p5^4 + 2 p0^3 p1 p2 p3^2 p4 p5^4 + 3 p0^2 p1^2 p2 p3^2 p4 p5^4 + 2 p0 p1^3 p2 p3^2 p4 p5^4 + 3 p0^2 p1 p2^2 p3^2 p4 p5^4 + 3 p0 p1^2 p2^2 p3^2 p4 p5^4 + 2 p0 p1 p2^3 p3^2 p4 p5^4 + p0^2 p1 p2 p3^3 p4 p5^4 + p0 p1^2 p2 p3^3 p4 p5^4 + p0 p1 p2^2 p3^3 p4 p5^4 + 2
p0^3 p1 p2 p4^2 p5^4 + 2 p0^2 p1^3 p2 p4^2 p5^4 + 2 p0^3 p1 p2^2 p4^2 p5^4 + 3 p0^2 p1^2 p2^2 p4^2 p5^4 + 2 p0 p1^3 p2^2 p4^2 p5^4 + 2 p0^2 p1 p2^3 p4^2 p5^4 + 2 p0 p1^2 p2^3 p4^2 p5^4 + 2 p0^3 p1 p2 p3 p4^2 p5^4 + 3 p0^2 p1^2 p2 p3 p4^2 p5^4 + 2 p0 p1^3 p2 p3 p4^2 p5^4 + 3 p0^2 p1 p2^2 p3 p4^2 p5^4
+ 3 p0 p1^2 p2^2 p3 p4^2 p5^4 + 2 p0 p1 p2^3 p3 p4^2 p5^4 + 2 p0^2 p1 p2 p3^2 p4^2 p5^4 + 2 p0 p1^2 p2 p3^2 p4^2 p5^4 + 2 p0 p1 p2^2 p3^2 p4^2 p5^4 + p0^2 p1^2 p2 p4^3 p5^4 + p0^2 p1 p2^2 p4^3 p5^4 + p0 p1^2 p2^2 p4^3 p5^4 + p0^2 p1 p2 p3 p4^3 p5^4 + p0 p1^2 p2 p3 p4^3 p5^4 + p0 p1 p2^2 p3 p4^3 p5^4

