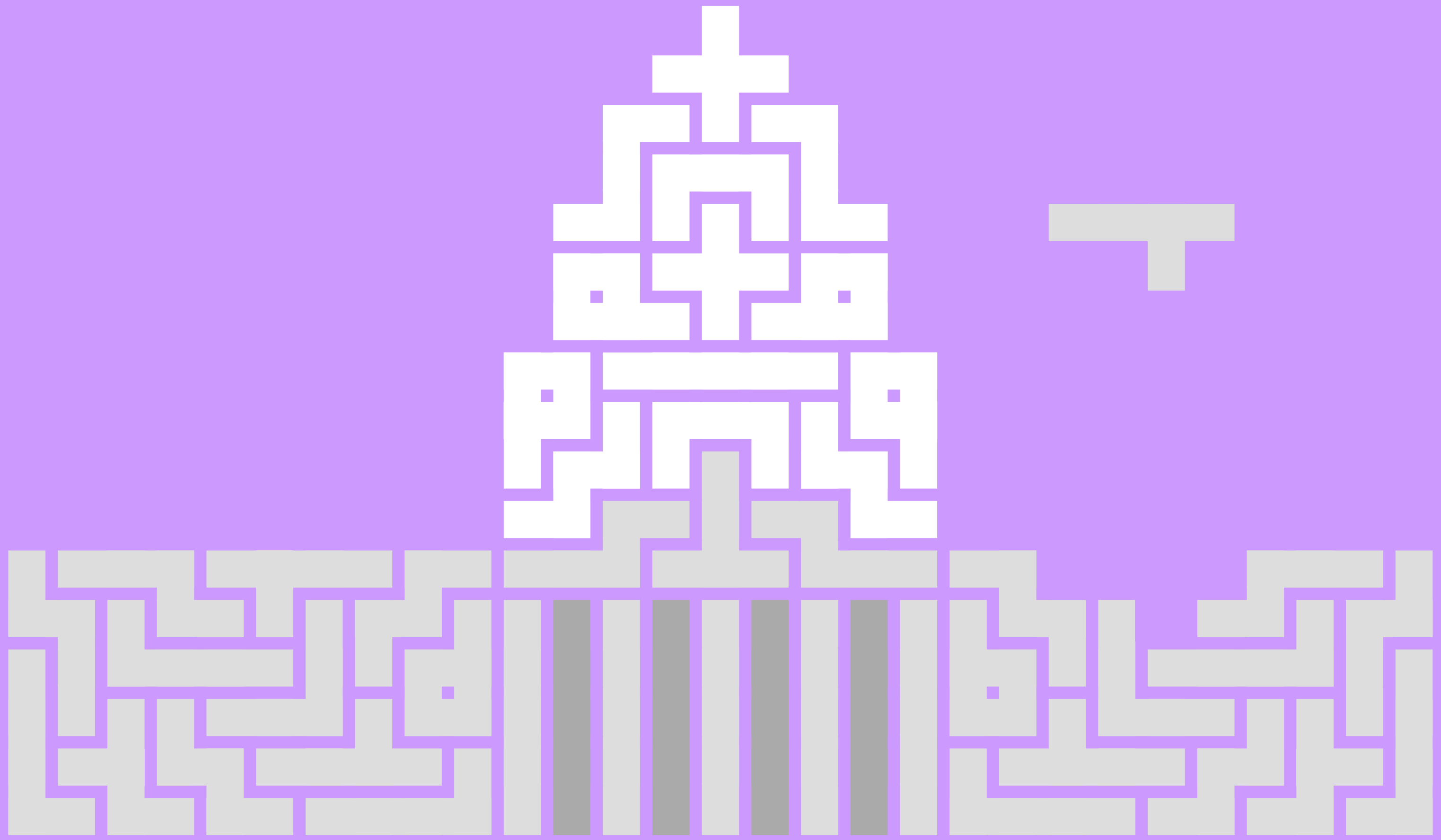

Algorithms For Democratic Decision-Making

Jamie Tucker-Foltz • Yale University • Spring 2026

Lecture 23: **Condorcet Winners**

Condorcet winning sets

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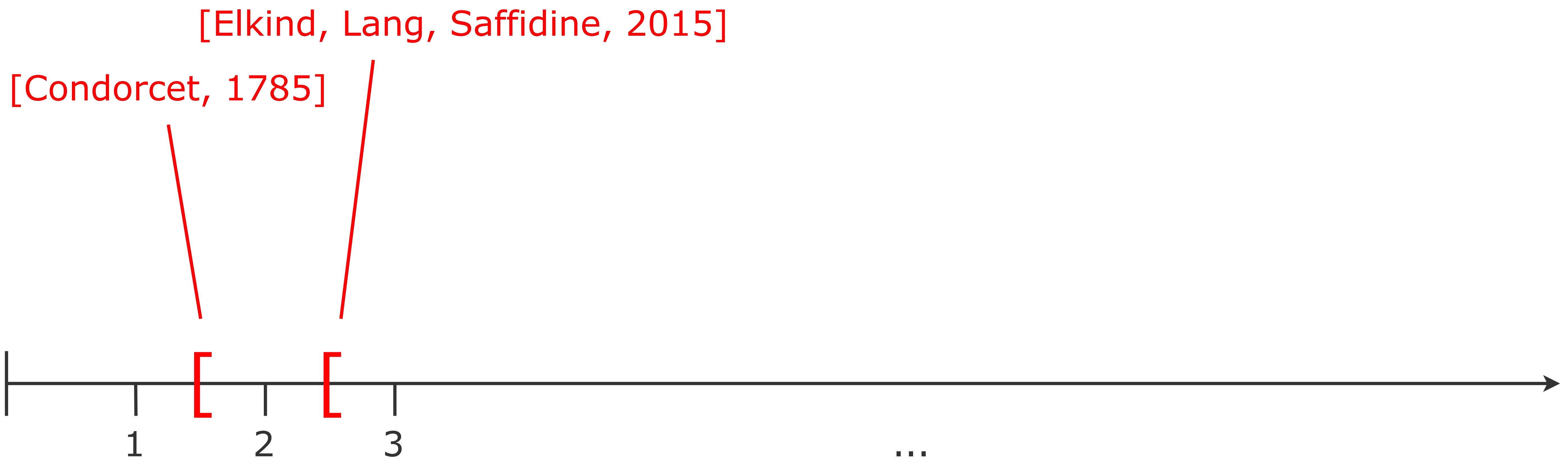

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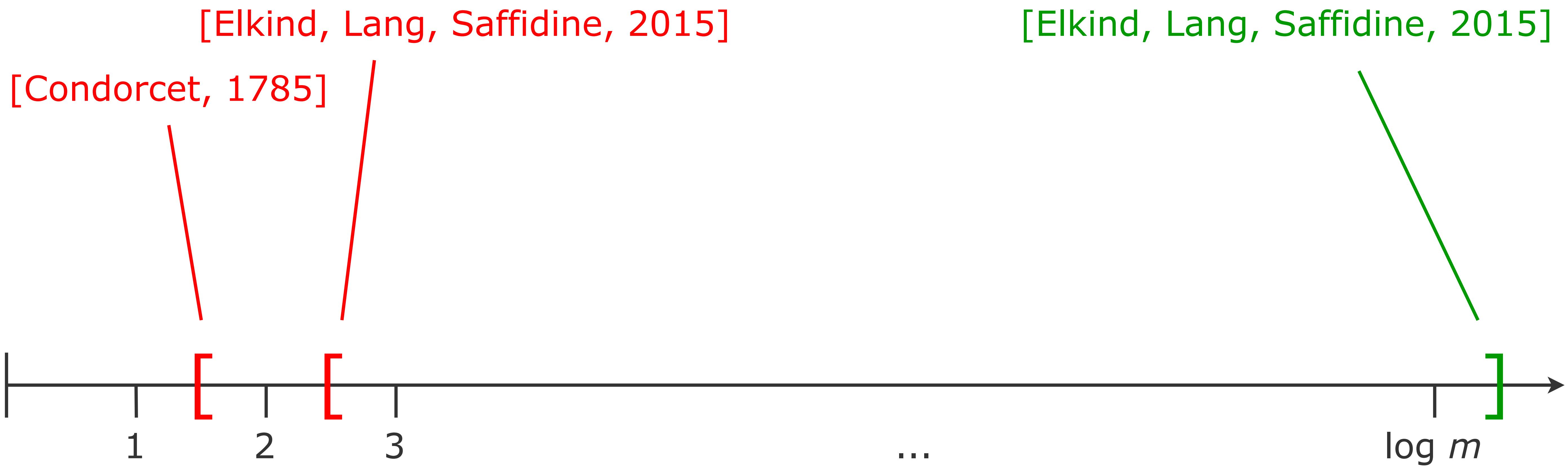