p0^4 p1^4 p2^2 p3^2 + 2 p0^4 p1^3 p2^3 p3^2 + 2 p0^3 p1^4 p2^3 p3^2 + p0^4 p1^2 p2^4 p3^2 + 2 p0^3 p1^3 p2^4 p3^2 + p0^2 p1^4 p2^4 p3^2 + p0^4 p1^3 p2^2 p3^3 + p0^3 p1^4 p2^2 p3^3 + p0^4 p1^2 p2^3 p3^3 + 3 p0^3 p1^3 p2^3 p3^3 + p0^2 p1^4 p2^3 p3^3 + p0^3 p1^2 p2^4 p3^3 + p0^2 p1^3 p2^4 p3^3 + 2
p0^4 p1^4 p2^2 p3 p4 + 4 p0^4 p1^3 p2^3 p3 p4 + 4 p0^3 p1^4 p2^3 p3 p4 + 2 p0^4 p1^2 p2^4 p3 p4 + 4 p0^3 p1^3 p2^4 p3 p4 + 2 p0^2 p1^4 p2^4 p3 p4 + p0^4 p1^4 p2 p3^2 p4 + 4 p0^4 p1^3 p2^2 p3^2 p4 + 5 p0^3 p1^4 p2^2 p3^2 p4 + 4 p0^4 p1^2 p2^3 p3^2 p4 + 11 p0^3 p1^3 p2^3 p3^2 p4 + 6 p0^2 p1^4 p2^3 p3^2
p4 + p0^4 p1 p2^4 p3^2 p4 + 5 p0^3 p1^2 p2^4 p3^2 p4 + 6 p0^2 p1^3 p2^4 p3^2 p4 + 2 p0 p1^4 p2^4 p3^2 p4 + p0^4 p1^3 p2 p3^3 p4 + p0^3 p1^4 p2 p3^3 p4 + 2 p0^4 p1^2 p2^2 p3^3 p4 + 5 p0^3 p1^3 p2^2 p3^3 p4 + 3 p0^2 p1^4 p2^2 p3^3 p4 + p0^4 p1 p2^3 p3^3 p4 + 5 p0^3 p1^2 p2^3 p3^3 p4 + 7 p0^2 p1^3 p2^3
p3^3 p4 + 2 p0 p1^4 p2^3 p3^3 p4 + p0^3 p1 p2^4 p3^3 p4 + 3 p0^2 p1^2 p2^4 p3^3 p4 + 2 p0 p1^3 p2^4 p3^3 p4 + p0^4 p1^4 p2^2 p4^2 + 2 p0^4 p1^3 p2^3 p4^2 + 2 p0^3 p1^4 p2^3 p4^2 + p0^4 p1^2 p2^4 p4^2 + 2 p0^3 p1^3 p2^4 p4^2 + p0^2 p1^4 p2^4 p4^2 + p0^4 p1^4 p2 p3 p4^2 + 4 p0^4 p1^3 p2^2 p3 p4^2 + 5
p0^3 p1^4 p2^2 p3 p4^2 + 4 p0^4 p1^2 p2^3 p3 p4^2 + 11 p0^3 p1^3 p2^3 p3 p4^2 + 6 p0^2 p1^4 p2^3 p3 p4^2 + p0^4 p1 p2^4 p3 p4^2 + 5 p0^3 p1^2 p2^4 p3 p4^2 + 6 p0^2 p1^3 p2^4 p3 p4^2 + 2 p0 p1^4 p2^4 p3 p4^2 + 2 p0^4 p1^3 p2 p3^2 p4^2 + p0^3 p1^4 p2 p3^2 p4^2 + 4 p0^4 p1^2 p2^2 p3^2 p4^2 + 13 p0^3
p1^3 p2^2 p3^2 p4^2 + 5 p0^2 p1^4 p2^2 p3^2 p4^2 + 2 p0^4 p1 p2^3 p3^2 p4^2 + 13 p0^3 p1^2 p2^3 p3^2 p4^2 + 18 p0^2 p1^3 p2^3 p3^2 p4^2 + 5 p0 p1^4 p2^3 p3^2 p4^2 + p0^3 p1 p2^4 p3^2 p4^2 + 5 p0^2 p1^2 p2^4 p3^2 p4^2 + 5 p0 p1^3 p2^4 p3^2 p4^2 + p1^4 p2^4 p3^2 p4^2 + 2 p0^4 p1^2 p2 p3^3 p4^2 + 2
p0^3 p1^3 p2 p3^3 p4^2 + p0^2 p1^4 p2 p3^3 p4^2 + 2 p0^4 p1 p2^2 p3^3 p4^2 + 6 p0^3 p1^2 p2^2 p3^3 p4^2 + 6 p0^2 p1^3 p2^2 p3^3 p4^2 + 2 p0 p1^4 p2^2 p3^3 p4^2 + 2 p0^3 p1 p2^3 p3^3 p4^2 + 6 p0^2 p1^2 p2^3 p3^3 p4^2 + 5 p0 p1^3 p2^3 p3^3 p4^2 + p1^4 p2^3 p3^3 p4^2 + p0^2 p1 p2^4 p3^3 p4^2 + 2 p0
p1^2 p2^4 p3^3 p4^2 + p1^3 p2^4 p3^3 p4^2 + p0^4 p1^3 p2^2 p4^3 + p0^3 p1^4 p2^2 p4^3 + p0^4 p1^2 p2^3 p4^3 + 3 p0^3 p1^3 p2^3 p4^3 + p0^2 p1^4 p2^3 p4^3 + p0^3 p1^2 p2^4 p4^3 + p0^2 p1^3 p2^4 p4^3 + p0^4 p1^3 p2 p3 p4^3 + p0^3 p1^4 p2 p3 p4^3 + 2 p0^4 p1^2 p2^2 p3 p4^3 + 5 p0^3 p1^3 p2^2 p3 p4^3 +
3 p0^2 p1^4 p2^2 p3 p4^3 + p0^4 p1 p2^3 p3 p4^3 + 5 p0^3 p1^2 p2^3 p3 p4^3 + 7 p0^2 p1^3 p2^3 p3 p4^3 + 2 p0 p1^4 p2^3 p3 p4^3 + p0^3 p1 p2^4 p3 p4^3 + 3 p0^2 p1^2 p2^4 p3 p4^3 + 2 p0 p1^3 p2^4 p3 p4^3 + 2 p0^4 p1^2 p2 p3^2 p4^3 + 2 p0^3 p1^3 p2 p3^2 p4^3 + p0^2 p1^4 p2 p3^2 p4^3 + 2 p0^4 p1 p2^2 p3^2
p4^3 + 6 p0^3 p1^2 p2^2 p3^2 p4^3 + 6 p0^2 p1^3 p2^2 p3^2 p4^3 + 2 p0 p1^4 p2^2 p3^2 p4^3 + 2 p0^3 p1 p2^3 p3^2 p4^3 + 6 p0^2 p1^2 p2^3 p3^2 p4^3 + 5 p0 p1^3 p2^3 p3^2 p4^3 + p1^4 p2^3 p3^2 p4^3 + p0^2 p1 p2^4 p3^2 p4^3 + 2 p0 p1^2 p2^4 p3^2 p4^3 + p1^3 p2^4 p3^2 p4^3 + 2 p0^3 p1^2 p2 p3^3 p4^3 +
p0^2 p1^3 p2 p3^3 p4^3 + 2 p0^3 p1 p2^2 p3^3 p4^3 + 4 p0^2 p1^2 p2^2 p3^3 p4^3 + 2 p0 p1^3 p2^2 p3^3 p4^3 + p0^2 p1 p2^3 p3^3 p4^3 + 2 p0 p1^2 p2^3 p3^3 p4^3 + p1^3 p2^3 p3^3 p4^3 + 2 p0^4 p1^4 p2^2 p3 p5 + 4 p0^4 p1^3 p2^3 p3 p5 + 4 p0^3 p1^4 p2^3 p3 p5 + 2 p0^4 p1^2 p2^4 p3 p5 + 4 p0^3 p1^3 p2^4 p3
p5 + 2 p0^2 p1^4 p2^4 p3 p5 + p0^4 p1^4 p2 p3^2 p5 + 5 p0^4 p1^3 p2^2 p3^2 p5 + 6 p0^3 p1^4 p2^2 p3^2 p5 + 5 p0^4 p1^2 p2^3 p3^2 p5 + 14 p0^3 p1^3 p2^3 p3^2 p5 + 7 p0^2 p1^4 p2^3 p3^2 p5 + p0^4 p1 p2^4 p3^2 p5 + 6 p0^3 p1^2 p2^4 p3^2 p5 + 7 p0^2 p1^3 p2^4 p3^2 p5 + 2 p0 p1^4 p2^4 p3^2 p5 + p0^4 p1^3
p2 p3^3 p5 + p0^3 p1^4 p2 p3^3 p5 + p0^4 p1^2 p2^2 p3^3 p5 + 5 p0^3 p1^3 p2^2 p3^3 p5 + 2 p0^2 p1^4 p2^2 p3^3 p5 + p0^4 p1 p2^3 p3^3 p5 + 5 p0^3 p1^2 p2^3 p3^3 p5 + 7 p0^2 p1^3 p2^3 p3^3 p5 + 2 p0 p1^4 p2^3 p3^3 p5 + p0^3 p1 p2^4 p3^3 p5 + 2 p0^2 p1^2 p2^4 p3^3 p5 + 2 p0 p1^3 p2^4 p3^3 p5 + 2 p0^4
p1^4 p2^2 p4 p5 + 4 p0^4 p1^3 p2^3 p4 p5 + 4 p0^3 p1^4 p2^3 p4 p5 + 2 p0^4 p1^2 p2^4 p4 p5 + 4 p0^3 p1^3 p2^4 p4 p5 + 2 p0^2 p1^4 p2^4 p4 p5 + 3 p0^4 p1^4 p2 p3 p4 p5 + 11 p0^4 p1^3 p2^2 p3 p4 p5 + 14 p0^3 p1^4 p2^2 p3 p4 p5 + 11 p0^4 p1^2 p2^3 p3 p4 p5 + 30 p0^3 p1^3 p2^3 p3 p4 p5 + 17 p0^2 p1^4 p2^3
p3 p4 p5 + 3 p0^4 p1 p2^4 p3 p4 p5 + 14 p0^3 p1^2 p2^4 p3 p4 p5 + 17 p0^2 p1^3 p2^4 p3 p4 p5 + 6 p0 p1^4 p2^4 p3 p4 p5 + 5 p0^4 p1^3 p2 p3^2 p4 p5 + 5 p0^3 p1^4 p2 p3^2 p4 p5 + 13 p0^4 p1^2 p2^2 p3^2 p4 p5 + 31 p0^3 p1^3 p2^2 p3^2 p4 p5 + 17 p0^2 p1^4 p2^2 p3^2 p4 p5 + 5 p0^4 p1 p2^3 p3^2 p4 p5 + 31
p0^3 p1^2 p2^3 p3^2 p4 p5 + 39 p0^2 p1^3 p2^3 p3^2 p4 p5 + 11 p0 p1^4 p2^3 p3^2 p4 p5 + 5 p0^3 p1 p2^4 p3^2 p4 p5 + 17 p0^2 p1^2 p2^4 p3^2 p4 p5 + 11 p0 p1^3 p2^4 p3^2 p4 p5 + 2 p1^4 p2^4 p3^2 p4 p5 + 2 p0^4 p1^2 p2 p3^3 p4 p5 + 4 p0^3 p1^3 p2 p3^3 p4 p5 + 2 p0^2 p1^4 p2 p3^3 p4 p5 + 2 p0^4 p1 p2^2
p3^3 p4 p5 + 16 p0^3 p1^2 p2^2 p3^3 p4 p5 + 17 p0^2 p1^3 p2^2 p3^3 p4 p5 + 5 p0 p1^4 p2^2 p3^3 p4 p5 + 4 p0^3 p1 p2^3 p3^3 p4 p5 + 17 p0^2 p1^2 p2^3 p3^3 p4 p5 + 10 p0 p1^3 p2^3 p3^3 p4 p5 + 2 p1^4 p2^3 p3^3 p4 p5 + 2 p0^2 p1 p2^4 p3^3 p4 p5 + 5 p0 p1^2 p2^4 p3^3 p4 p5 + 2 p1^3 p2^4 p3^3 p4 p5 + p0^4
p1^4 p2 p4^2 p5 + 5 p0^4 p1^3 p2^2 p4^2 p5 + 6 p0^3 p1^4 p2^2 p4^2 p5 + 5 p0^4 p1^2 p2^3 p4^2 p5 + 14 p0^3 p1^3 p2^3 p4^2 p5 + 7 p0^2 p1^4 p2^3 p4^2 p5 + p0^4 p1 p2^4 p4^2 p5 + 6 p0^3 p1^2 p2^4 p4^2 p5 + 7 p0^2 p1^3 p2^4 p4^2 p5 + 2 p0 p1^4 p2^4 p4^2 p5 + 5 p0^4 p1^3 p2 p3 p4^2 p5 + 5 p0^3 p1^4 p2 p3
p4^2 p5 + 13 p0^4 p1^2 p2^2 p3 p4^2 p5 + 31 p0^3 p1^3 p2^2 p3 p4^2 p5 + 17 p0^2 p1^4 p2^2 p3 p4^2 p5 + 5 p0^4 p1 p2^3 p3 p4^2 p5 + 31 p0^3 p1^2 p2^3 p3 p4^2 p5 + 39 p0^2 p1^3 p2^3 p3 p4^2 p5 + 11 p0 p1^4 p2^3 p3 p4^2 p5 + 5 p0^3 p1 p2^4 p3 p4^2 p5 + 17 p0^2 p1^2 p2^4 p3 p4^2 p5 + 11 p0 p1^3 p2^4 p3
p4^2 p5 + 2 p1^4 p2^4 p3 p4^2 p5 + 6 p0^4 p1^2 p2 p3^2 p4^2 p5 + 16 p0^3 p1^3 p2 p3^2 p4^2 p5 + 7 p0^2 p1^4 p2 p3^2 p4^2 p5 + 6 p0^4 p1 p2^2 p3^2 p4^2 p5 + 32 p0^3 p1^2 p2^2 p3^2 p4^2 p5 + 39 p0^2 p1^3 p2^2 p3^2 p4^2 p5 + 12 p0 p1^4 p2^2 p3^2 p4^2 p5 + 16 p0^3 p1 p2^3 p3^2 p4^2 p5 + 39 p0^2 p1^2 p2^3
p3^2 p4^2 p5 + 30 p0 p1^3 p2^3 p3^2 p4^2 p5 + 5 p1^4 p2^3 p3^2 p4^2 p5 + 7 p0^2 p1 p2^4 p3^2 p4^2 p5 + 12 p0 p1^2 p2^4 p3^2 p4^2 p5 + 5 p1^3 p2^4 p3^2 p4^2 p5 + 2 p0^4 p1 p2 p3^3 p4^2 p5 + 6 p0^3 p1^2 p2 p3^3 p4^2 p5 + 7 p0^2 p1^3 p2 p3^3 p4^2 p5 + 2 p0 p1^4 p2 p3^3 p4^2 p5 + 6 p0^3 p1 p2^2 p3^3 p4^2
p5 + 17 p0^2 p1^2 p2^2 p3^3 p4^2 p5 + 11 p0 p1^3 p2^2 p3^3 p4^2 p5 + 2 p1^4 p2^2 p3^3 p4^2 p5 + 7 p0^2 p1 p2^3 p3^3 p4^2 p5 + 11 p0 p1^2 p2^3 p3^3 p4^2 p5 + 3 p1^3 p2^3 p3^3 p4^2 p5 + 2 p0 p1 p2^4 p3^3 p4^2 p5 + 2 p1^2 p2^4 p3^3 p4^2 p5 + p0^4 p1^3 p2 p4^3 p5 + p0^3 p1^4 p2 p4^3 p5 + p0^4 p1^2 p2^2
p4^3 p5 + 5 p0^3 p1^3 p2^2 p4^3 p5 + 2 p0^2 p1^4 p2^2 p4^3 p5 + p0^4 p1 p2^3 p4^3 p5 + 5 p0^3 p1^2 p2^3 p4^3 p5 + 7 p0^2 p1^3 p2^3 p4^3 p5 + 2 p0 p1^4 p2^3 p4^3 p5 + p0^3 p1 p2^4 p4^3 p5 + 2 p0^2 p1^2 p2^4 p4^3 p5 + 2 p0 p1^3 p2^4 p4^3 p5 + 2 p0^4 p1^2 p2 p3 p4^3 p5 + 4 p0^3 p1^3 p2 p3 p4^3 p5 + 2
p0^2 p1^4 p2 p3 p4^3 p5 + 2 p0^4 p1 p2^2 p3 p4^3 p5 + 16 p0^3 p1^2 p2^2 p3 p4^3 p5 + 17 p0^2 p1^3 p2^2 p3 p4^3 p5 + 5 p0 p1^4 p2^2 p3 p4^3 p5 + 4 p0^3 p1 p2^3 p3 p4^3 p5 + 17 p0^2 p1^2 p2^3 p3 p4^3 p5 + 10 p0 p1^3 p2^3 p3 p4^3 p5 + 2 p1^4 p2^3 p3 p4^3 p5 + 2 p0^2 p1 p2^4 p3 p4^3 p5 + 5 p0 p1^2 p2^4 p3
p4^3 p5 + 2 p1^3 p2^4 p3 p4^3 p5 + 2 p0^4 p1 p2 p3^2 p4^3 p5 + 6 p0^3 p1^2 p2 p3^2 p4^3 p5 + 7 p0^2 p1^3 p2 p3^2 p4^3 p5 + 2 p0 p1^4 p2 p3^2 p4^3 p5 + 6 p0^3 p1 p2^2 p3^2 p4^3 p5 + 17 p0^2 p1^2 p2^2 p3^2 p4^3 p5 + 11 p0 p1^3 p2^2 p3^2 p4^3 p5 + 2 p1^4 p2^2 p3^2 p4^3 p5 + 7 p0^2 p1 p2^3 p3^2 p4^3 p5 +
11 p0 p1^2 p2^3 p3^2 p4^3 p5 + 3 p1^3 p2^3 p3^2 p4^3 p5 + 2 p0 p1 p2^4 p3^2 p4^3 p5 + 2 p1^2 p2^4 p3^2 p4^3 p5 + 2 p0^3 p1 p2 p3^3 p4^3 p5 + 2 p0^2 p1^2 p2 p3^3 p4^3 p5 + 2 p0 p1^3 p2 p3^3 p4^3 p5 + 2 p0^2 p1 p2^2 p3^3 p4^3 p5 + 4 p0 p1^2 p2^2 p3^3 p4^3 p5 + 2 p1^3 p2^2 p3^3 p4^3 p5 + 2 p0 p1 p2^3 p3^3
p4^3 p5 + 2 p1^2 p2^3 p3^3 p4^3 p5 + p0^4 p1^4 p2^2 p5^2 + 2 p0^4 p1^3 p2^3 p5^2 + 2 p0^3 p1^4 p2^3 p5^2 + p0^4 p1^2 p2^4 p5^2 + 2 p0^3 p1^3 p2^4 p5^2 + p0^2 p1^4 p2^4 p5^2 + p0^4 p1^4 p2 p3 p5^2 + 6 p0^4 p1^3 p2^2 p3 p5^2 + 7 p0^3 p1^4 p2^2 p3 p5^2 + 6 p0^4 p1^2 p2^3 p3 p5^2 + 17 p0^3 p1^3 p2^3 p3
p5^2 + 8 p0^2 p1^4 p2^3 p3 p5^2 + p0^4 p1 p2^4 p3 p5^2 + 7 p0^3 p1^2 p2^4 p3 p5^2 + 8 p0^2 p1^3 p2^4 p3 p5^2 + 2 p0 p1^4 p2^4 p3 p5^2 + 3 p0^4 p1^3 p2 p3^2 p5^2 + 2 p0^3 p1^4 p2 p3^2 p5^2 + 5 p0^4 p1^2 p2^2 p3^2 p5^2 + 17 p0^3 p1^3 p2^2 p3^2 p5^2 + 6 p0^2 p1^4 p2^2 p3^2 p5^2 + 3 p0^4 p1 p2^3 p3^2
p5^2 + 17 p0^3 p1^2 p2^3 p3^2 p5^2 + 22 p0^2 p1^3 p2^3 p3^2 p5^2 + 6 p0 p1^4 p2^3 p3^2 p5^2 + 2 p0^3 p1 p2^4 p3^2 p5^2 + 6 p0^2 p1^2 p2^4 p3^2 p5^2 + 6 p0 p1^3 p2^4 p3^2 p5^2 + p1^4 p2^4 p3^2 p5^2 + p0^4 p1^2 p2 p3^3 p5^2 + 2 p0^3 p1^3 p2 p3^3 p5^2 + p0^4 p1 p2^2 p3^3 p5^2 + 7 p0^3 p1^2 p2^2 p3^3
p5^2 + 7 p0^2 p1^3 p2^2 p3^3 p5^2 + p0 p1^4 p2^2 p3^3 p5^2 + 2 p0^3 p1 p2^3 p3^3 p5^2 + 7 p0^2 p1^2 p2^3 p3^3 p5^2 + 5 p0 p1^3 p2^3 p3^3 p5^2 + p1^4 p2^3 p3^3 p5^2 + p0 p1^2 p2^4 p3^3 p5^2 + p1^3 p2^4 p3^3 p5^2 + p0^4 p1^4 p2 p4 p5^2 + 6 p0^4 p1^3 p2^2 p4 p5^2 + 7 p0^3 p1^4 p2^2 p4 p5^2 + 6 p0^4
p1^2 p2^3 p4 p5^2 + 17 p0^3 p1^3 p2^3 p4 p5^2 + 8 p0^2 p1^4 p2^3 p4 p5^2 + p0^4 p1 p2^4 p4 p5^2 + 7 p0^3 p1^2 p2^4 p4 p5^2 + 8 p0^2 p1^3 p2^4 p4 p5^2 + 2 p0 p1^4 p2^4 p4 p5^2 + 7 p0^4 p1^3 p2 p3 p4 p5^2 + 7 p0^3 p1^4 p2 p3 p4 p5^2 + 18 p0^4 p1^2 p2^2 p3 p4 p5^2 + 39 p0^3 p1^3 p2^2 p3 p4 p5^2 + 22 p0^2
p1^4 p2^2 p3 p4 p5^2 + 7 p0^4 p1 p2^3 p3 p4 p5^2 + 39 p0^3 p1^2 p2^3 p3 p4 p5^2 + 47 p0^2 p1^3 p2^3 p3 p4 p5^2 + 13 p0 p1^4 p2^3 p3 p4 p5^2 + 7 p0^3 p1 p2^4 p3 p4 p5^2 + 22 p0^2 p1^2 p2^4 p3 p4 p5^2 + 13 p0 p1^3 p2^4 p3 p4 p5^2 + 2 p1^4 p2^4 p3 p4 p5^2 + 6 p0^4 p1^2 p2 p3^2 p4 p5^2 + 17 p0^3 p1^3 p2
p3^2 p4 p5^2 + 7 p0^2 p1^4 p2 p3^2 p4 p5^2 + 6 p0^4 p1 p2^2 p3^2 p4 p5^2 + 39 p0^3 p1^2 p2^2 p3^2 p4 p5^2 + 46 p0^2 p1^3 p2^2 p3^2 p4 p5^2 + 12 p0 p1^4 p2^2 p3^2 p4 p5^2 + 17 p0^3 p1 p2^3 p3^2 p4 p5^2 + 46 p0^2 p1^2 p2^3 p3^2 p4 p5^2 + 31 p0 p1^3 p2^3 p3^2 p4 p5^2 + 3 p1^4 p2^3 p3^2 p4 p5^2 + 7 p0^2
p1 p2^4 p3^2 p4 p5^2 + 12 p0 p1^2 p2^4 p3^2 p4 p5^2 + 3 p1^3 p2^4 p3^2 p4 p5^2 + p0^4 p1 p2 p3^3 p4 p5^2 + 7 p0^3 p1^2 p2 p3^3 p4 p5^2 + 8 p0^2 p1^3 p2 p3^3 p4 p5^2 + p0 p1^4 p2 p3^3 p4 p5^2 + 7 p0^3 p1 p2^2 p3^3 p4 p5^2 + 20 p0^2 p1^2 p2^2 p3^3 p4 p5^2 + 12 p0 p1^3 p2^2 p3^3 p4 p5^2 + 2 p1^4 p2^2
p3^3 p4 p5^2 + 8 p0^2 p1 p2^3 p3^3 p4 p5^2 + 12 p0 p1^2 p2^3 p3^3 p4 p5^2 + 3 p1^3 p2^3 p3^3 p4 p5^2 + p0 p1 p2^4 p3^3 p4 p5^2 + 2 p1^2 p2^4 p3^3 p4 p5^2 + 3 p0^4 p1^3 p2 p4^2 p5^2 + 2 p0^3 p1^4 p2 p4^2 p5^2 + 5 p0^4 p1^2 p2^2 p4^2 p5^2 + 17 p0^3 p1^3 p2^2 p4^2 p5^2 + 6 p0^2 p1^4 p2^2 p4^2 p5^2 + 3
p0^4 p1 p2^3 p4^2 p5^2 + 17 p0^3 p1^2 p2^3 p4^2 p5^2 + 22 p0^2 p1^3 p2^3 p4^2 p5^2 + 6 p0 p1^4 p2^3 p4^2 p5^2 + 2 p0^3 p1 p2^4 p4^2 p5^2 + 6 p0^2 p1^2 p2^4 p4^2 p5^2 + 6 p0 p1^3 p2^4 p4^2 p5^2 + p1^4 p2^4 p4^2 p5^2 + 6 p0^4 p1^2 p2 p3 p4^2 p5^2 + 17 p0^3 p1^3 p2 p3 p4^2 p5^2 + 7 p0^2 p1^4 p2 p3 p4^2
p5^2 + 6 p0^4 p1 p2^2 p3 p4^2 p5^2 + 39 p0^3 p1^2 p2^2 p3 p4^2 p5^2 + 46 p0^2 p1^3 p2^2 p3 p4^2 p5^2 + 12 p0 p1^4 p2^2 p3 p4^2 p5^2 + 17 p0^3 p1 p2^3 p3 p4^2 p5^2 + 46 p0^2 p1^2 p2^3 p3 p4^2 p5^2 + 31 p0 p1^3 p2^3 p3 p4^2 p5^2 + 3 p1^4 p2^3 p3 p4^2 p5^2 + 7 p0^2 p1 p2^4 p3 p4^2 p5^2 + 12 p0 p1^2
p2^4 p3 p4^2 p5^2 + 3 p1^3 p2^4 p3 p4^2 p5^2 + 4 p0^4 p1 p2 p3^2 p4^2 p5^2 + 17 p0^3 p1^2 p2 p3^2 p4^2 p5^2 + 20 p0^2 p1^3 p2 p3^2 p4^2 p5^2 + 7 p0 p1^4 p2 p3^2 p4^2 p5^2 + 17 p0^3 p1 p2^2 p3^2 p4^2 p5^2 + 45 p0^2 p1^2 p2^2 p3^2 p4^2 p5^2 + 29 p0 p1^3 p2^2 p3^2 p4^2 p5^2 + 3 p1^4 p2^2 p3^2 p4^2 p5^2
+ 20 p0^2 p1 p2^3 p3^2 p4^2 p5^2 + 29 p0 p1^2 p2^3 p3^2 p4^2 p5^2 + 12 p1^3 p2^3 p3^2 p4^2 p5^2 + 7 p0 p1 p2^4 p3^2 p4^2 p5^2 + 3 p1^2 p2^4 p3^2 p4^2 p5^2 + 2 p0^3 p1 p2 p3^3 p4^2 p5^2 + 6 p0^2 p1^2 p2 p3^3 p4^2 p5^2 + 6 p0 p1^3 p2 p3^3 p4^2 p5^2 + p1^4 p2 p3^3 p4^2 p5^2 + 6 p0^2 p1 p2^2 p3^3 p4^2
p5^2 + 11 p0 p1^2 p2^2 p3^3 p4^2 p5^2 + 3 p1^3 p2^2 p3^3 p4^2 p5^2 + 6 p0 p1 p2^3 p3^3 p4^2 p5^2 + 3 p1^2 p2^3 p3^3 p4^2 p5^2 + p1 p2^4 p3^3 p4^2 p5^2 + p0^4 p1^2 p2 p4^3 p5^2 + 2 p0^3 p1^3 p2 p4^3 p5^2 + p0^4 p1 p2^2 p4^3 p5^2 + 7 p0^3 p1^2 p2^2 p4^3 p5^2 + 7 p0^2 p1^3 p2^2 p4^3 p5^2 + p0 p1^4 p2^2
p4^3 p5^2 + 2 p0^3 p1 p2^3 p4^3 p5^2 + 7 p0^2 p1^2 p2^3 p4^3 p5^2 + 5 p0 p1^3 p2^3 p4^3 p5^2 + p1^4 p2^3 p4^3 p5^2 + p0 p1^2 p2^4 p4^3 p5^2 + p1^3 p2^4 p4^3 p5^2 + p0^4 p1 p2 p3 p4^3 p5^2 + 7 p0^3 p1^2 p2 p3 p4^3 p5^2 + 8 p0^2 p1^3 p2 p3 p4^3 p5^2 + p0 p1^4 p2 p3 p4^3 p5^2 + 7 p0^3 p1 p2^2 p3 p4^3
p5^2 + 20 p0^2 p1^2 p2^2 p3 p4^3 p5^2 + 12 p0 p1^3 p2^2 p3 p4^3 p5^2 + 2 p1^4 p2^2 p3 p4^3 p5^2 + 8 p0^2 p1 p2^3 p3 p4^3 p5^2 + 12 p0 p1^2 p2^3 p3 p4^3 p5^2 + 3 p1^3 p2^3 p3 p4^3 p5^2 + p0 p1 p2^4 p3 p4^3 p5^2 + 2 p1^2 p2^4 p3 p4^3 p5^2 + 2 p0 p1 p2^3 p3^2 p4^3 p5^2 + 6 p0^2 p1^2 p2 p3^2 p4^3 p5^2 +
6 p0 p1^3 p2 p3^2 p4^3 p5^2 + p1^4 p2 p3^2 p4^3 p5^2 + 6 p0^2 p1 p2^2 p3^2 p4^3 p5^2 + 11 p0 p1^2 p2^2 p3^2 p4^3 p5^2 + 3 p1^3 p2^2 p3^2 p4^3 p5^2 + 6 p0 p1 p2^3 p3^2 p4^3 p5^2 + 3 p1^2 p2^3 p3^2 p4^3 p5^2 + p1 p2^4 p3^2 p4^3 p5^2 - p0^2 p1 p2^3 p4^3 p5^2 + p1^3 p2 p3^3 p4^3 p5^2 + 2 p1^2 p2^2
p3^3 p4^3 p5^2 + p1 p2^3 p3^3 p4^3 p5^2 + 2 p0^4 p1^3 p2^2 p5^3 + 2 p0^3 p1^4 p2^2 p5^3 + 2 p0^4 p1^2 p2^3 p5^3 + 6 p0^3 p1^3 p2^3 p5^3 + 2 p0^2 p1^4 p2^3 p5^3 + 2 p0^3 p1^2 p2^4 p5^3 + 2 p0^2 p1^3 p2^4 p5^3 + 2 p0^4 p1^3 p2 p3 p5^3 + 2 p0^3 p1^4 p2 p3 p5^3 + 5 p0^4 p1^2 p2^2 p3 p5^3 + 11 p0^3 p1^3
p2^2 p3 p5^3 + 6 p0^2 p1^4 p2^2 p3 p5^3 + 2 p0^4 p1 p2^3 p3 p5^3 + 11 p0^3 p1^2 p2^3 p3 p5^3 + 13 p0^2 p1^3 p2^3 p3 p5^3 + 3 p0 p1^4 p2^3 p3 p5^3 + 2 p0^3 p1 p2^4 p3 p5^3 + 6 p0^2 p1^2 p2^4 p3 p5^3 + 3 p0 p1^3 p2^4 p3 p5^3 + p0^4 p1^2 p2 p3^2 p5^3 + 5 p0^3 p1^3 p2 p3^2 p5^3 + p0^2 p1^4 p2 p3^2 p5^3
+ 2 p0^4 p1 p2^2 p3^2 p5^3 + 12 p0^3 p1^2 p2^2 p3^2 p5^3 + 12 p0^2 p1^3 p2^2 p3^2 p5^3 + 2 p0 p1^4 p2^2 p3^2 p5^3 + 5 p0^3 p1 p2^3 p3^2 p5^3 + 12 p0^2 p1^2 p2^3 p3^2 p5^3 + 8 p0 p1^3 p2^3 p3^2 p5^3 + p1^4 p2^3 p3^2 p5^3 + p0 p1 p2^4 p3^2 p5^3 + 2 p0 p1^2 p2^4 p3^2 p5^3 + p1^3 p2^4 p3^2 p5^3 + 2
p0^3 p1^2 p2 p3^3 p5^3 + p0^2 p1^3 p2 p3^3 p5^3 + 2 p0^3 p1 p2^2 p3^3 p5^3 + 7 p0^2 p1^2 p2^2 p3^3 p5^3 + 2 p0 p1^3 p2^2 p3^3 p5^3 + p0^2 p1 p2^3 p3^3 p5^3 + 2 p0 p1^2 p2^3 p3^3 p5^3 + p1^3 p2^3 p3^3 p5^3 + 2 p0^4 p1^3 p2 p4 p5^3 + 2 p0^3 p1^4 p2 p4 p5^3 + 5 p0^4 p1^2 p2^2 p4 p5^3 + 11 p0^3 p1^3 p2^2
p4 p5^3 + 6 p0^2 p1^4 p2^2 p4 p5^3 + 2 p0^4 p1 p2^3 p4 p5^3 + 11 p0^3 p1^2 p2^3 p4 p5^3 + 13 p0^2 p1^3 p2^3 p4 p5^3 + 3 p0 p1^4 p2^3 p4 p5^3 + 2 p0^3 p1 p2^4 p4 p5^3 + 6 p0^2 p1^2 p2^4 p4 p5^3 + 3 p0 p1^3 p2^4 p4 p5^3 + 5 p0^4 p1^2 p2 p3 p4 p5^3 + 10 p0^3 p1^3 p2 p3 p4 p5^3 + 5 p0^2 p1^4 p2 p3 p4 p5^3
+ 5 p0^4 p1 p2^2 p3 p4 p5^3 + 30 p0^3 p1^2 p2^2 p3 p4 p5^3 + 31 p0^2 p1^3 p2^2 p3 p4 p5^3 + 8 p0 p1^4 p2^2 p3 p4 p5^3 + 10 p0^3 p1 p2^3 p3 p4 p5^3 + 31 p0^2 p1^2 p2^3 p3 p4 p5^3 + 16 p0 p1^3 p2^3 p3 p4 p5^3 + 2 p1^4 p2^3 p3 p4 p5^3 + 5 p0^2 p1 p2^4 p3 p4 p5^3 + 8 p0 p1^2 p2^4 p3 p4 p5^3 + 2 p1^3 p2^4
p3 p4 p5^3 + 2 p0^4 p1 p2 p3^2 p4 p5^3 + 11 p0^3 p1^2 p2 p3^2 p4 p5^3 + 12 p0^2 p1^3 p2 p3^2 p4 p5^3 + 2 p0 p1^4 p2 p3^2 p4 p5^3 + 11 p0^3 p1 p2^2 p3^2 p4 p5^3 + 29 p0^2 p1^2 p2^2 p3^2 p4 p5^3 + 16 p0 p1^3 p2^2 p3^2 p4 p5^3 + 2 p1^4 p2^2 p3^2 p4 p5^3 + 12 p0^2 p1 p2^3 p3^2 p4 p5^3 + 16 p0 p1^2 p2^3
p3^2 p4 p5^3 + 3 p1^3 p2^3 p3^2 p4 p5^3 + 2 p0 p1 p2^4 p3^2 p4 p5^3 + 2 p1^2 p2^4 p3^2 p4 p5^3 + 2 p0^3 p1 p2 p3^3 p4 p5^3 + 6 p0^2 p1^2 p2 p3^3 p4 p5^3 + 2 p0 p1^3 p2 p3^3 p4 p5^3 + 6 p0^2 p1 p2^2 p3^3 p4 p5^3 + 8 p0 p1^2 p2^2 p3^3 p4 p5^3 + 2 p1^3 p2^2 p3^3 p4 p5^3 + 2 p0 p1 p2^3 p3^3 p4 p5^3 + 2
p1^2 p2^3 p3^3 p4 p5^3 + 2 p0^4 p1^2 p2 p4^2 p5^3 + 5 p0^3 p1^3 p2 p4^2 p5^3 + p0^2 p1^4 p2 p4^2 p5^3 + 2 p0^4 p1 p2^2 p4^2 p5^3 + 12 p0^3 p1^2 p2^2 p4^2 p5^3 + 12 p0^2 p1^3 p2^2 p4^2 p5^3 + 2 p0 p1^4 p2^2 p4^2 p5^3 + 5 p0^3 p1 p2^3 p4^2 p5^3 + 12 p0^2 p1^2 p2^3 p4^2 p5^3 + 8 p0 p1^3 p2^3 p4^2 p5^3
+ p1^4 p2^3 p4^2 p5^3 + p0^2 p1 p2^4 p4^2 p5^3 + 2 p0 p1^2 p2^4 p4^2 p5^3 + p1^3 p2^4 p4^2 p5^3 + 2 p0^4 p1 p2 p3 p4^2 p5^3 + 11 p0^3 p1^2 p2 p3 p4^2 p5^3 + 12 p0^2 p1^3 p2 p3 p4^2 p5^3 + 2 p0 p1^4 p2 p3 p4^2 p5^3 + 11 p0^3 p1 p2^2 p3 p4^2 p5^3 + 29 p0^2 p1^2 p2^2 p3 p4^2 p5^3 + 16 p0 p1^3 p2^2 p3
p4^2 p5^3 + 2 p1^4 p2^2 p3 p4^2 p5^3 + 12 p0^2 p1 p2^3 p3 p4^2 p5^3 + 16 p0 p1^2 p2^3 p3 p4^2 p5^3 + 3 p1^3 p2^3 p3 p4^2 p5^3 + 2 p0 p1 p2^4 p3 p4^2 p5^3 + 2 p1^2 p2^4 p3 p4^2 p5^3 + 4 p0^3 p1 p2 p3^2 p4^2 p5^3 + 11 p0^2 p1^2 p2 p3^2 p4^2 p5^3 + 8 p0 p1^3 p2 p3^2 p4^2 p5^3 + p1^4 p2 p3^2 p4^2 p5^3 +
11 p0^2 p1 p2^2 p3^2 p4^2 p5^3 + 16 p0 p1^2 p2^2 p3^2 p4^2 p5^3 + 3 p1^3 p2^2 p3^2 p4^2 p5^3 + 8 p0 p1 p2^3 p3^2 p4^2 p5^3 + 3 p1^2 p2^3 p3^2 p4^2 p5^3 + p1 p2^4 p3^2 p4^2 p5^3 + p0 p1^2 p2 p3^3 p4^2 p5^3 + p0 p1 p2^2 p3^3 p4^2 p5^3 + 2 p1^2 p2^2 p3^3 p4^2 p5^3 + p1 p2^3
p3^3 p4^2 p5^3 + 2 p0^3 p1^2 p2 p4^3 p5^3 + p0^2 p1^3 p2 p4^3 p5^3 + 2 p0^3 p1 p2^2 p4^3 p5^3 + 7 p0^2 p1^2 p2^2 p4^3 p5^3 + 2 p0 p1^3 p2^2 p4^3 p5^3 + p0^2 p1 p2^3 p4^3 p5^3 + 2 p0 p1^2 p2^3 p4^3 p5^3 + p1^3 p2^3 p4^3 p5^3 + 2 p0^3 p1 p2 p3 p4^3 p5^3 + 6 p0^2 p1^2 p2 p3 p4^3 p5^3 + 2 p0 p1^3 p2 p3
p4^3 p5^3 + 6 p0^2 p1 p2^2 p3 p4^3 p5^3 + 8 p0 p1^2 p2^2 p3 p4^3 p5^3 + 2 p1^3 p2^2 p3 p4^3 p5^3 + 2 p0 p1 p2^3 p3 p4^3 p5^3 + 2 p1^2 p2^3 p3 p4^3 p5^3 + p0 p1^2 p2 p3^2 p4^3 p5^3 + p1^3 p2 p3^2 p4^3 p5^3 + p0 p1 p2^2 p3^2 p4^3 p5^3 + 2 p1^2 p2^2 p3^2 p4^3 p5^3 + p1 p2^3 p3^2 p4^3 p5^3 + p0^4 p1^2
p2^2 p5^4 + 2 p0^3 p1^3 p2^2 p5^4 + p0^2 p1^4 p2^2 p5^4 + 2 p0^3 p1^2 p2^3 p5^4 + 2 p0^2 p1^3 p2^3 p5^4 + p0^2 p1^2 p2^4 p5^4 + p0^4 p1 p2^2 p3 p5^4 + 2 p0^3 p1^2 p2 p3 p5^4 + p0^2 p1^4 p2 p3 p5^4 + p0^4 p1 p2^2 p3 p5^4 + 3 p0^3 p1^2 p2^2 p3 p5^4 + 3 p0^2 p1^3 p2^2 p3 p5^4 + p0 p1^4 p2^2 p3 p5^4 + 2
p0^3 p1 p2^3 p3 p5^4 + 3 p0^2 p1^2 p2^3 p3 p5^4 + 2 p0 p1^3 p2^3 p3 p5^4 + p0^2 p1 p2^4 p3 p5^4 + p0 p1^2 p2^4 p3 p5^4 + 2 p0^3 p1^2 p2 p3^2 p5^4 + 2 p0^2 p1^3 p2 p3^2 p5^4 + 2 p0^3 p1 p2^2 p3^2 p5^4 + 3 p0^2 p1^2 p2^2 p3^2 p5^4 + 2 p0 p1^3 p2^2 p3^2 p5^4 + 2 p0^2 p1 p2^3 p3^2 p5^4 + 2 p0 p1^2 p2^3
p3^2 p5^4 + p0^2 p1^2 p2 p3^3 p5^4 + p0^2 p1 p2^2 p3^3 p5^4 + p0 p1^2 p2^2 p3^3 p5^4 + p0^4 p1^2 p2 p4 p5^4 + 2 p0^3 p1^3 p2 p4 p5^4 + p0^2 p1^4 p2 p4 p5^4 + p0^4 p1 p2^2 p4 p5^4 + 3 p0^3 p1^2 p2^2 p4 p5^4 + 3 p0^2 p1^3 p2^2 p4 p5^4 + p0 p1^4 p2^2 p4 p5^4 + 2 p0^3 p1 p2^3 p4 p5^4 + 3 p0^2 p1^2 p2^3
p4 p5^4 + 2 p0 p1^3 p2^3 p4 p5^4 + p0^2 p1 p2^4 p4 p5^4 + p0 p1^2 p2^4 p4 p5^4 + p0^4 p1 p2 p3 p4 p5^4 + 3 p0^3 p1^2 p2 p3 p4 p5^4 + 3 p0^2 p1^3 p2 p3 p4 p5^4 + p0 p1^4 p2 p3 p4 p5^4 + 3 p0^3 p1 p2^2 p3 p4 p5^4 + 12 p0^2 p1^2 p2^2 p3 p4 p5^4 + 3 p0 p1^3 p2^2 p3 p4 p5^4 + 3 p0^2 p1 p2^3 p3 p4 p5^4 + 3
p0 p1^2 p2^3 p3 p4 p5^4 + p0 p1 p2^4 p3 p4 p5^4 + 2 p0^3 p1 p2 p3^2 p4 p5^4 + 3 p0^2 p1^2 p2 p3^2 p4 p5^4 + 2 p0 p1^3 p2 p3^2 p4 p5^4 + 3 p0^2 p1 p2^2 p3^2 p4 p5^4 + 3 p0 p1^2 p2^2 p3^2 p4 p5^4 + 2 p0 p1 p2^3 p3^2 p4 p5^4 + p0^2 p1 p2 p3^3 p4 p5^4 + p0 p1^2 p2 p3^3 p4 p5^4 + p0 p1 p2^2 p3^3 p4 p5^4 + 2
p0^3 p1^2 p2 p4^2 p5^4 + 2 p0^2 p1^3 p2 p4^2 p5^4 + 2 p0^3 p1 p2^2 p4^2 p5^4 + 3 p0^2 p1^2 p2^2 p4^2 p5^4 + 2 p0 p1^3 p2^2 p4^2 p5^4 + 2 p0^2 p1 p2^3 p4^2 p5^4 + 2 p0 p1^2 p2^3 p4^2 p5^4 + 2 p0^3 p1 p2 p3 p4^2 p5^4 + 3 p0^2 p1^2 p2 p3 p4^2 p5^4 + 2 p0 p1^3 p2 p3 p4^2 p5^4 + 3 p0^2 p1 p2^2 p3 p4^2 p5^4
+ 3 p0 p1^2 p2^2 p3 p4^2 p5^4 + 2 p0 p1 p2^3 p3 p4^2 p5^4 + 2 p0^2 p1 p2 p3^2 p4^2 p5^4 + 2 p0 p1^2 p2 p3^2 p4^2 p5^4 + 2 p0 p1 p2^2 p3^2 p4^2 p5^4 + p0^2 p1^2 p2 p4^3 p5^4 + p0^2 p1 p2^2 p4^3 p5^4 + p0 p1^2 p2^2 p4^3 p5^4 + p0^2 p1 p2 p3 p4^3 p5^4 + p0 p1^2 p2 p3 p4^3 p5^4 + p0 p1 p2^2 p3 p4^3 p5^4


+7 p0^3 p1^2
^2 p1^3 p2 p3 p4
p3 p4^3 p5^2 + 2 p0
4^3 p5^2 - p0^2 p1 p
p3 p5^3 + 2 p0^3 p1
^4 p1^2 p2 p3^2
p2^4 p3^2

Conditional Poisson rounding counterexample

$$r_1 = \frac{507253948259935192780968006562751533342883849690618065351660559666521755532151106261121543553 63727}{820344544708777686628383921246921807685840301005725685964582767932498825215270734395260460182489056}$$

$$r_2 = \frac{16995635448467104183757178014765702738718813669542372343274191569647329181122211511135709477229515}{820344544708777686628383921246921807685840301005725685964582767932498825215270734395260460182489056}$$

$$r_3 = \frac{16995635448467104183757178014765702738718813669542372343274191569647329181122211511135709477229515}{820344544708777686628383921246921807685840301005725685964582767932498825215270734395260460182489056}$$

$$r_4 = \frac{203732522694899282198523379819523750032272292460663481324189224126535182285019873726167269517595483}{205086136177194421657095980311730451921460075251431421491145691983124706303817683598815115045622264}$$

$$r_5 = \frac{203732522694899282198523379819523750032272292460663481324189224126535182285019873726167269517595483}{205086136177194421657095980311730451921460075251431421491145691983124706303817683598815115045622264}$$

$$r_6 = \frac{746456786844211074651353568498768863987616551023722656078520071679268183450193679728059651096880547}{820344544708777686628383921246921807685840301005725685964582767932498825215270734395260460182489056}$$

There is a non-monotone perturbation causing the probability of selecting {1, 2, 3} to decrease on the order of 10^{-26}.

Our actual hero: Sampford rounding

Sample k parties sequentially, *with* replacement.

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Theorem (Sampford, 1967)

Sampford rounding satisfies ex ante proportionality.