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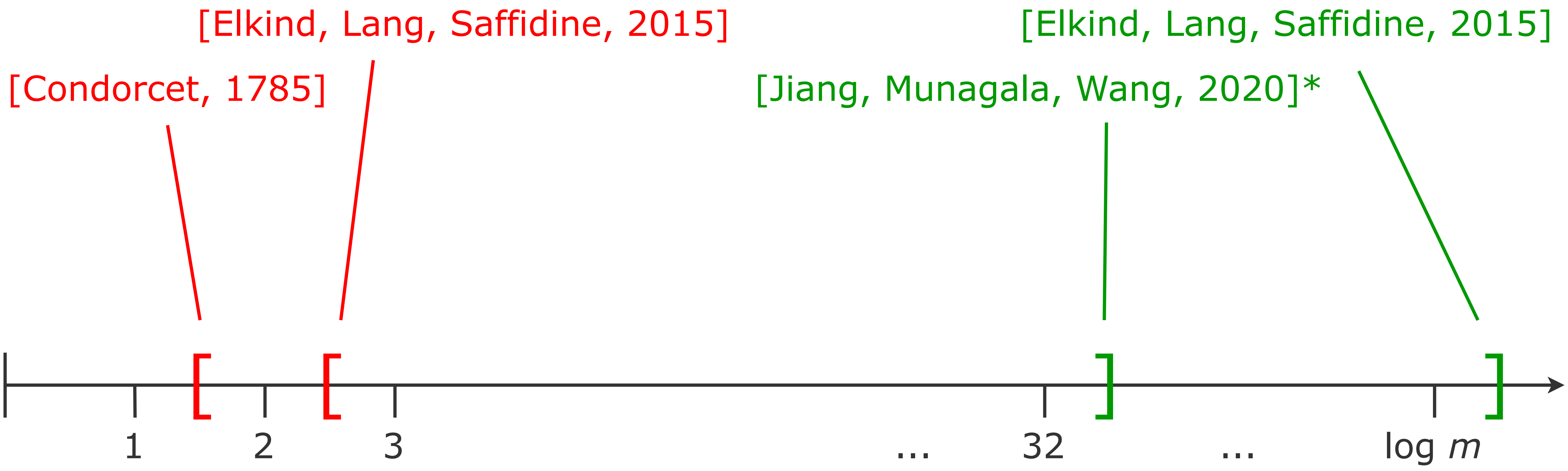

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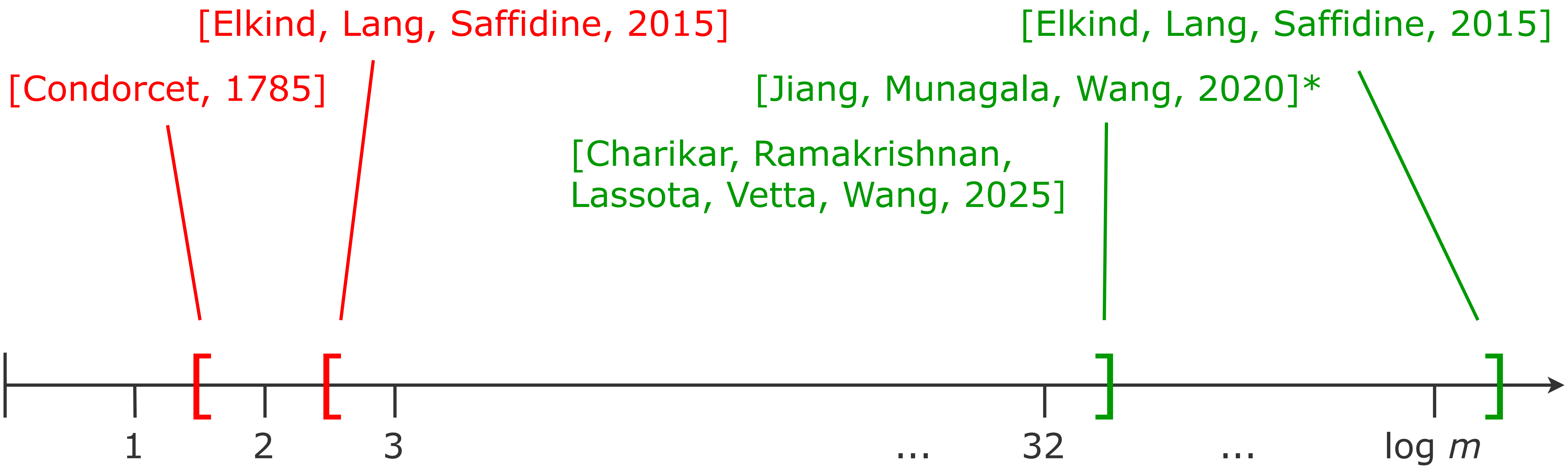

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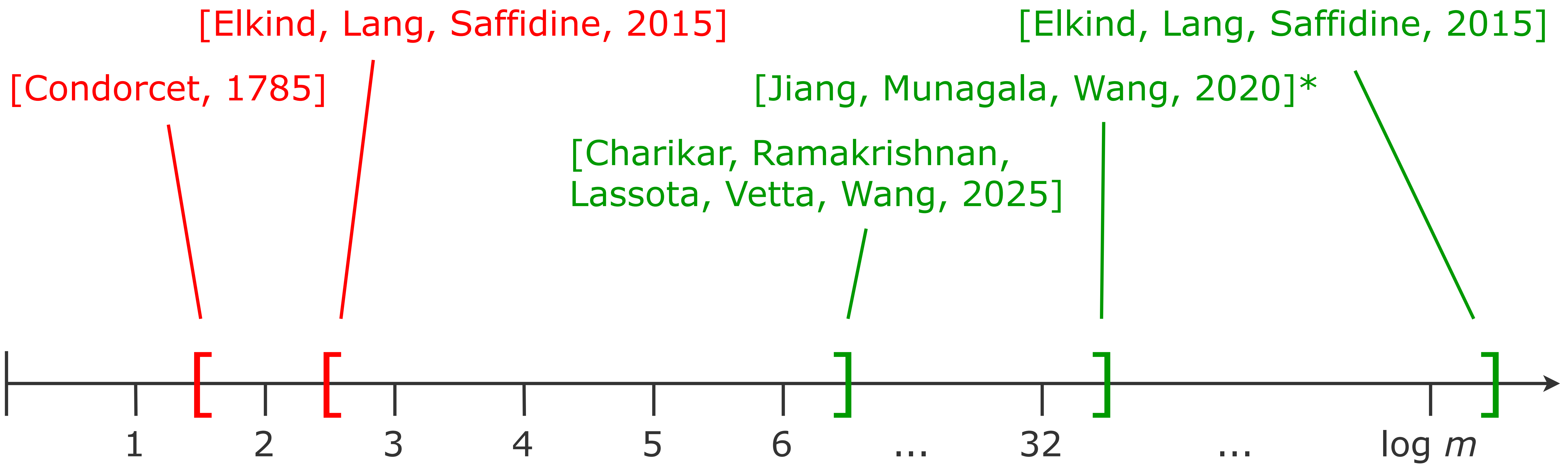

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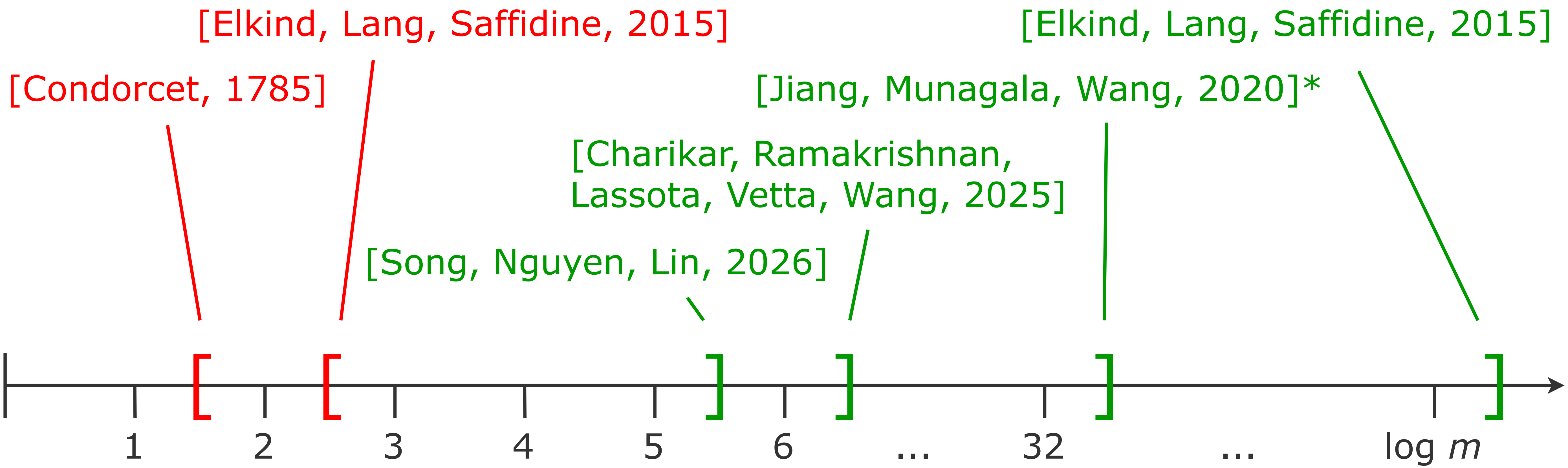

The best-known lower bound: 2

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
b	c	d	e	a	g	h	i	j	f	l	m	n	o	k
c	d	e	a	b	h	i	j	f	g	m	n	o	k	l
d	e	a	b	c	i	j	f	g	h	n	o	k	l	m
e	a	b	c	d	j	f	g	h	i	o	k	l	m	n
f	g	h	i	j	k	l	m	n	o	a	b	c	d	e
g	h	i	j	f	l	m	n	o	k	b	c	d	e	a
h	i	j	f	g	m	n	o	k	l	c	d	e	a	b
i	j	f	g	h	n	o	k	l	m	d	e	a	b	c
j	f	g	h	i	o	k	l	m	n	e	a	b	c	d
k	l	m	n	o	a	b	c	d	e	f	g	h	i	j
l	m	n	o	k	b	c	d	e	a	g	h	i	j	f
m	n	o	k	l	c	d	e	a	b	h	i	j	f	g
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Easy upper bound: O(log *m*)

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Open Question: What is the minimal c we can get for monotone preferences?
(You can't do c = 1.)