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Theorem (Sampford, 1967)

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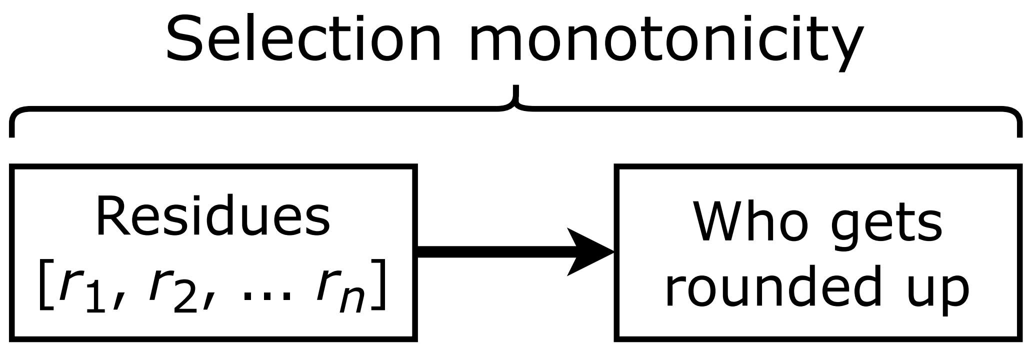
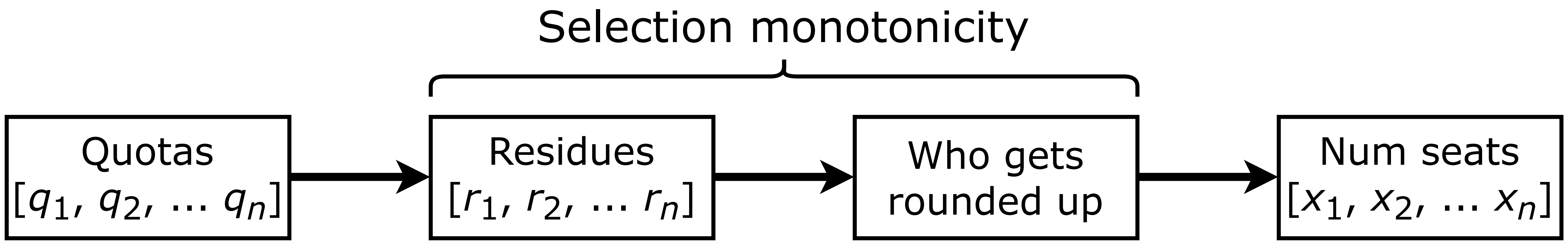
Theorem

Sampford rounding satisfies selection monotonicity.

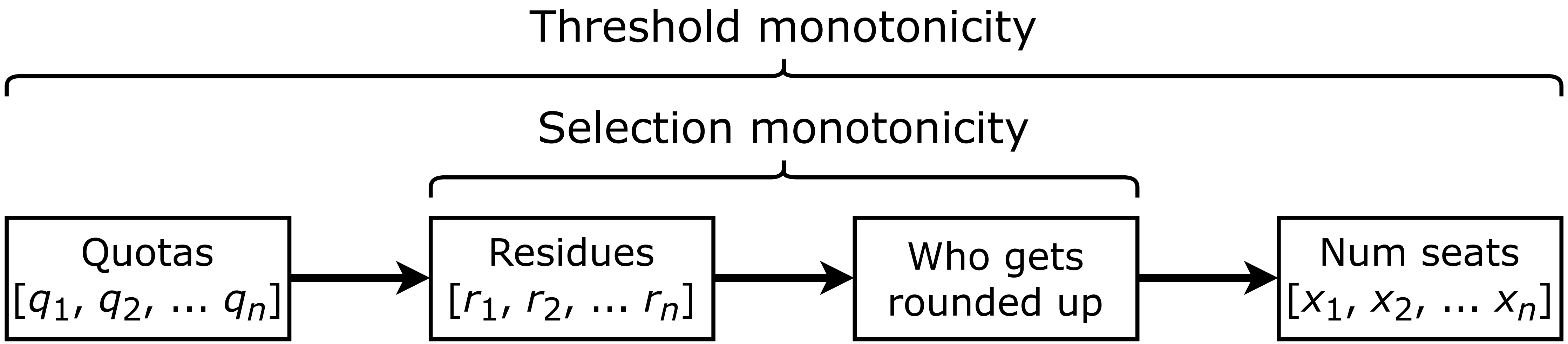
Stronger coalitional monotonicity axioms



Stronger coalitional monotonicity axioms

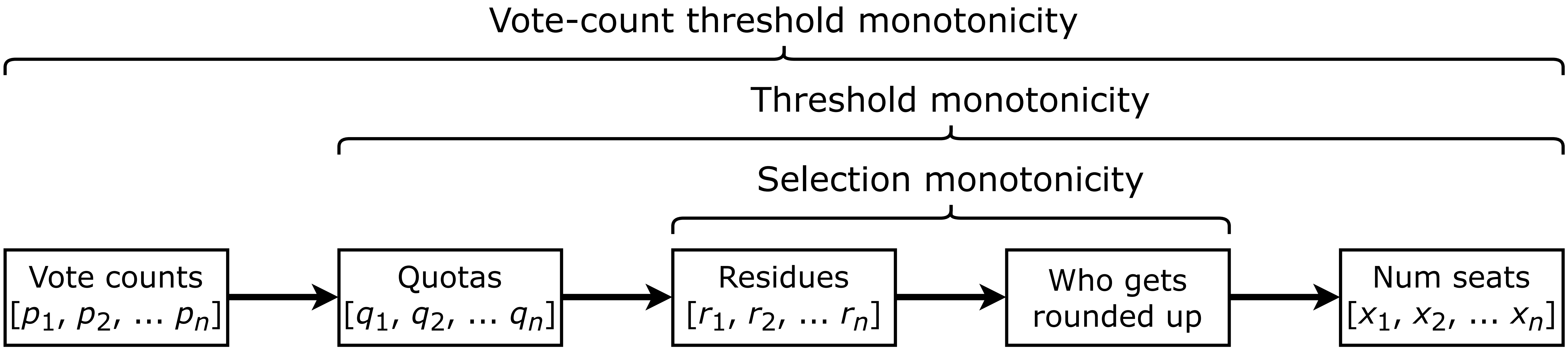


Stronger coalitional monotonicity axioms



Threshold monotonicity: For any threshold m, if shares of parties in T weakly increase and shares of parties outside of T weakly decrease, the probability of T controlling at least m seats weakly increases.

Stronger coalitional monotonicity axioms



Threshold monotonicity: For any threshold *m*, if shares of parties in *T* weakly increase and shares of parties outside of *T* weakly decrease, the probability of *T* controlling at least *m* seats weakly increases.

Vote-count threshold monotonicity: For any threshold *m*, if the numbers of votes for parties in *T* weakly increase and the numbers of votes for parties outside of *T* weakly decrease, the probability of *T* controlling at least *m* seats weakly increases.

What is known

Theorem

Grimmett's method satisfies threshold monotonicity for coalitions T of size 2.

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Conjecture

Sampford rounding satisfies threshold monotonicity.

What is known

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Theorem

No (ex ante proportional) apportionment method satisfies vote-count threshold monotonicity.

The Treaty of Rome

ARTICOLO 148

1. Salvo contrarie disposizioni del presente Trattato, le deliberazioni del Consiglio sono valide se approvate a maggioranza dei membri che lo compongono.

2. Per le deliberazioni del Consiglio che richiedono una maggioranza qualificata, ai voti dei membri è attribuita la seguente ponderazione:

Belgio	2
Germania	4
Francia	4
Italia	4
Lussemburgo	1
Paesi Bassi	2

Le deliberazioni sono valide se hanno raccolto almeno:

— dodici voti quando, in virtù del presente Trattato, debbono essere prese su proposta della Commissione,

— dodici voti che esprimano la votazione favorevole di almeno quattro membri, negli altri casi.

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Belgio	2
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Le deliberazioni sono valide se hanno raccolto almeno:

12 votes total

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Banzhaf Power Index

For a given player i, suppose all other players vote Yes independently with probability $\frac{1}{2}$. The *Banzhaf power index* $\beta_i \in [0, 1]$ is defined as the probability that player i's vote is pivotal.

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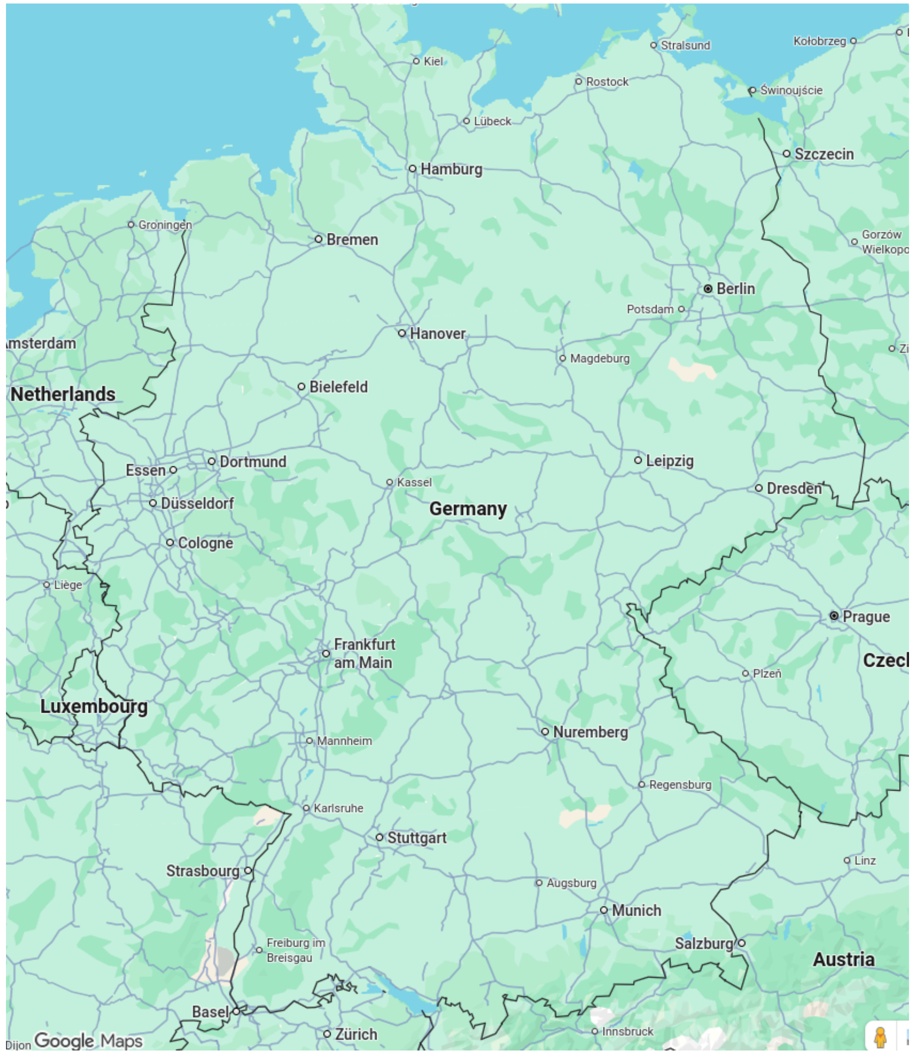
(12 needed to win)



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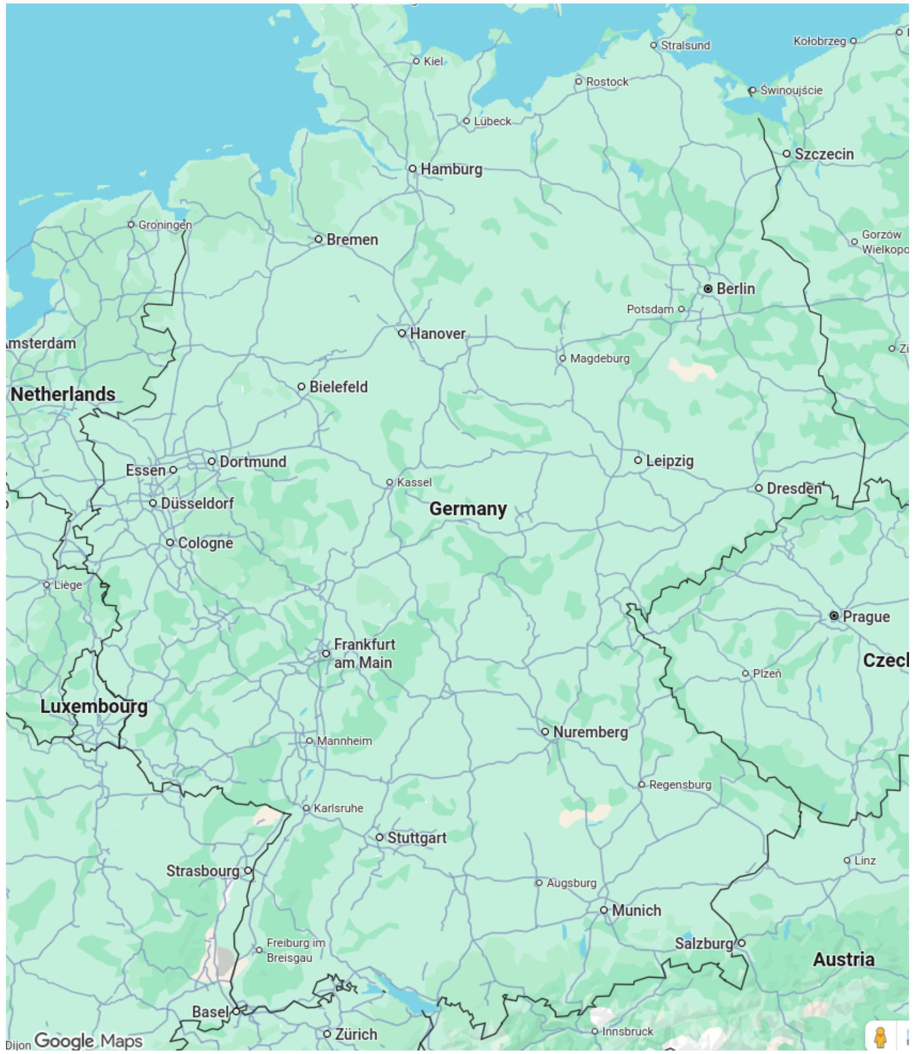

▸ **By what factor is the BPI of Germany (weight 4) greater than the BPI of Luxembourg (weight 1)?**

Respond at:

pollev.com/jtuckerfoltz255 or

bit.ly/jtfpoll or

text jtuckerfoltz255 to 37607

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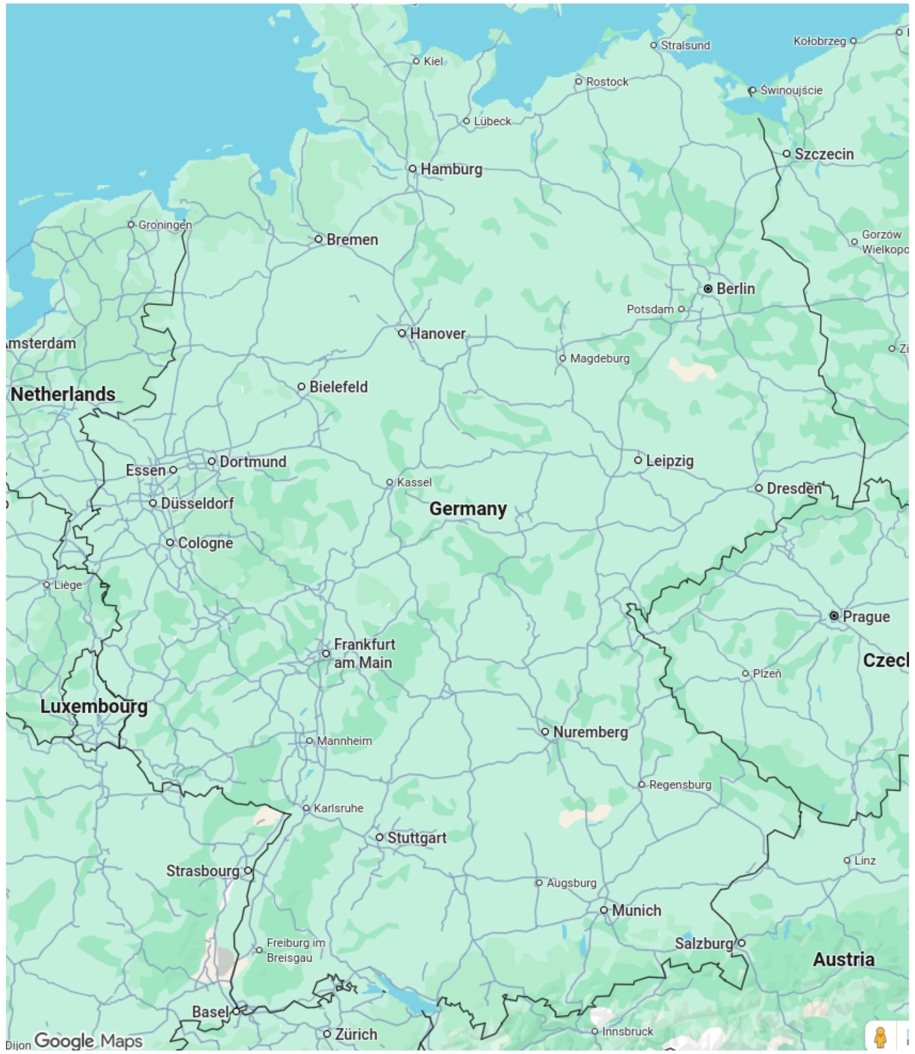
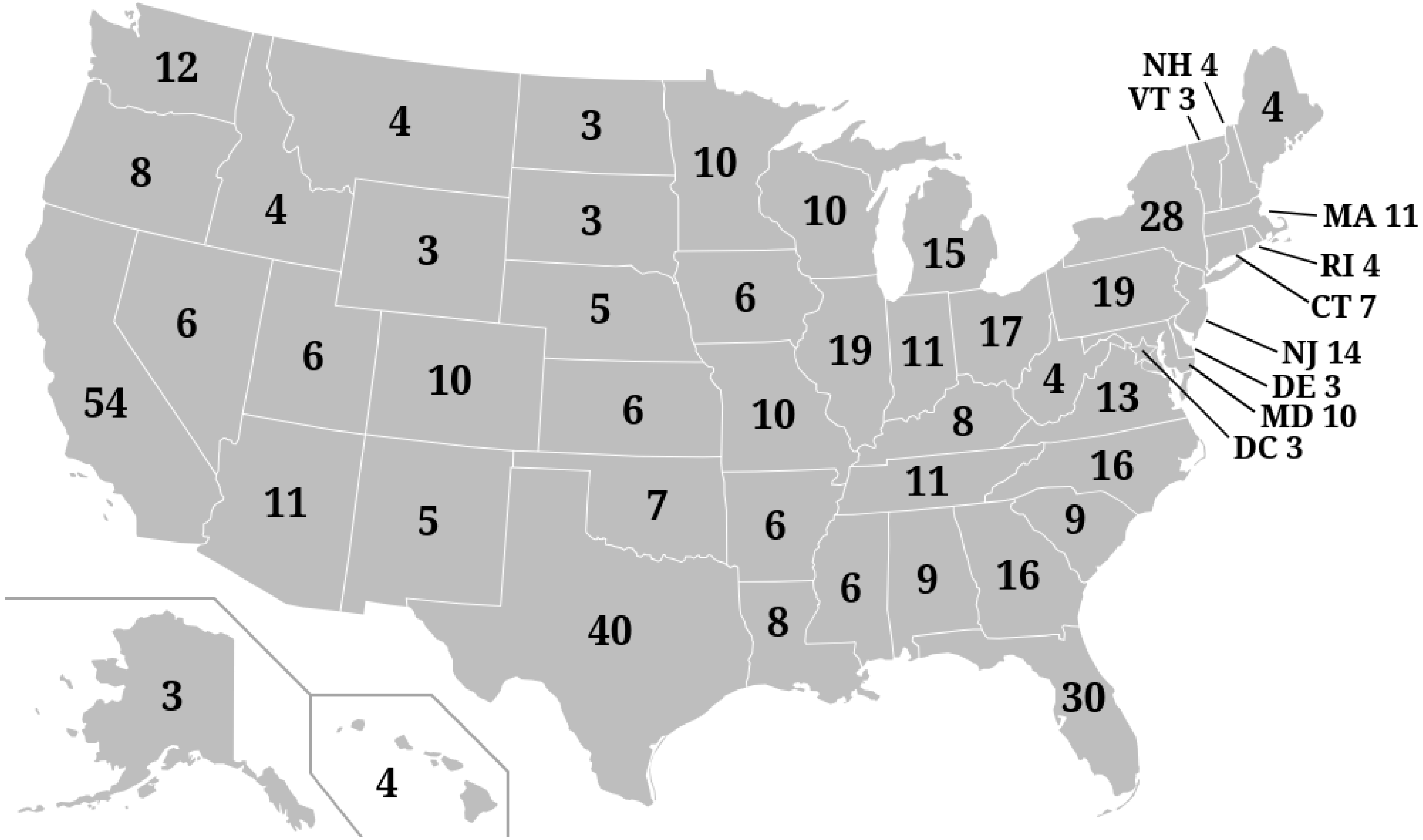


Answer: ∞!

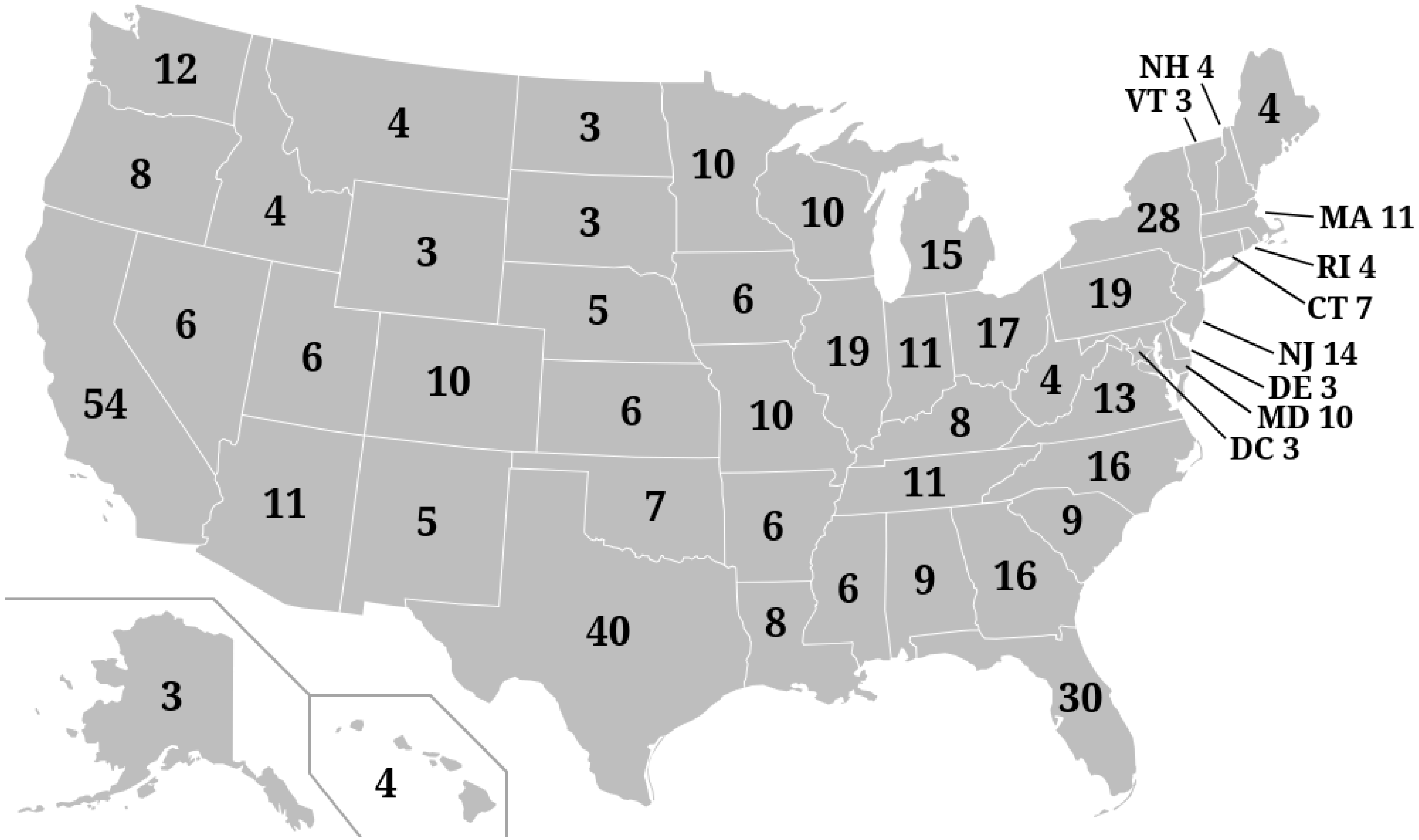
Luxembourg is a *dummy player*: no matter how other players vote, Luxembourg can never affect the outcome in this voting game, so it has power zero under any reasonable metric.

▸ By what factor is the BPI of Germany (weight 4) greater than the BPI of Luxembourg (weight 1)?

Other examples of weighted voting



Other examples of weighted voting



Town name	Adj. Pop.	Pop. share	$T = 1/2$ weight
Town of South Bristol	1641	0.01459	1488
Town of Canadice	1668	0.01483	1511
Town of Bristol	2284	0.02031	2106
Town of Naples	2403	0.02137	2211
Town of Seneca	2644	0.02351	2421
Town of West Bloomfield	2740	0.02437	2506
Town of Richmond	3360	0.02988	3123
Town of Geneva	3473	0.03088	3204
Town of East Bloomfield	3640	0.03237	3340
City of Geneva (5,6)	3679	0.03272	3373
City of Geneva (3,4)	3921	0.03487	3582
Town of Hopewell	3931	0.03496	3589
Town of Gorham	4106	0.03651	3740
City of Canandaigua (2,3)	5140	0.04571	4682
City of Geneva (1,2)	5210	0.04633	4740
City of Canandaigua (1,4)	5436	0.04834	4932
Town of Phelps	6637	0.05902	6000
Town of Manchester	9404	0.08362	8381
Town of Canandaigua	11109	0.09879	9748
Town of Farmington	14170	0.12600	12092
Town of Victor	15860	0.14103	13231

Iannuchi v. Board of Supervisors (1967)

"The principle of one man-one vote is violated, however, when the power of a representative to affect the passage of legislation by his vote... does not roughly correspond to the proportion of the population in his constituency."

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https://www.jamie.tuckerfoltz.com/Projects/Banzhaf/Demo-5.html?pop=1641-1668-2284-2403-2644-2740-3360-3473-3640-3679-3921-3931-4106-5140-5210-5436-6637-9404-11109-14170-15860

The Banzhaf distortion problem

Fix a vector of weights $w_1, w_2, \ldots, w_n$ summing to T, and consider the relative powers of i and j.

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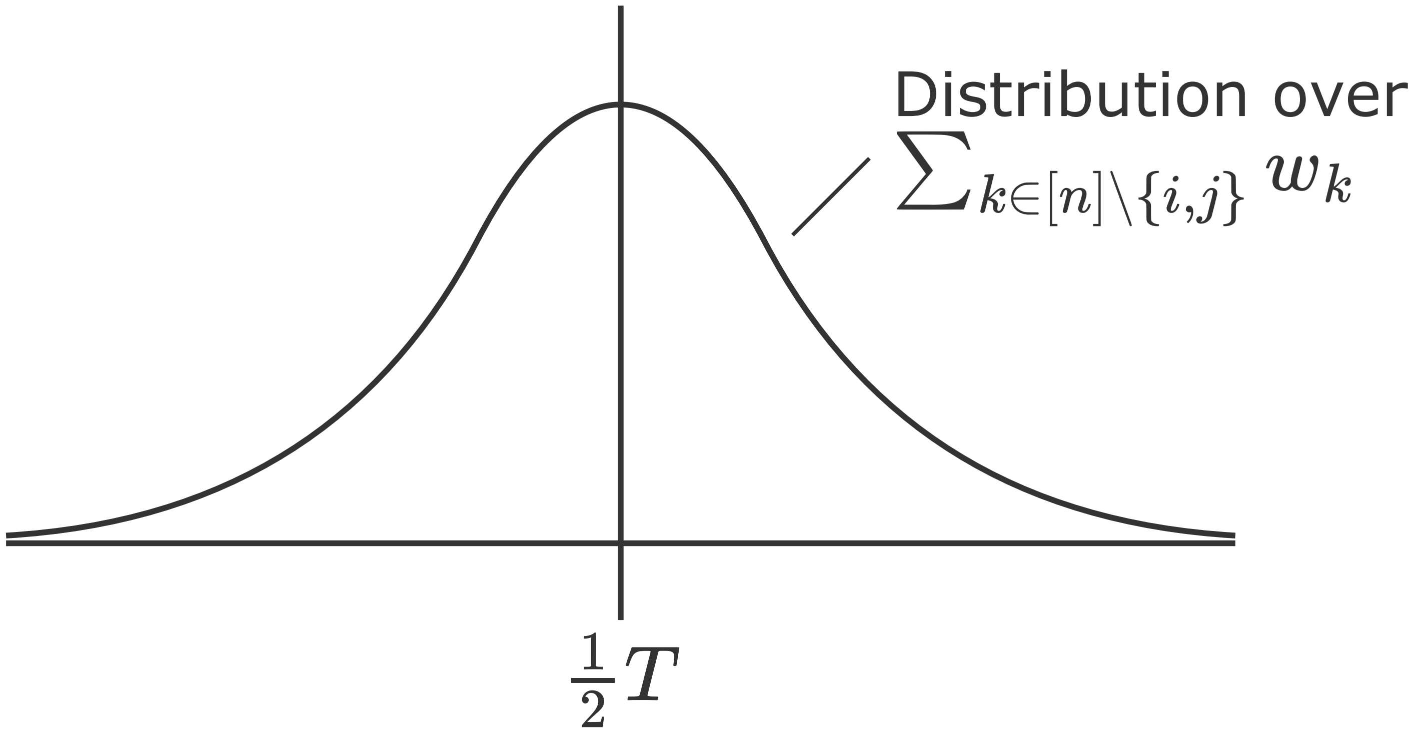

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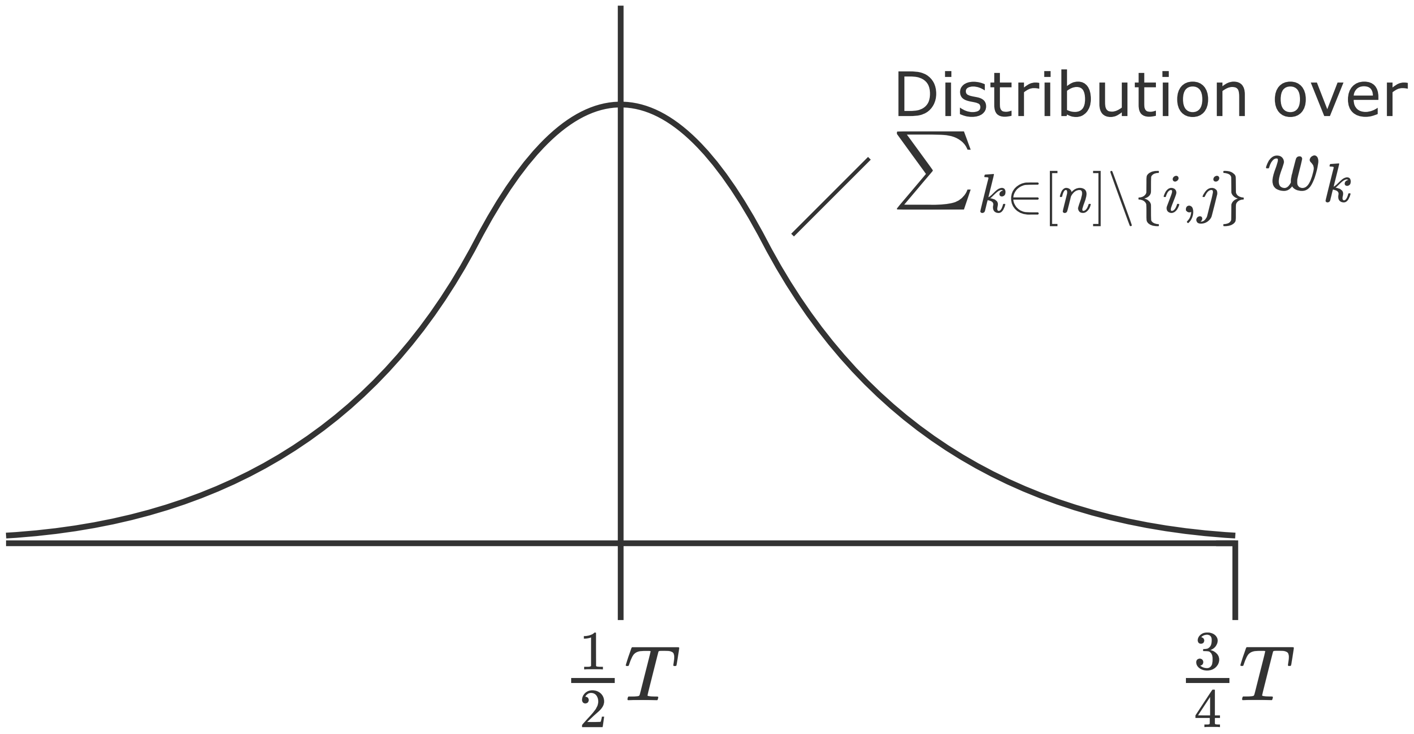