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Corollary

There is always a Condorcet winning set of size 32.

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Condorcet winning lotteries

Example

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a	b	c
b	c	a
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Condorcet winning lotteries

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▸ **Is there a lottery over candidates such that no alternative lottery over candidates is <u>strictly</u> preferred by each of a majority of voters?**

Respond at:

pollev.com/jtuckerfoltz255 or

bit.ly/jtfpoll or

text jtuckerfoltz255 to 37607

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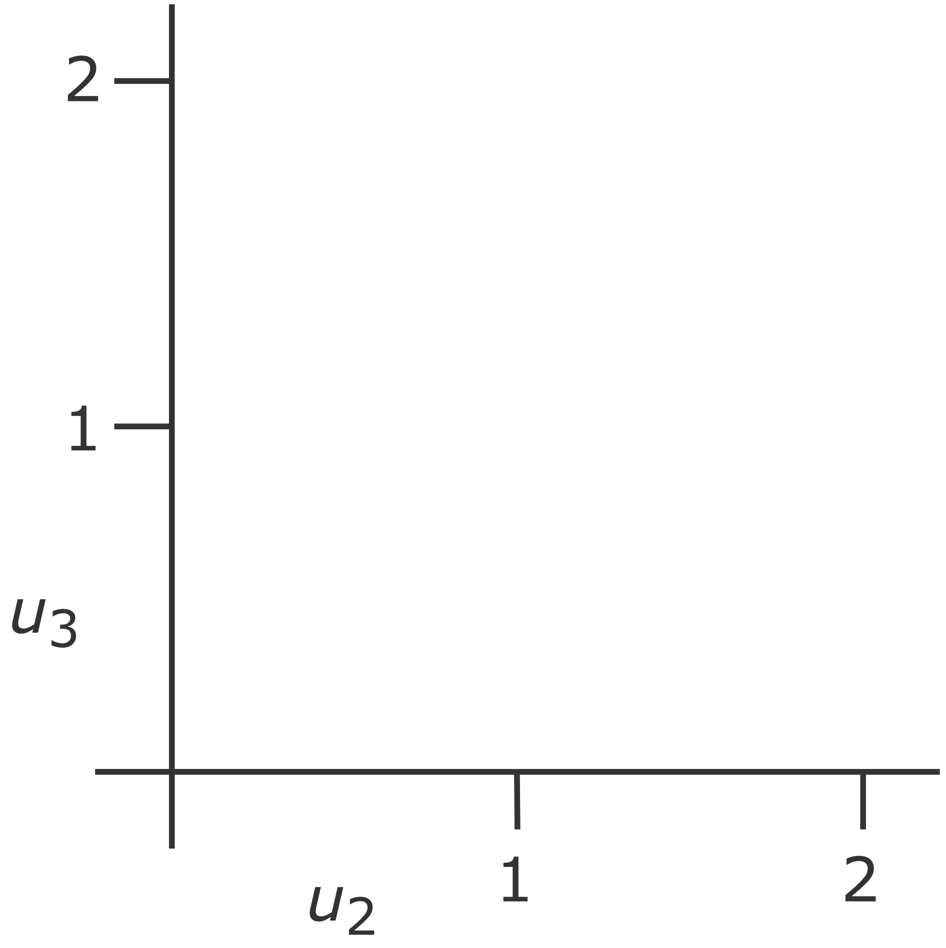

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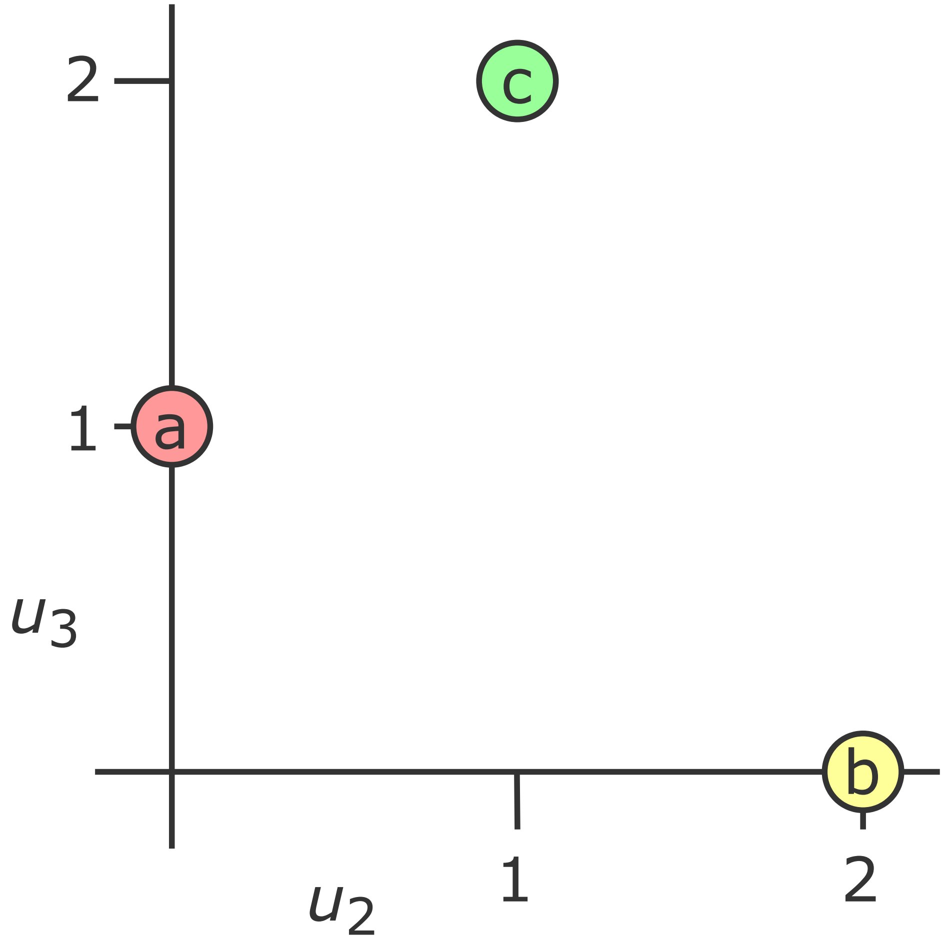

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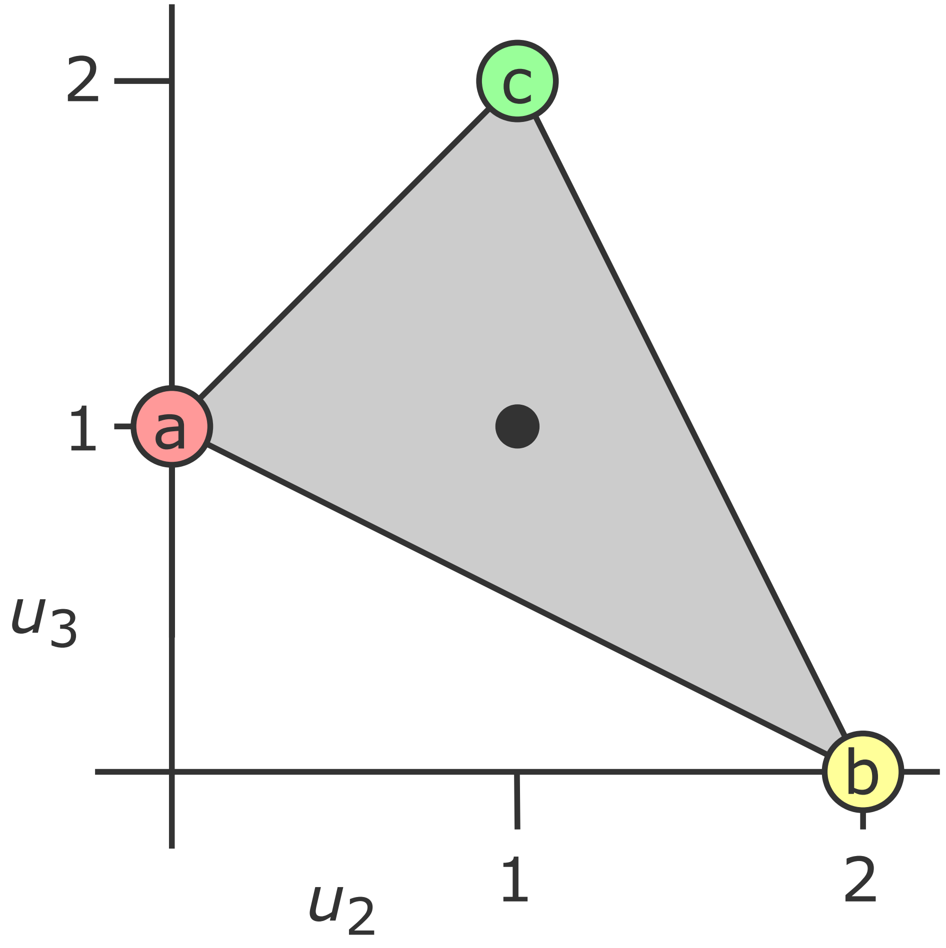