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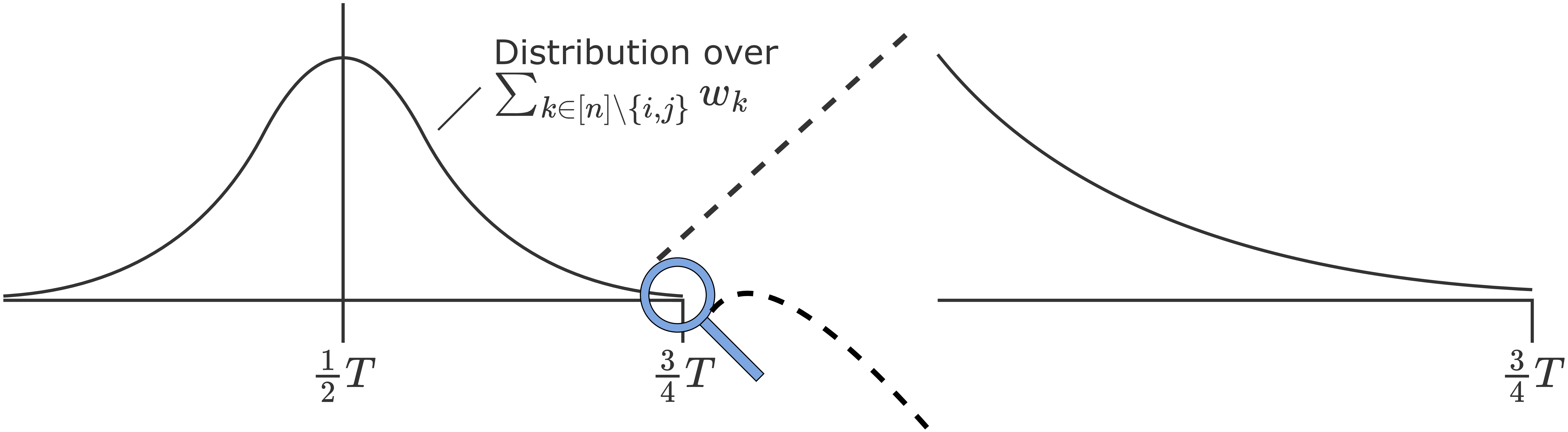

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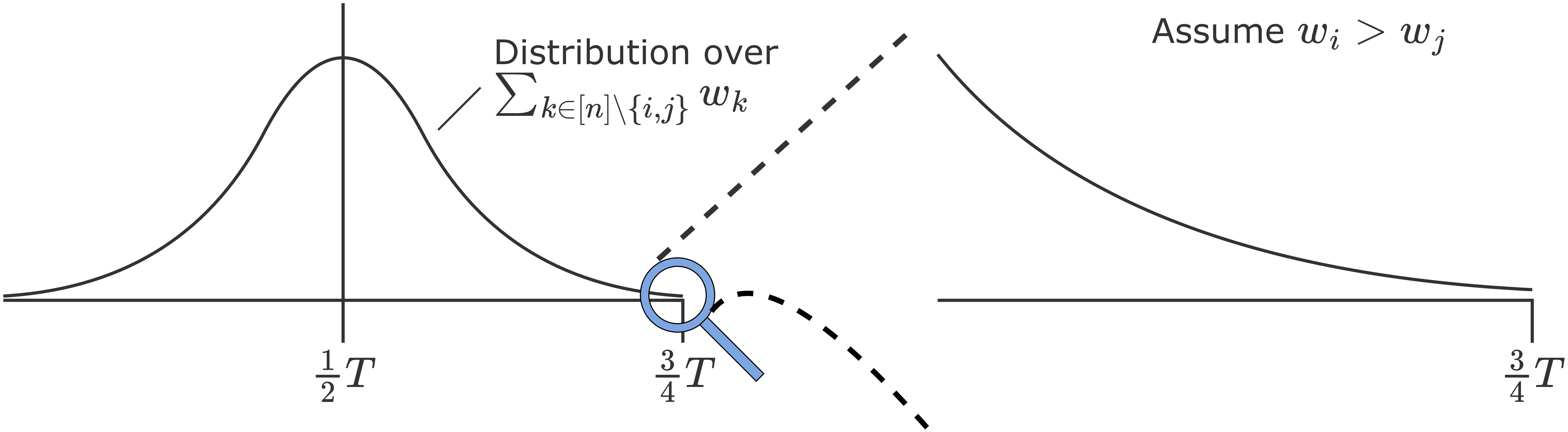

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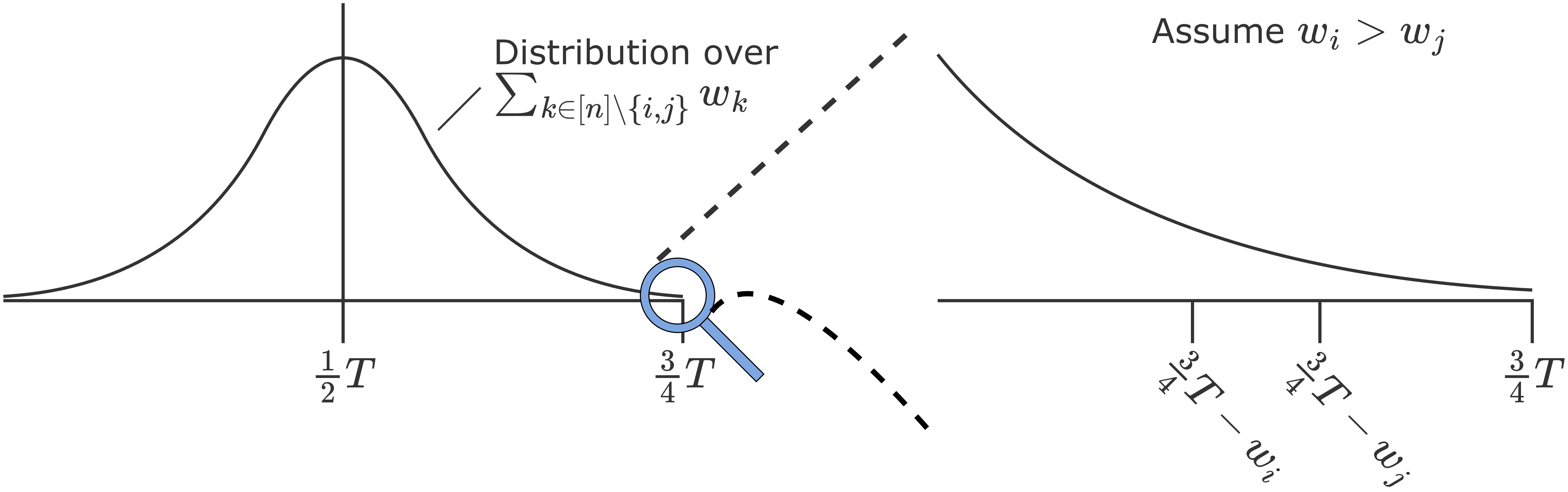

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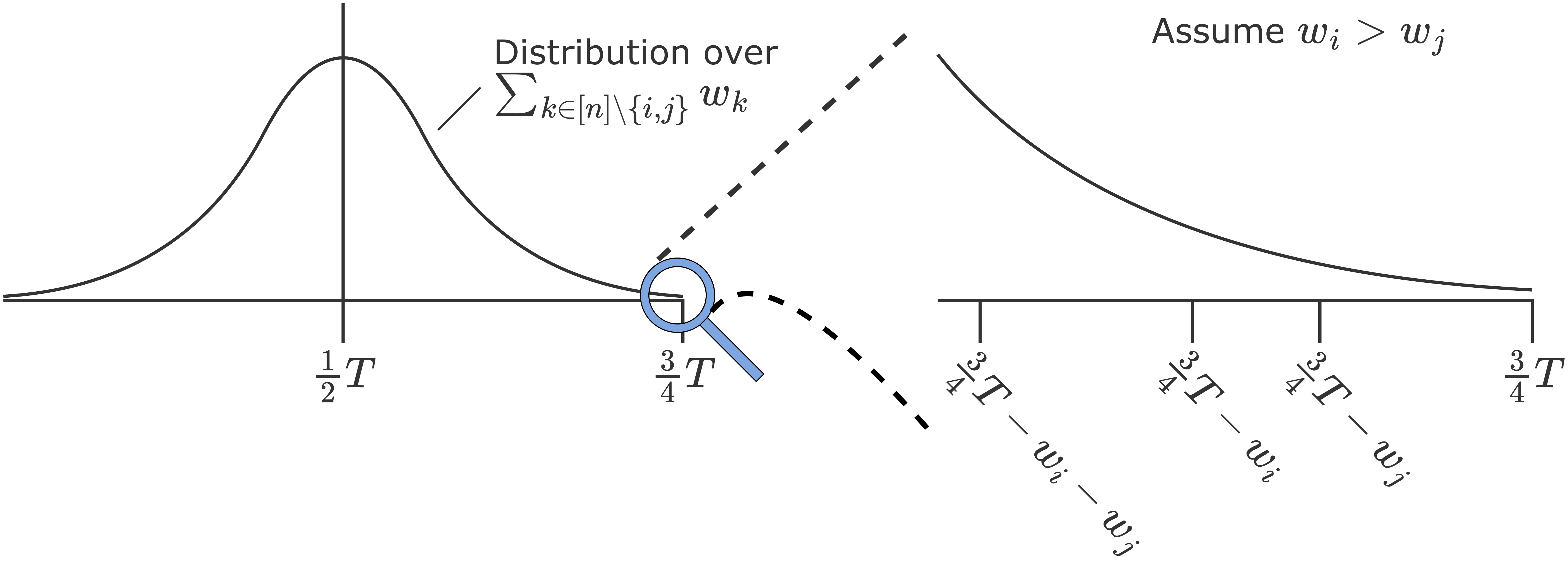

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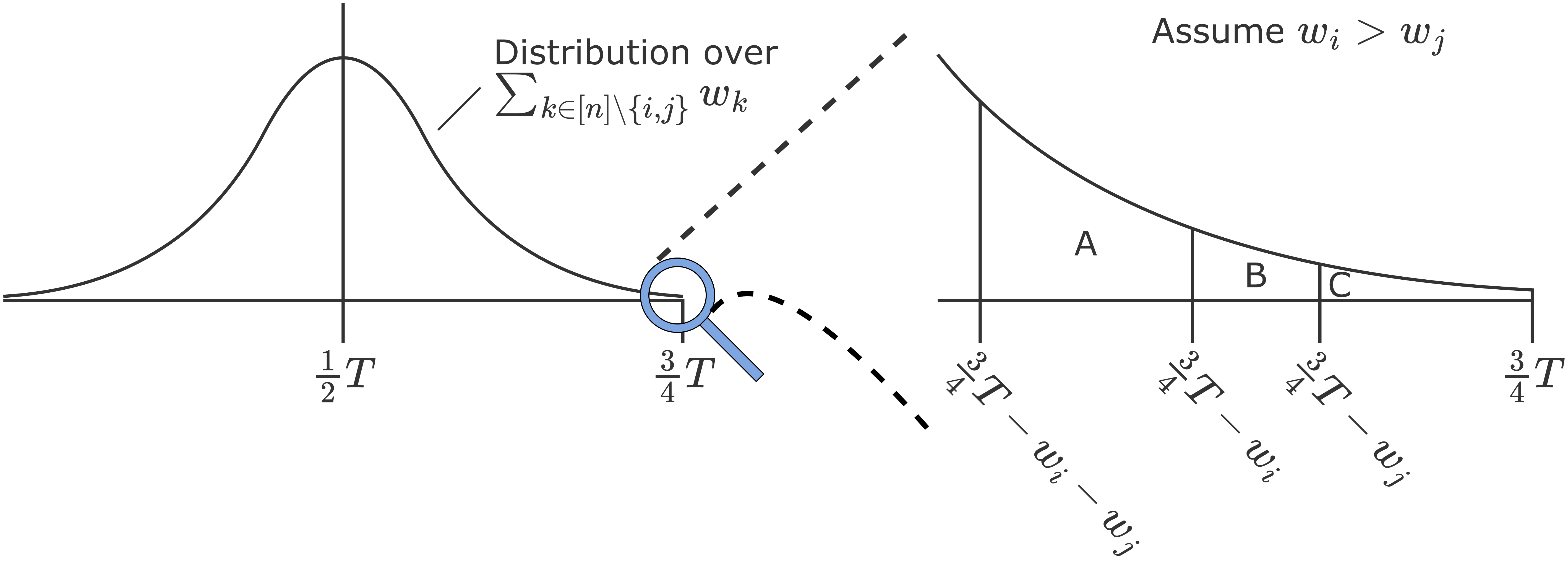

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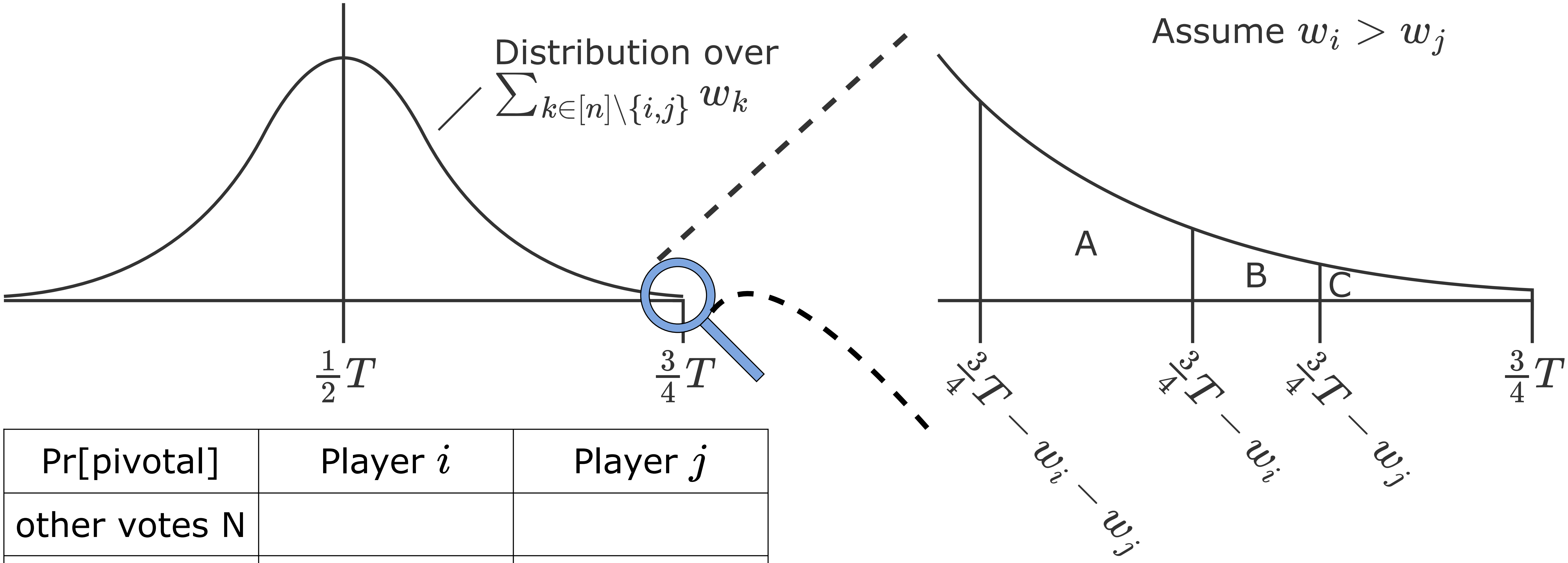

Pr[pivotal]	Player i	Player j
other votes N		
other votes Y		

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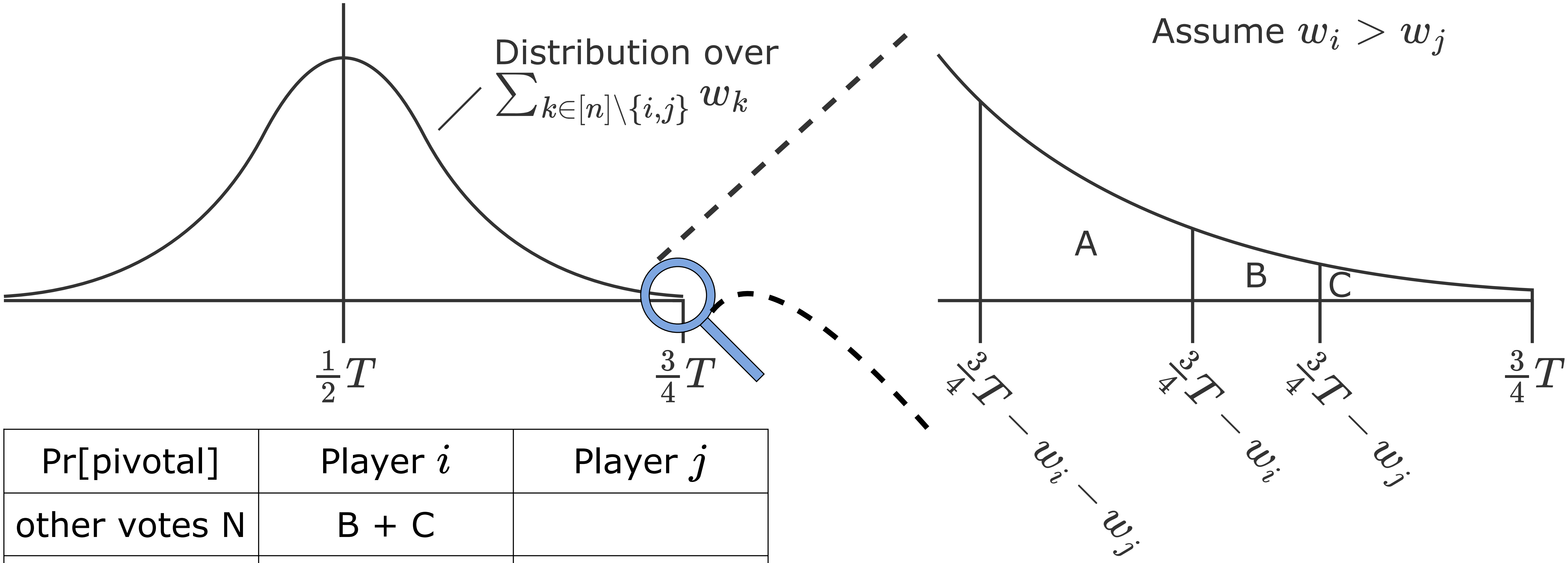

Pr[pivotal]	Player i	Player j
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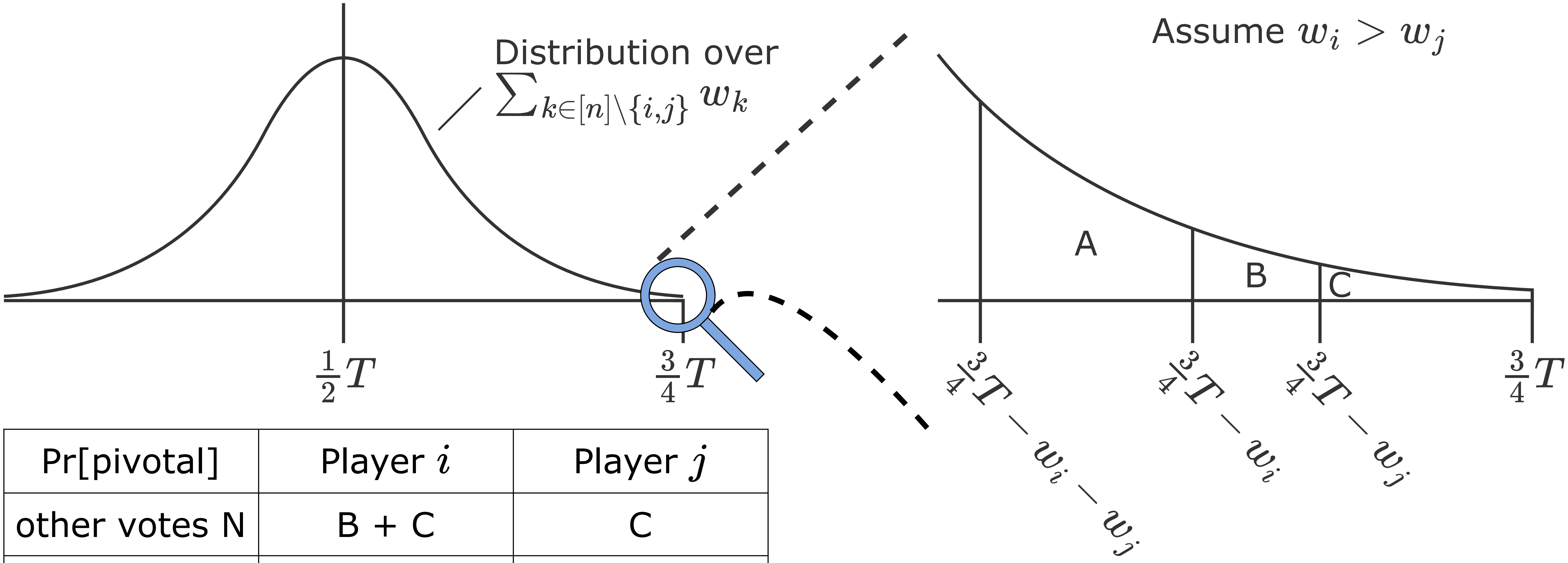

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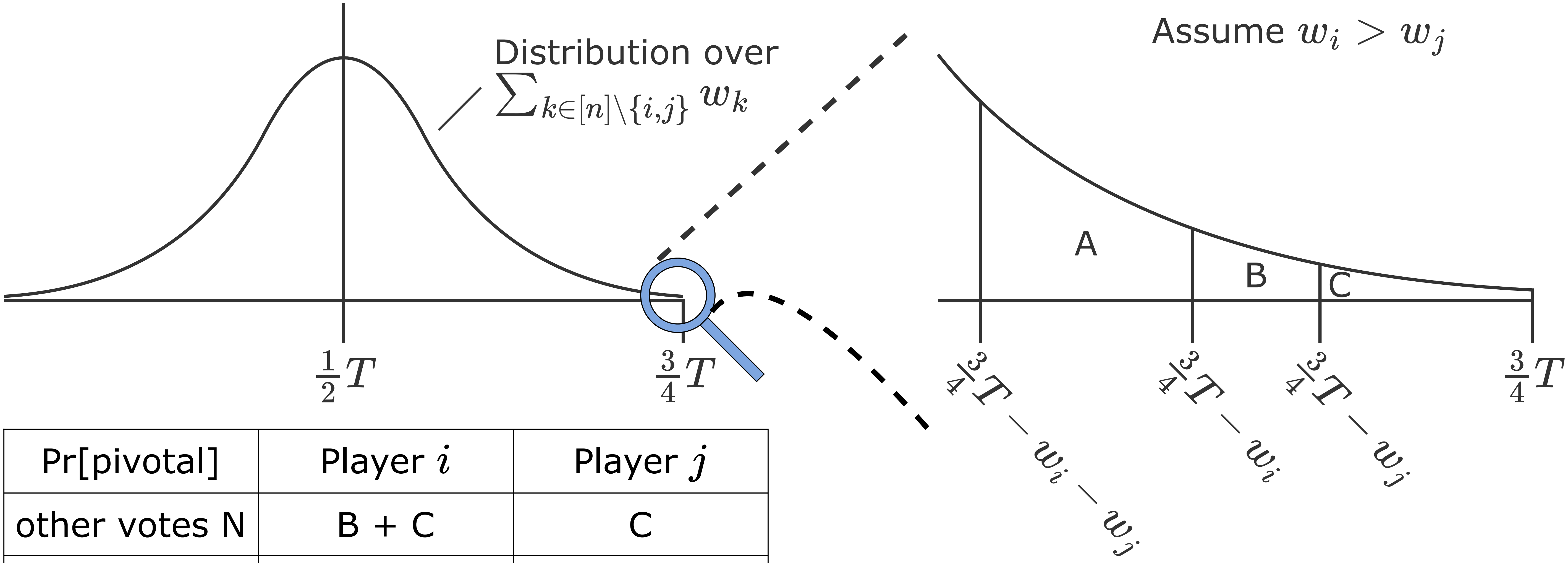

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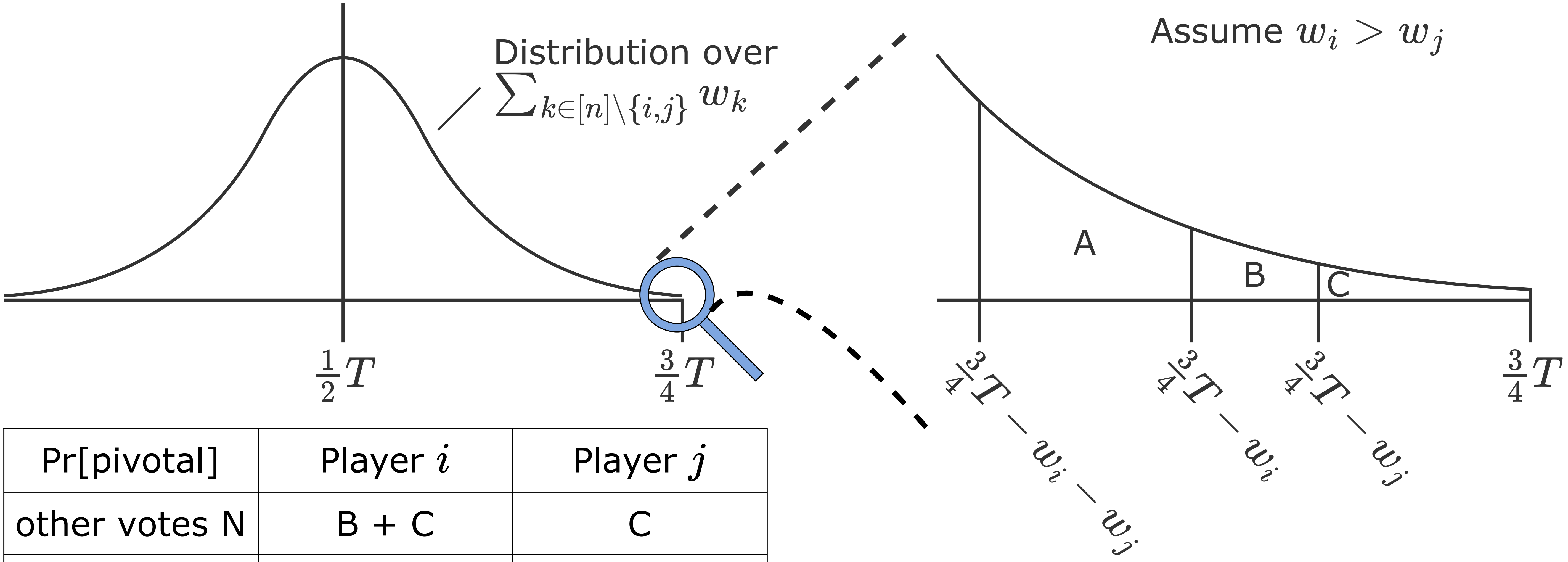

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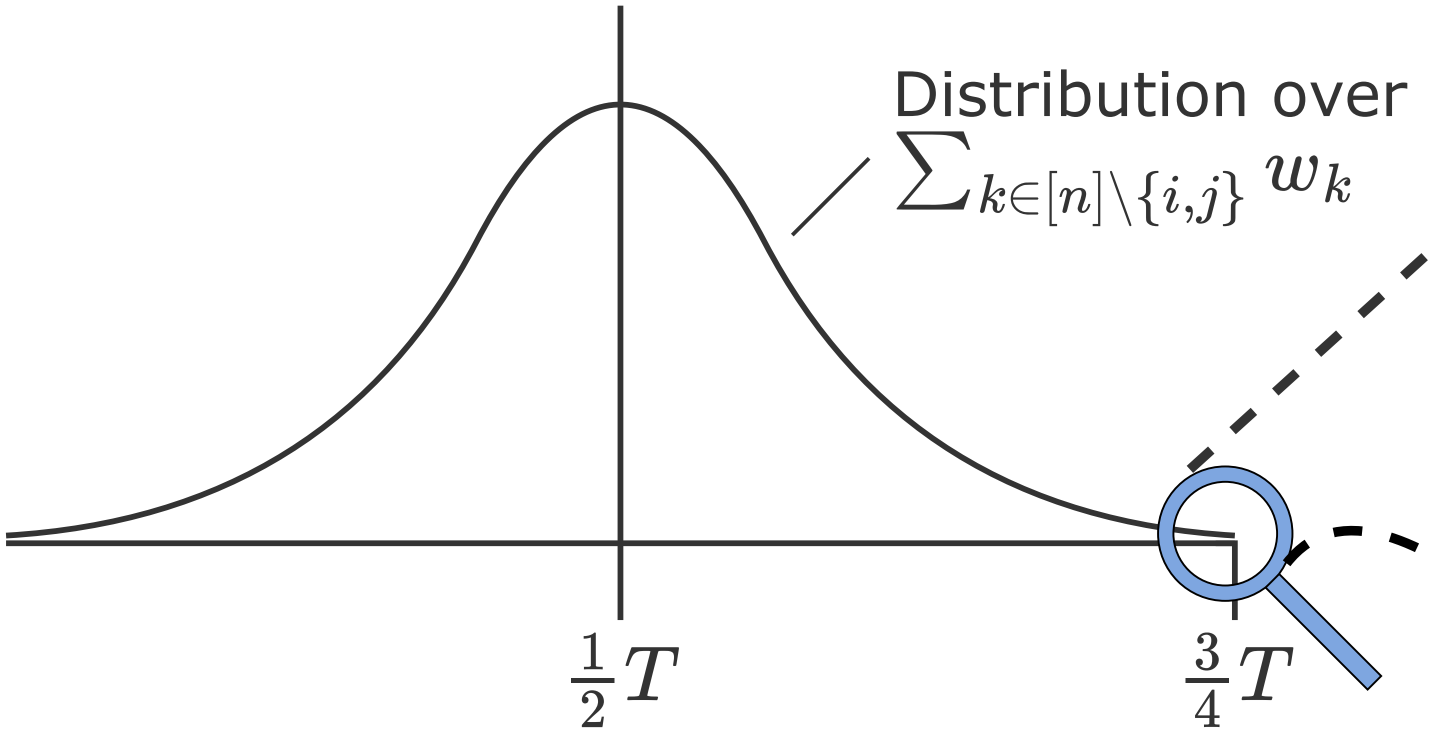

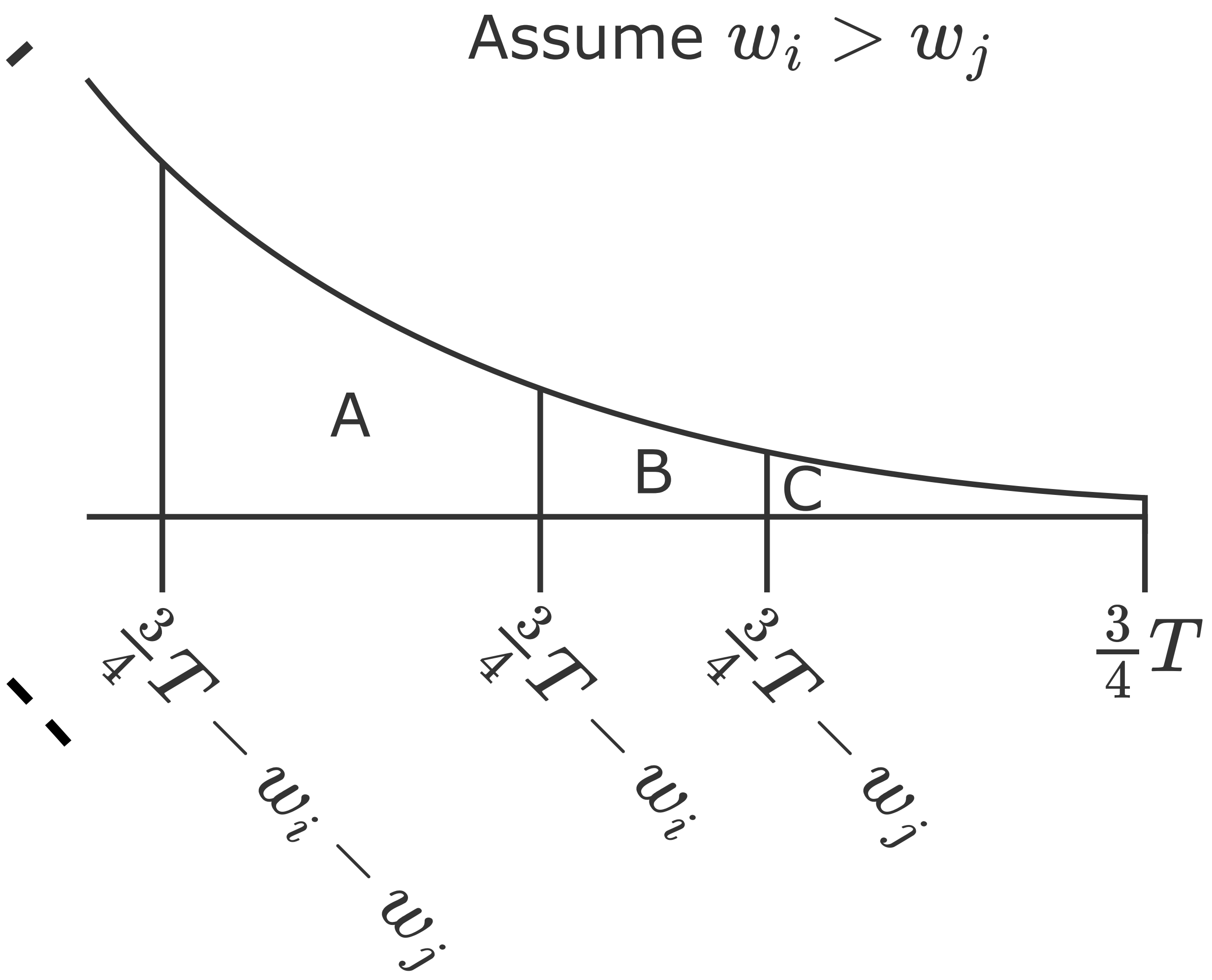

Pr[pivotal]	Player i	Player j
other votes N	(small)	(small)
other votes Y	A + B	A