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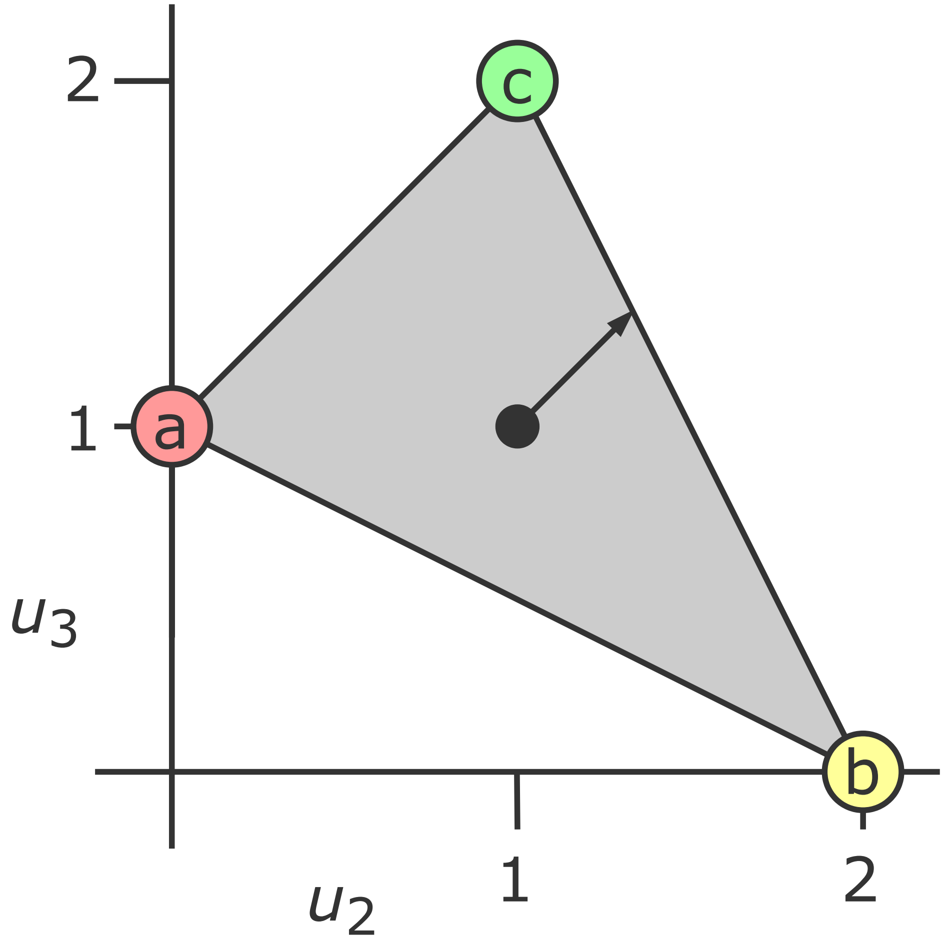

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Theorem (Zeckhauser, 1969)

This can happen even with single-peaked preferences!

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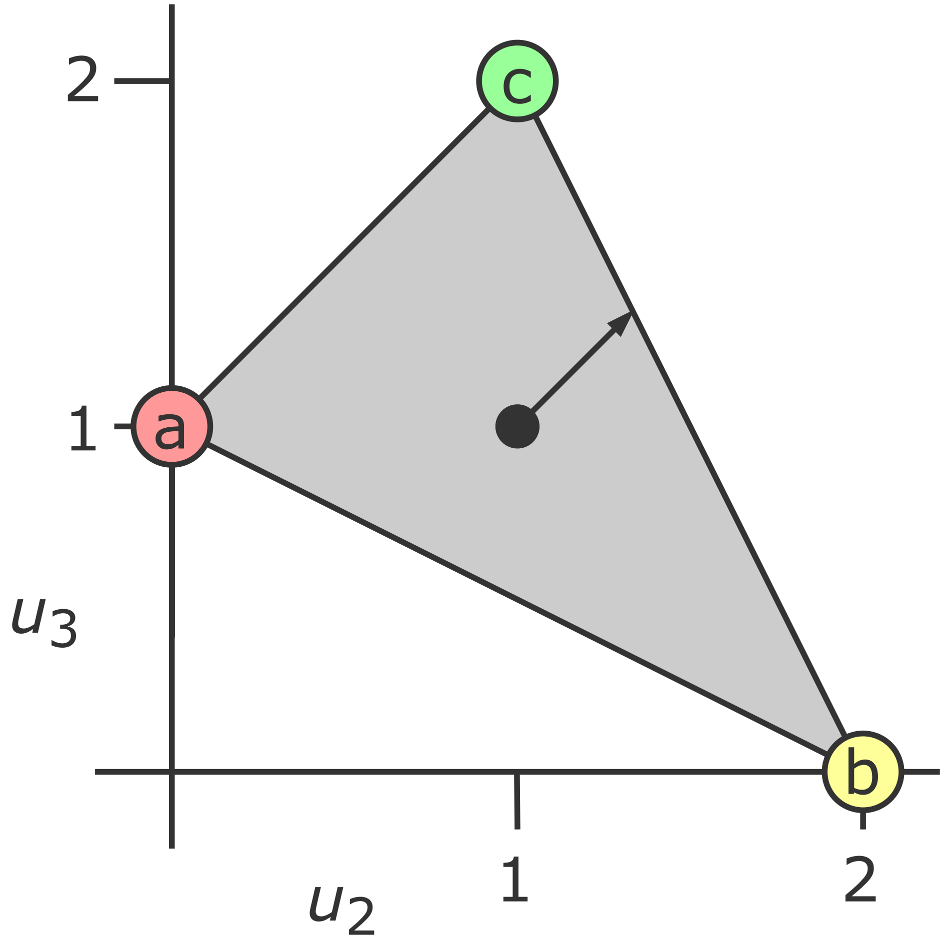

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Ex ante stability

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Example

Claim: $(0.1, 0.6, 0.3)$ is an equilibrium.

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Claim: $(0.1, 0.6, 0.3)$ is an equilibrium.

Prefers a

Prefers a with probability 0.6

Example

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Not ex ante stable!

Claim: $(0.1, 0.6, 0.3)$ is an equilibrium.

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Prefers a

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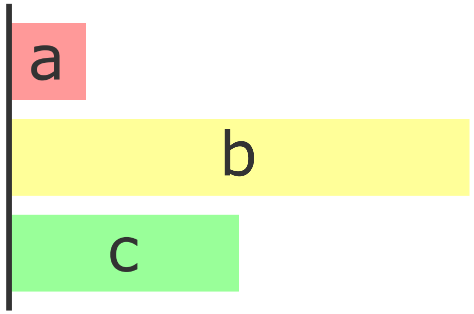

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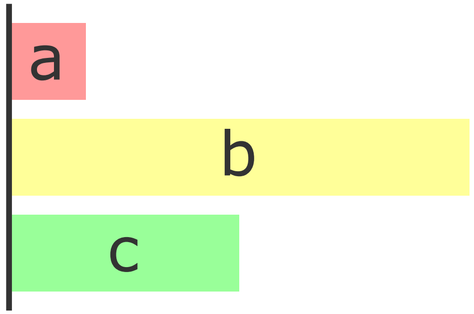

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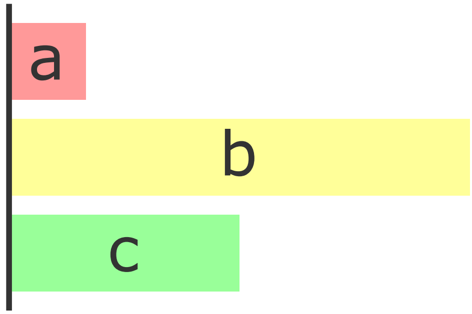
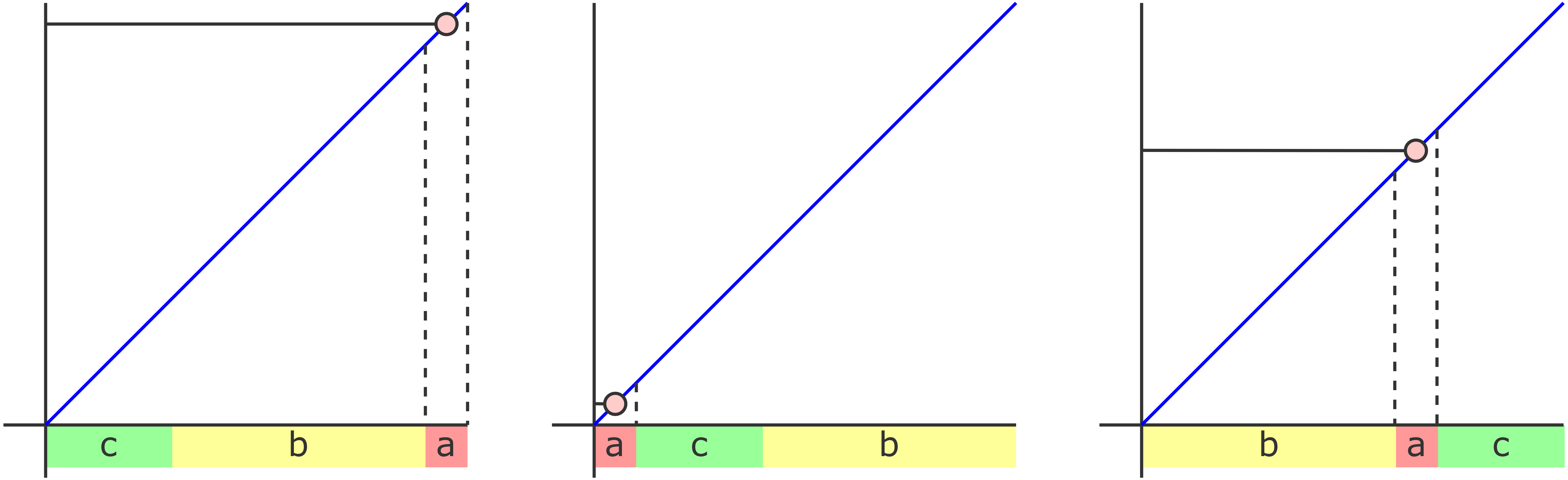
$p =$ 



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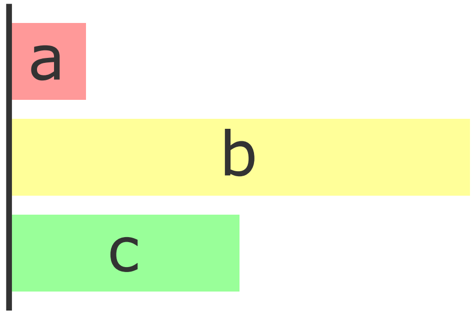
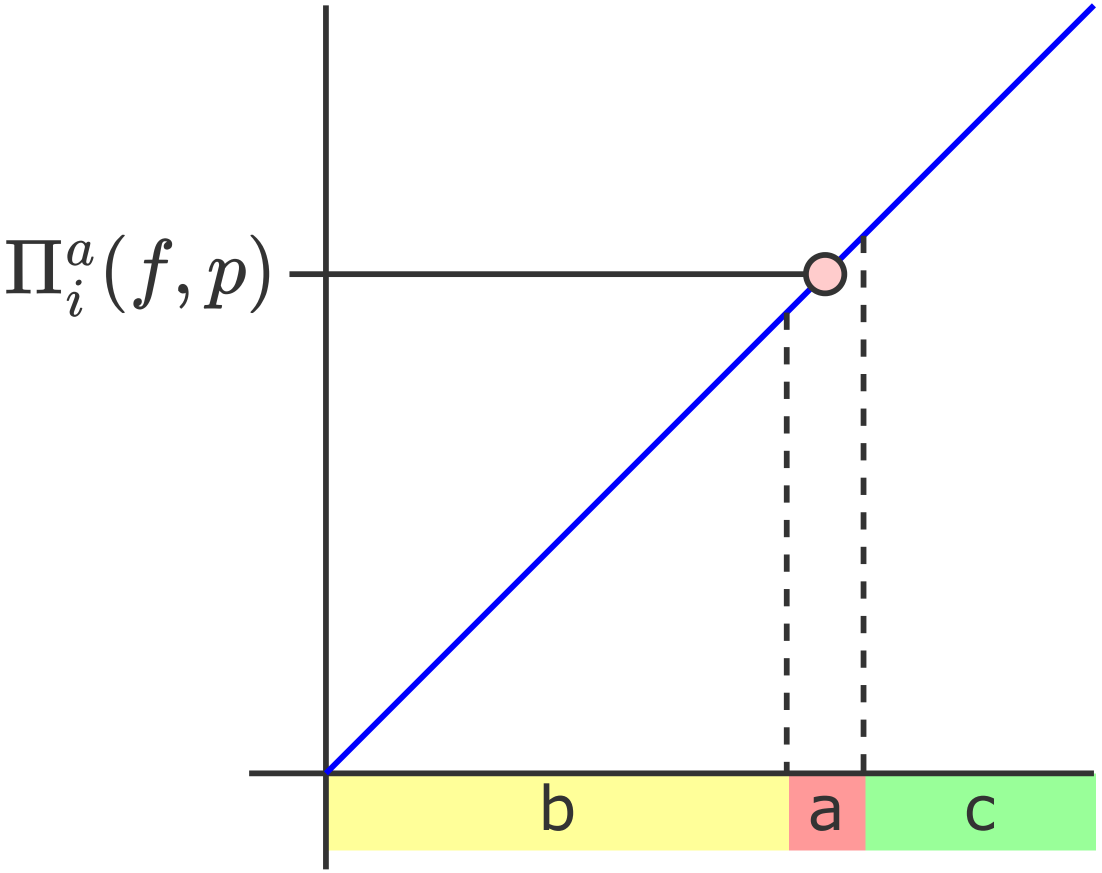
$p =$ 

Let $f : [0,1] \to [0,1]$ be a monotone function. An *equilibrium lottery* for f is a distribution over candidates $p \in \Delta([m])$ such that every candidate a in the support of p maximizes

$$\mathbb{E}_{i \sim [n]} [\Pi_i^a(f, p)].$$



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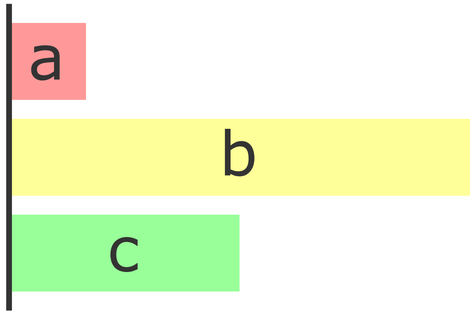

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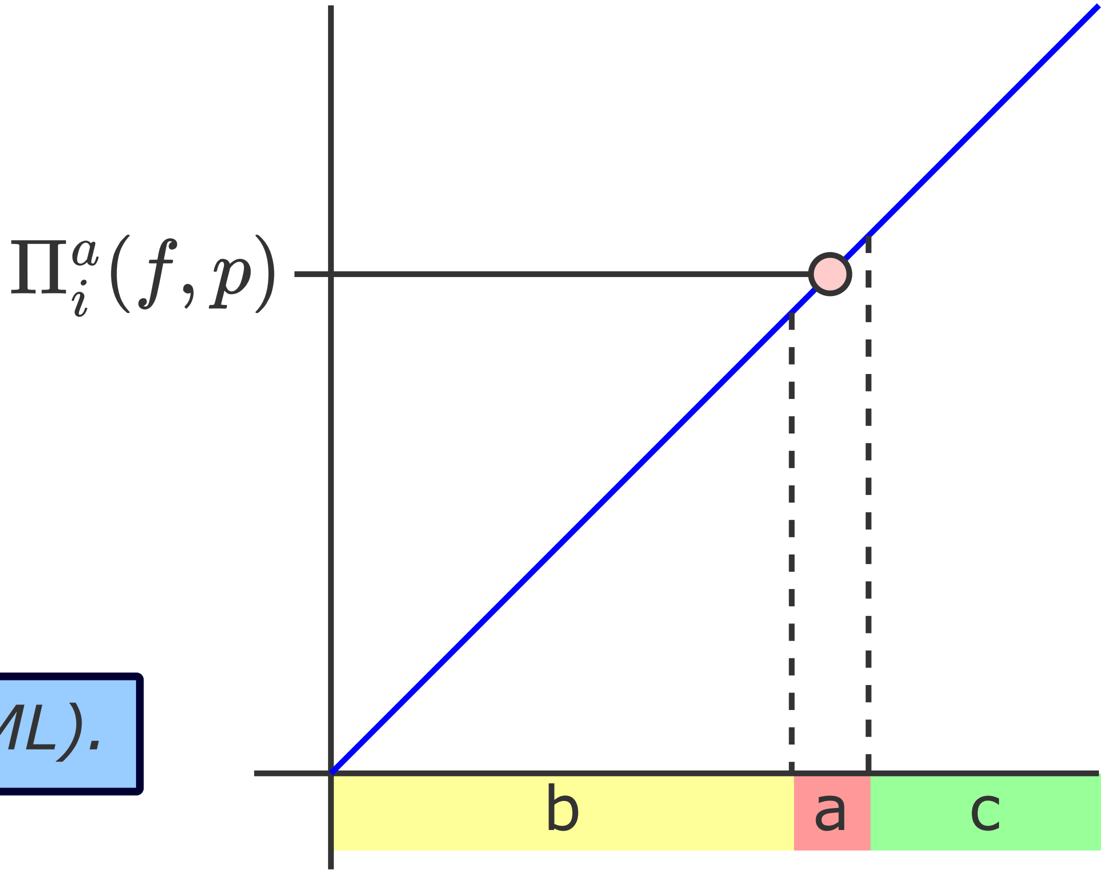

Proposition

An equilibrium lottery exists for $f(x) = x$ (namely, ML).