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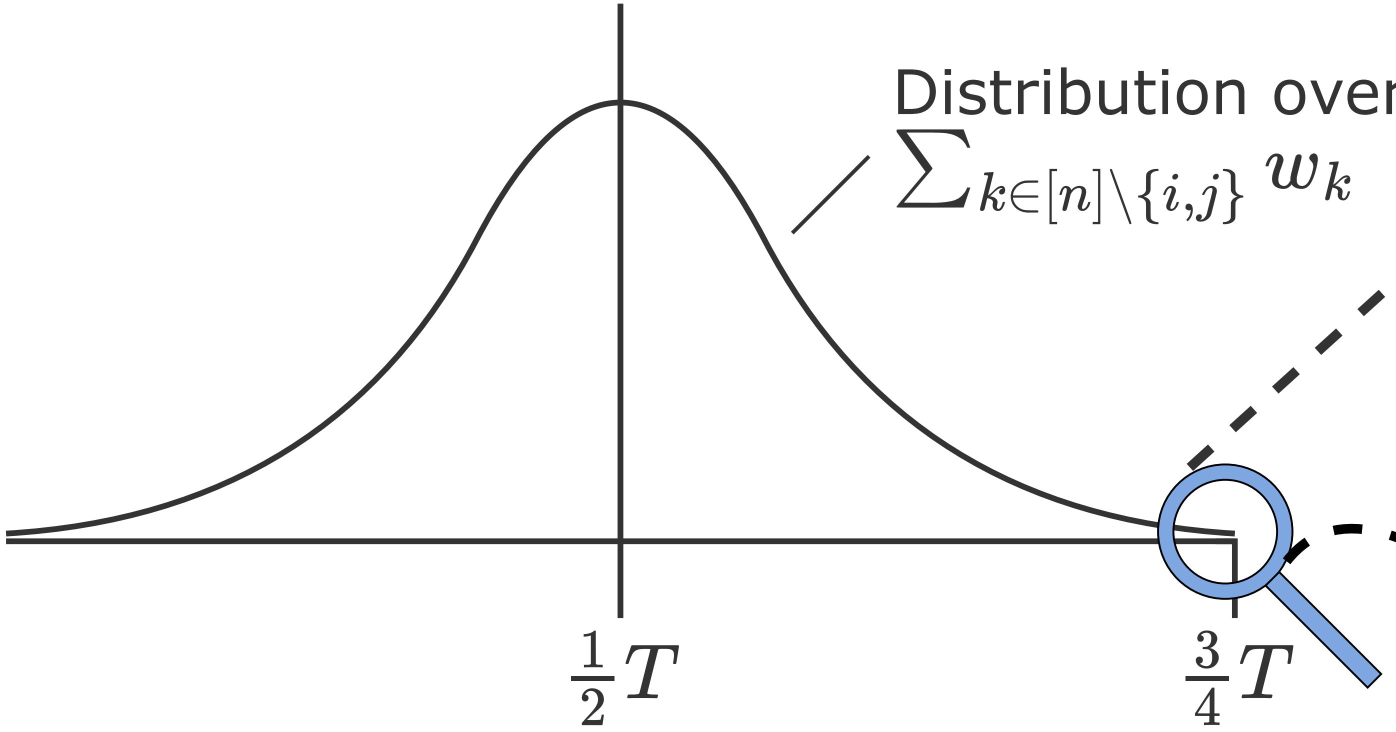

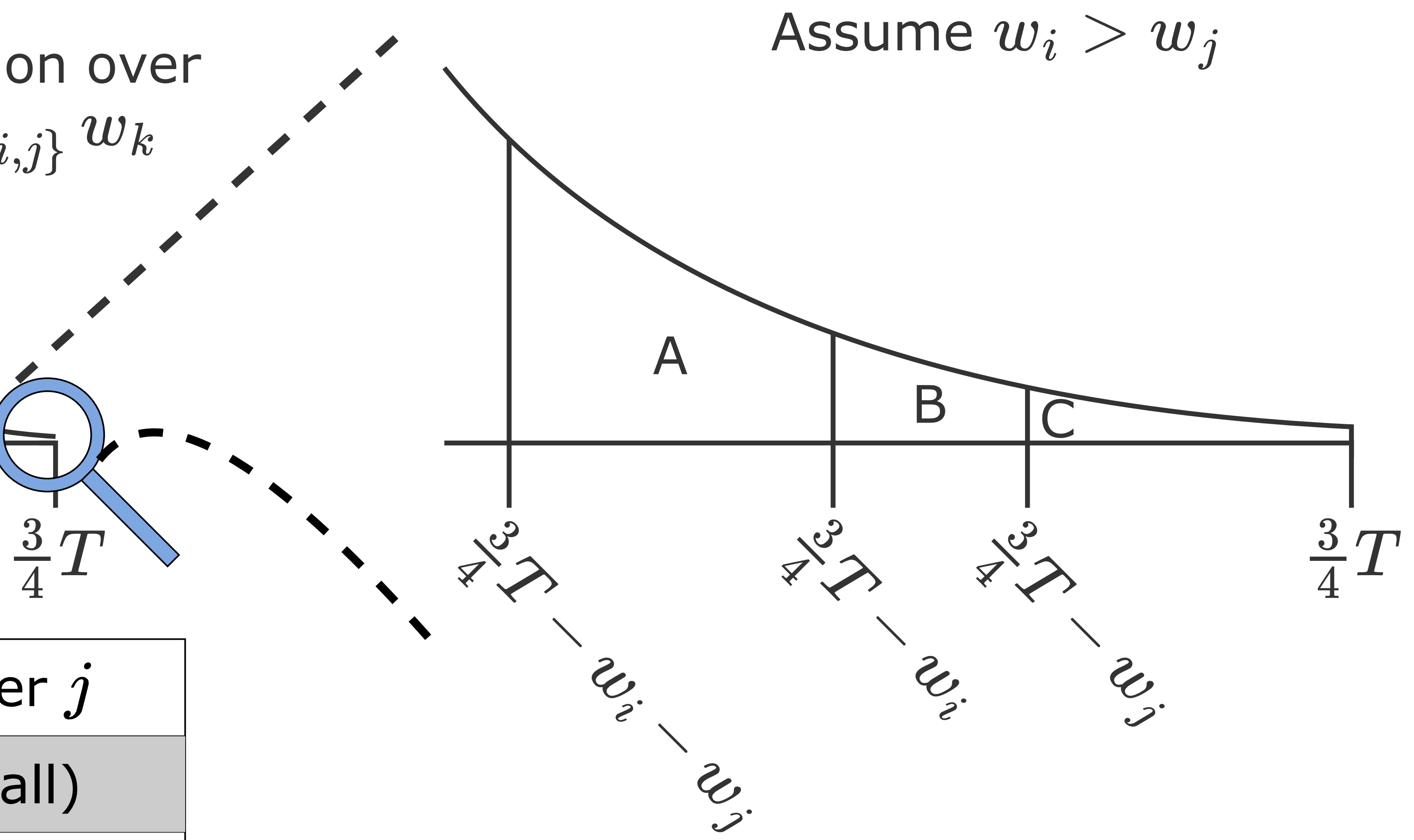

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Since A is artificially large, (A + B)/A is artificially close to 1 (powers are equalized).

Solution 1: *propensity-Banzhaf* (de Raaij, Duchin, Procaccia, T-F, 2026)

Theorem

Assume voters vote Yes with probability p. Under any sequence of voting games (satisfying some regularity assumptions) on increasing numbers of players n, the limiting ratio of p-propensity Banzhaf indices with threshold q is

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$$\lim_{n\to\infty} \frac{\beta_i}{\beta_j} = \frac{\left(\sum_{k=1}^{w_i} r^{k-1}\right)(1-p-pr^{w_j})}{\left(\sum_{k=1}^{w_j} r^{k-1}\right)(1-p-pr^{w_i})}$$

where r is the solution to

$$\sum_{w=1}^{\infty}\left(d_w \frac{pwr^2}{1-p+pr^2}\right) = q\sum_{w=1}^{\infty} d_w w.$$

Limiting density of weight w

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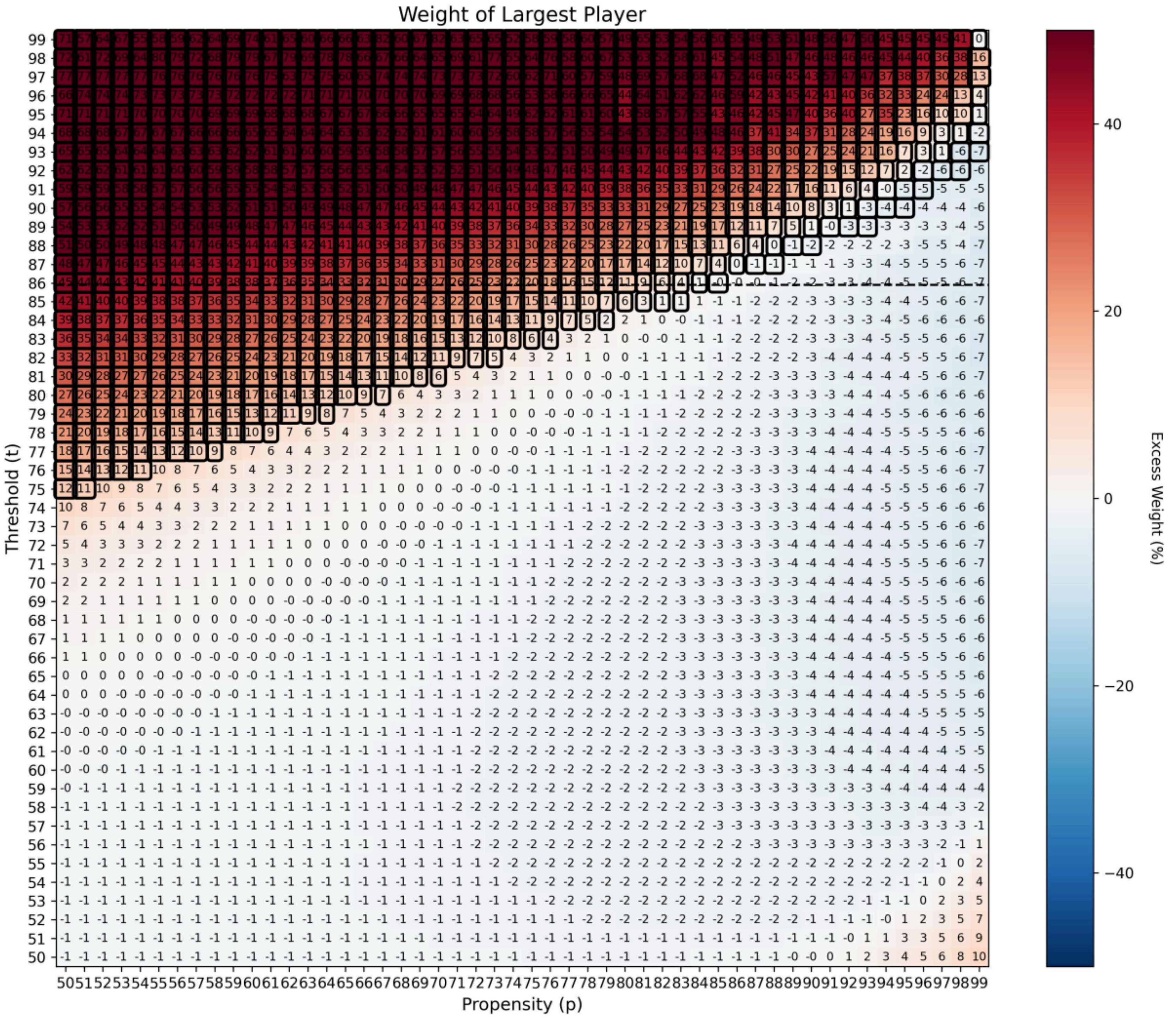
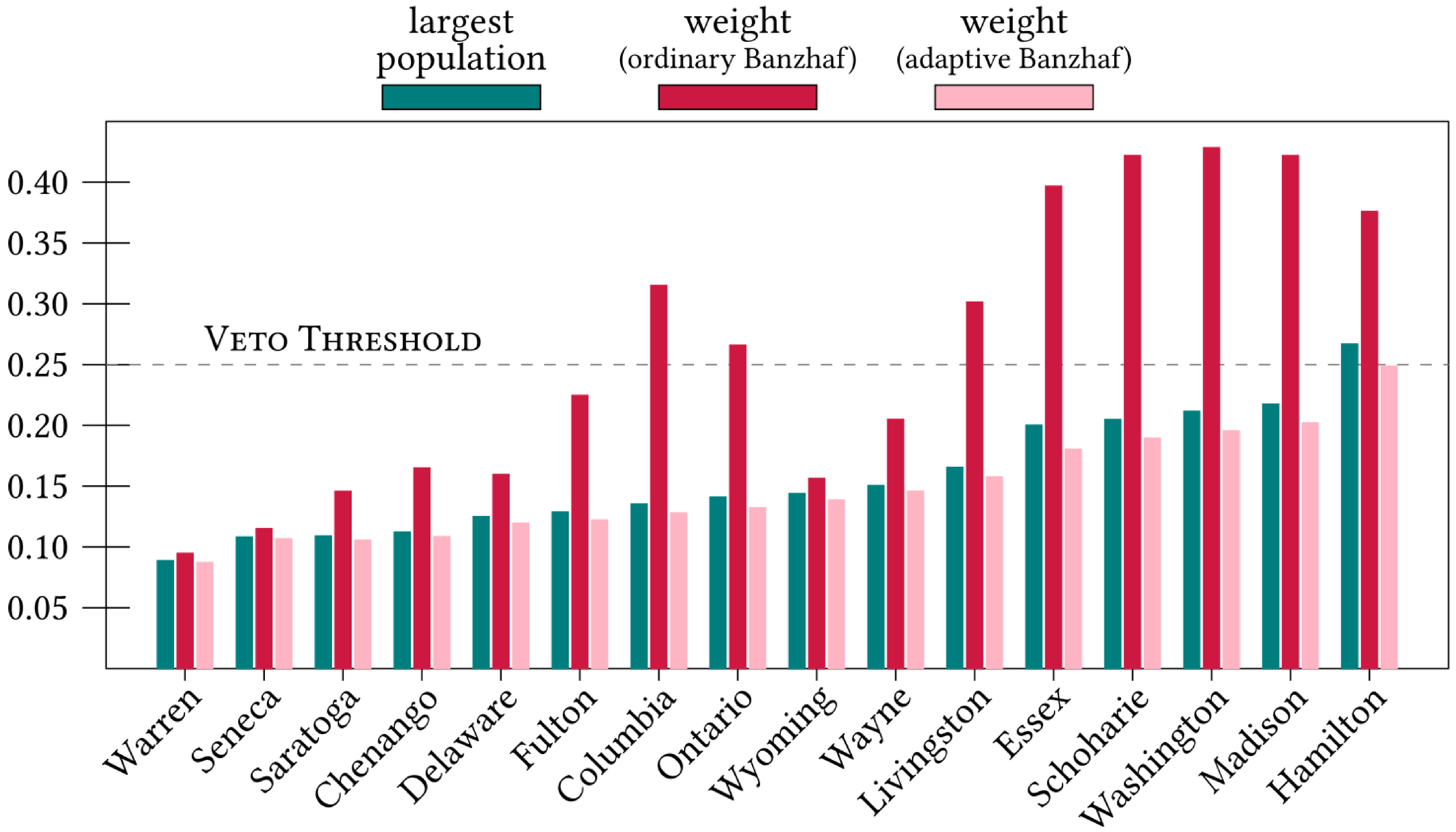
Powers converge to weights iff $p = q$.

Victor's weight with different thresholds



All 16 NY counties at $q = \frac{3}{4}$ and $p \in \{\frac{1}{2}, \frac{3}{4}\}$



Solution 2: Random ordering (Shapley, Shubik, 1954)

Banzhaf: $$\beta_i = \frac{1}{2^{n-1}} \sum_{S \subseteq [n] \setminus \{i\}} \mathbf{1}[qT - w_i \leq \Sigma_{s \in S} < qT]$$

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Note: Unlike Banzhaf and propensity-Banzhaf, the Shapley-Shubik indices naturally sum to one, as they represent probabilities in the same probability space:

Randomly order the players, and start counting weights according to the random order. The player who first causes the count to reach the threshold q is the pivotal one. The Shapley-Shubik index is the probability of being pivotal under this process.