Existence of equilibrium lotteries

Theorem (Nguyen, 2025)

An equilibrium lottery exists for any monotone and continuous scoring function.

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Proof. We seek to find $p \in \Delta([m])$ such that, for all a for which $p_a > 0$,

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Equivalently, p must be a best-response to the following two-step optimization problem:

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$$(\gamma_1', \gamma_2', \ldots, \gamma_m') = \left(\mathbb{E}_{i \sim [n]} \left[\Pi_i^1(f, p)\right], \mathbb{E}_{i \sim [n]} \left[\Pi_i^2(f, p)\right], \ldots, \mathbb{E}_{i \sim [n]} \left[\Pi_i^m(f, p)\right]\right).$$

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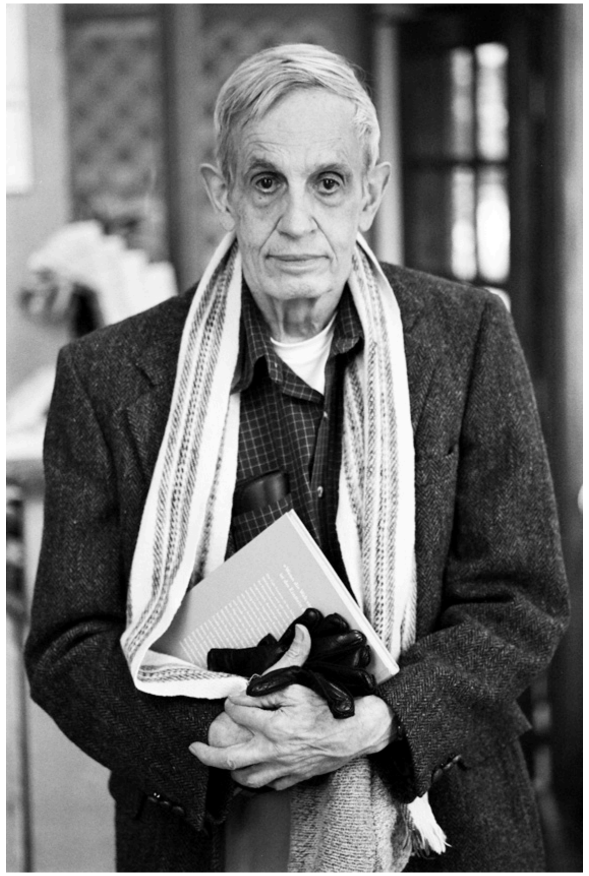
2. Then compute a best response $p' \in \Delta([m])$ to γ':

$$p' = \arg\max_{p'} (\gamma' \cdot p').$$

Existence of equilibrium lotteries, proof continued



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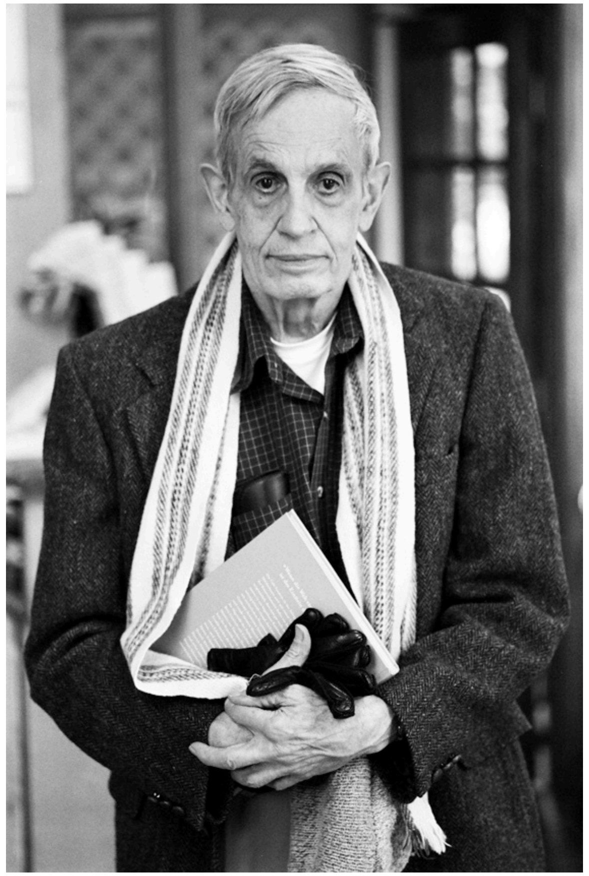

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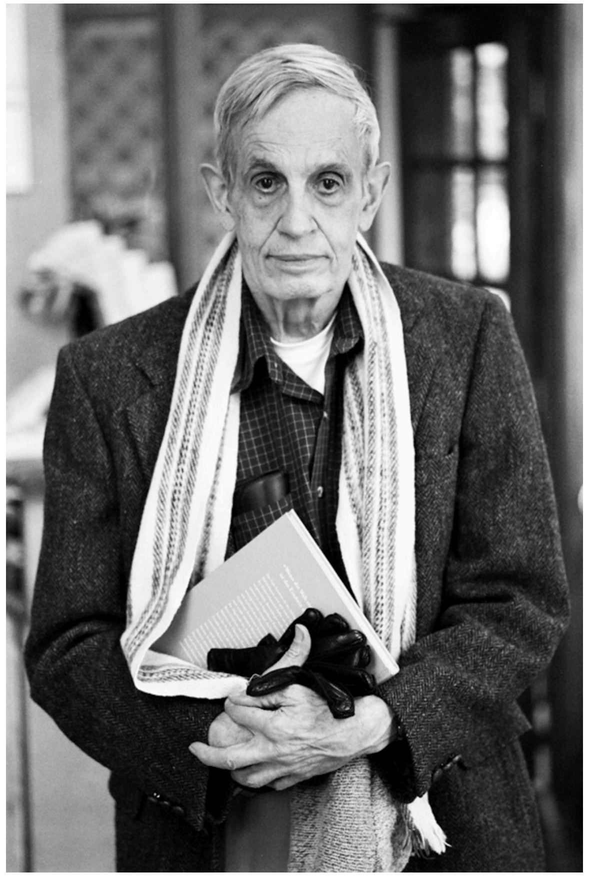

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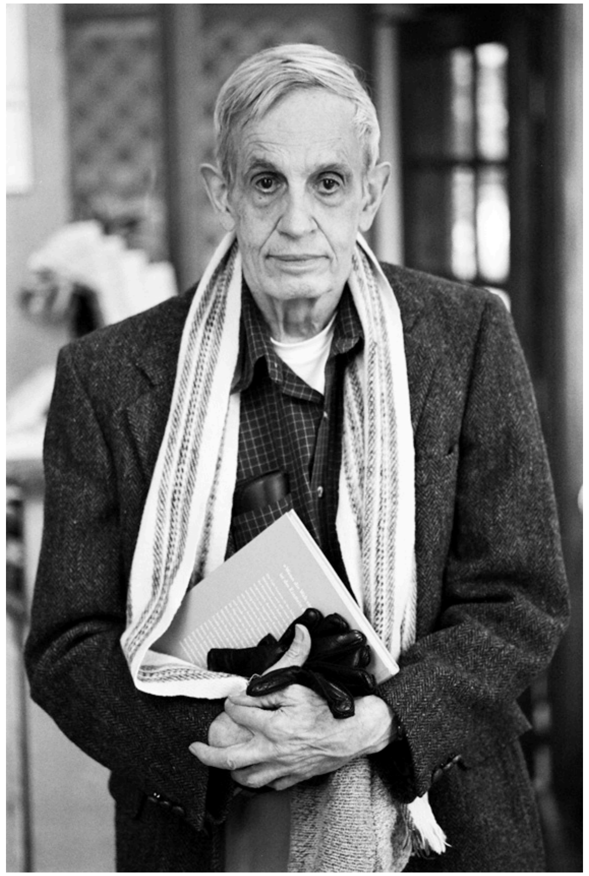

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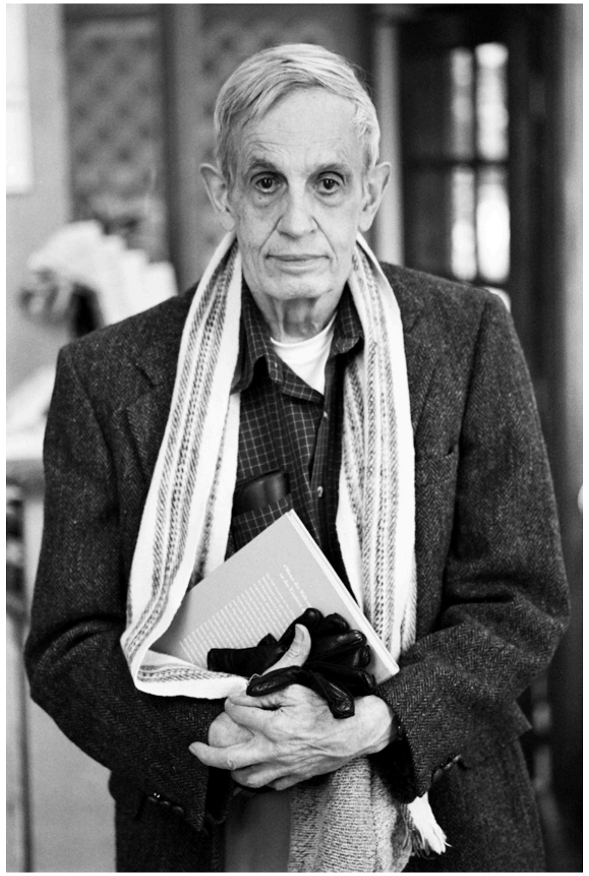

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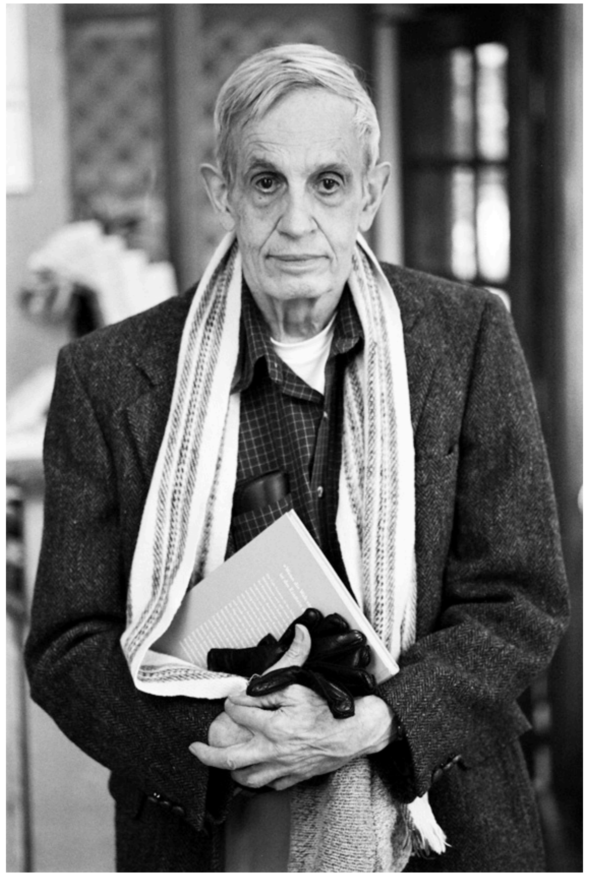

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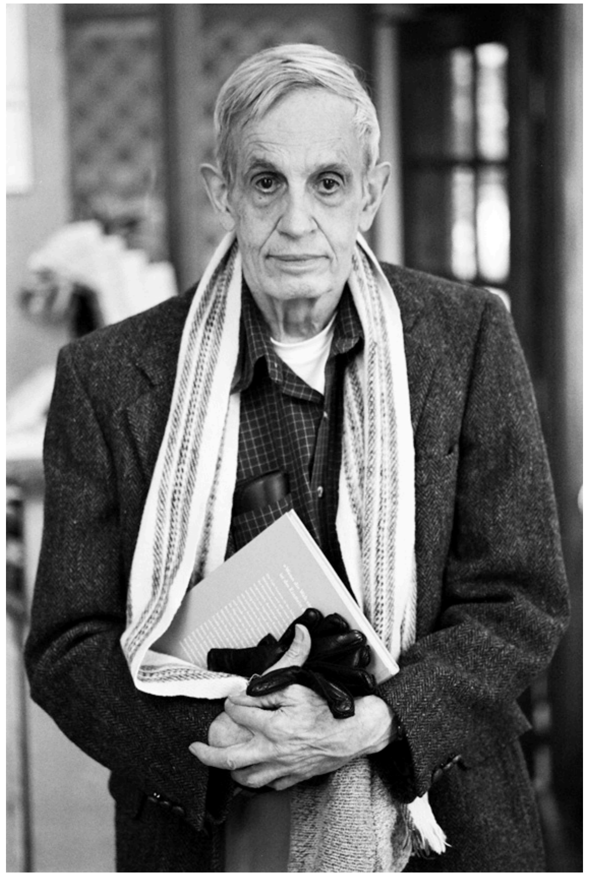

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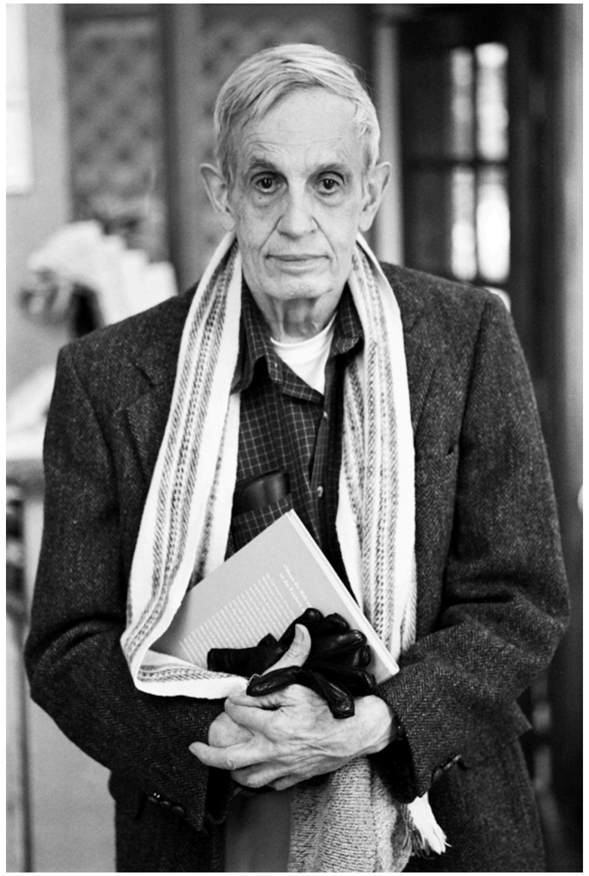

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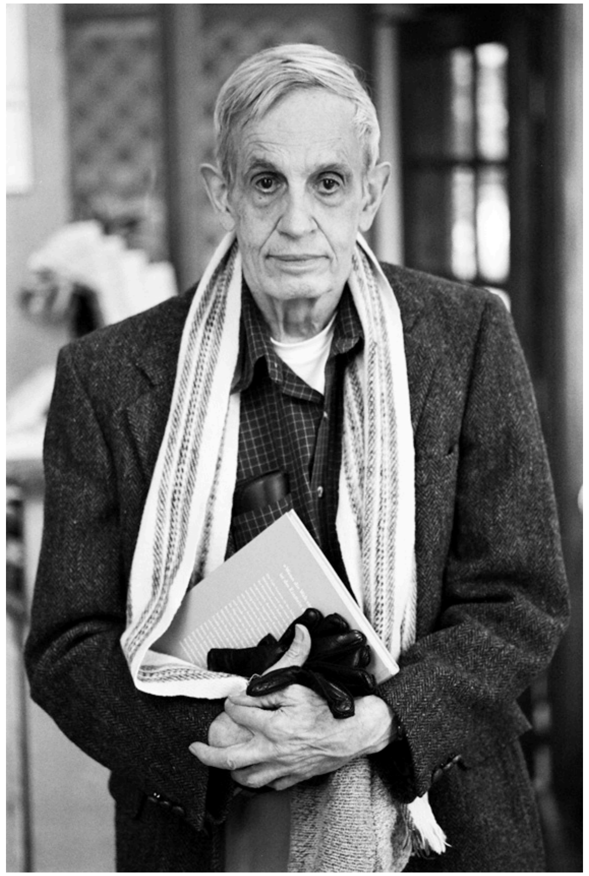

Existence of equilibrium lotteries, proof continued

We have a best response map $\Phi : \Sigma \to 2^\Sigma$
$$(p, \gamma) \mapsto \{\text{optimal } (p', \gamma')\}$$
satisfying the following properties:

1. $\Sigma = \Delta([m]) \times [0,1]^m$ is compact, convex, and nonempty.
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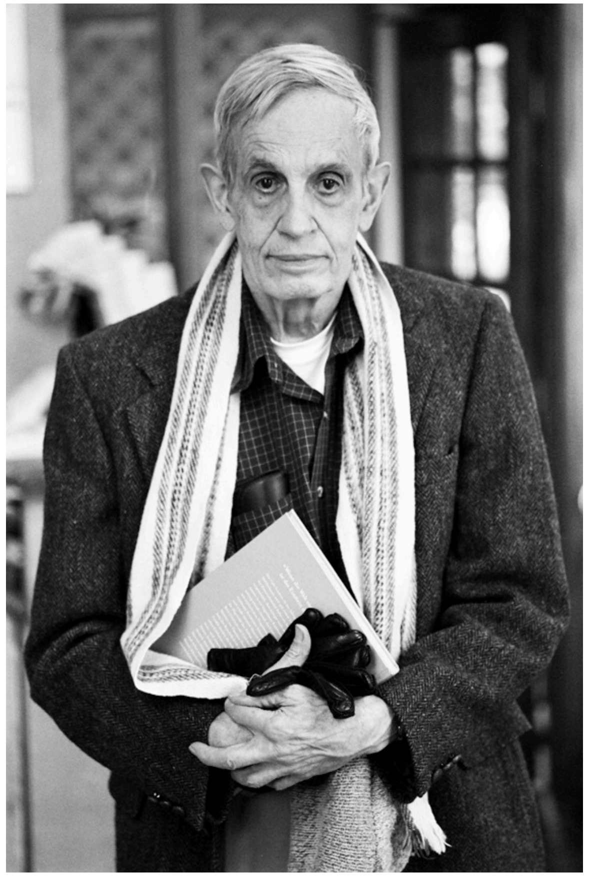
The theorem follows by Kakutani's fixed-point Theorem. ■



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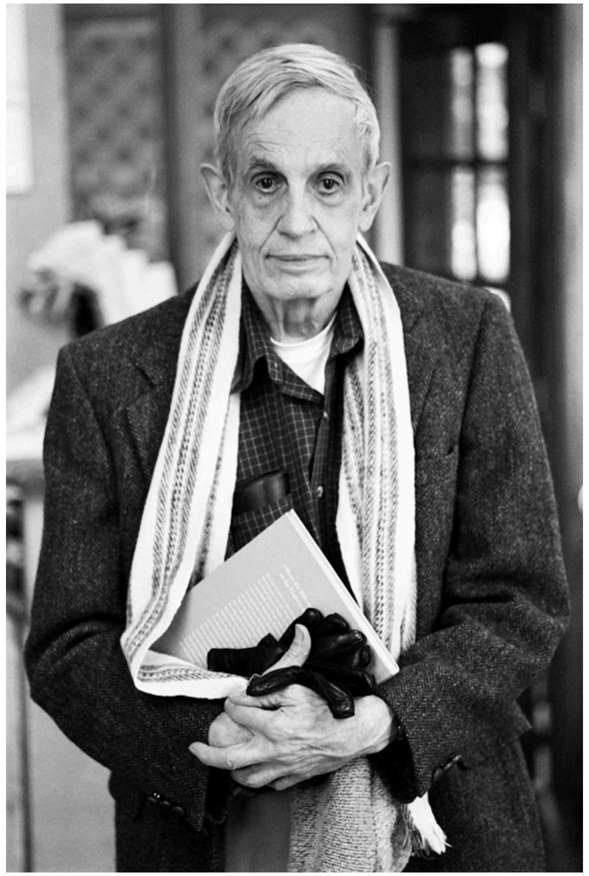
Theorem (Kakutani, 1941)

Any set-valued best-response function satisfying the above properties has a fixed point, i.e., a tuple (p, γ) that is a best response to itself.



Existence of stable lotteries

Corollary

An equilibrium lottery exists even for the discontinuous step function:

$$h(x) = \begin{cases} 0 & \text{if } x < 1/2 \\ 1 & \text{if } x \geq 1/2 \end{cases}$$

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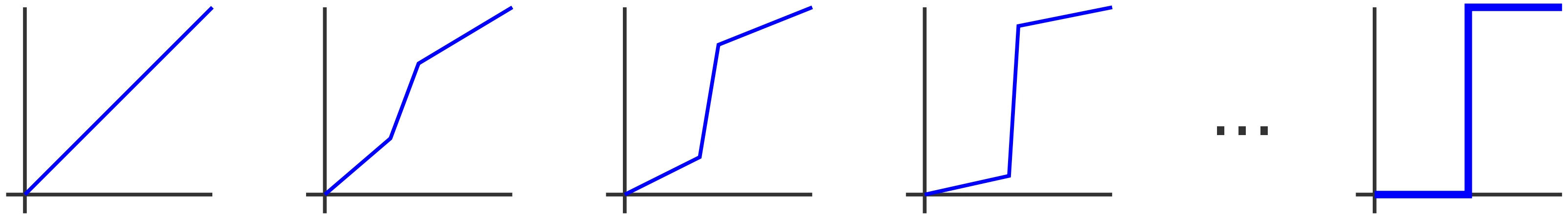

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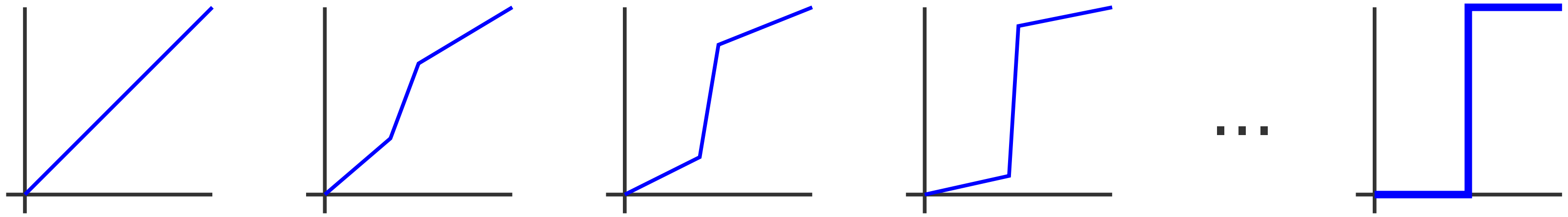

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An equilibrium lottery for h is ex ante stable.

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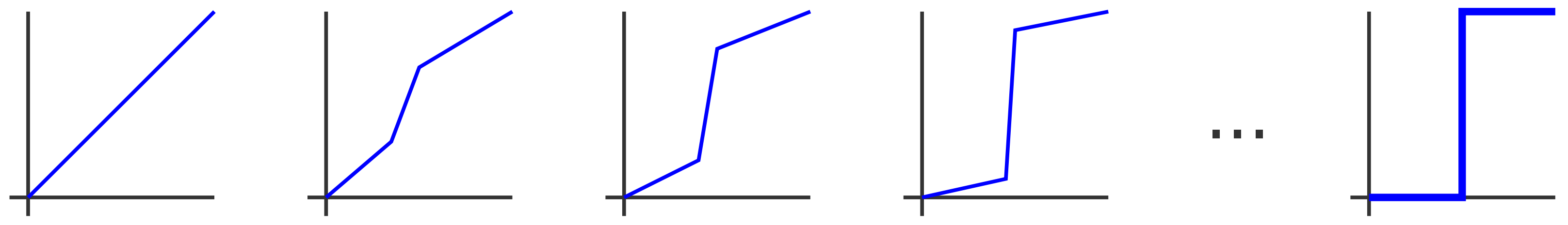

Theorem (Nguyen, 2025)

An equilibrium lottery for h is ***ex ante and ex post*** *stable.*

Existence of stable lotteries, proof

Lemma

For any score function f*, if* p *is an equilibrium lottery, then the maximum score of any candidate (attained by each candidate in the support of* p*) is*

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Lemma

For any score function f, if p is an equilibrium lottery, then the maximum score of any candidate (attained by each candidate in the support of p) is

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Proof. Since p is only supported on score-maximizing candidates,

$$\max_{a\in[m]}\mathbb{E}_{i\sim[n]}\left[\Pi_i^a(f,p)\right] = \mathbb{E}_{a\sim p}\mathbb{E}_{i\sim[n]}\left[\Pi_i^a(f,p)\right]$$

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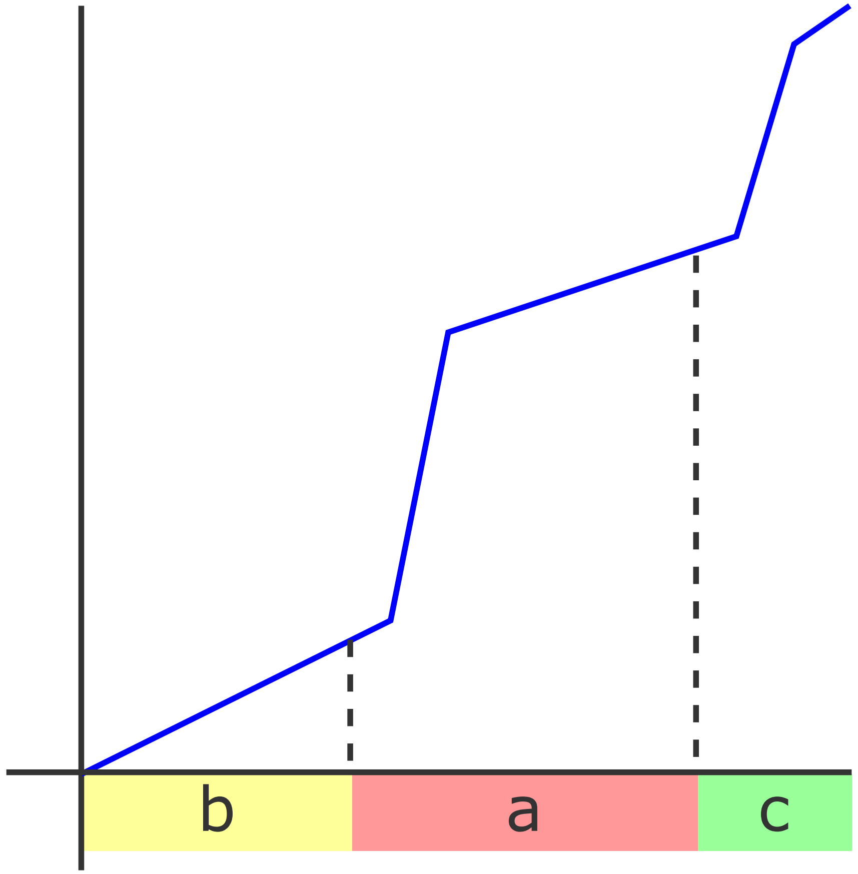

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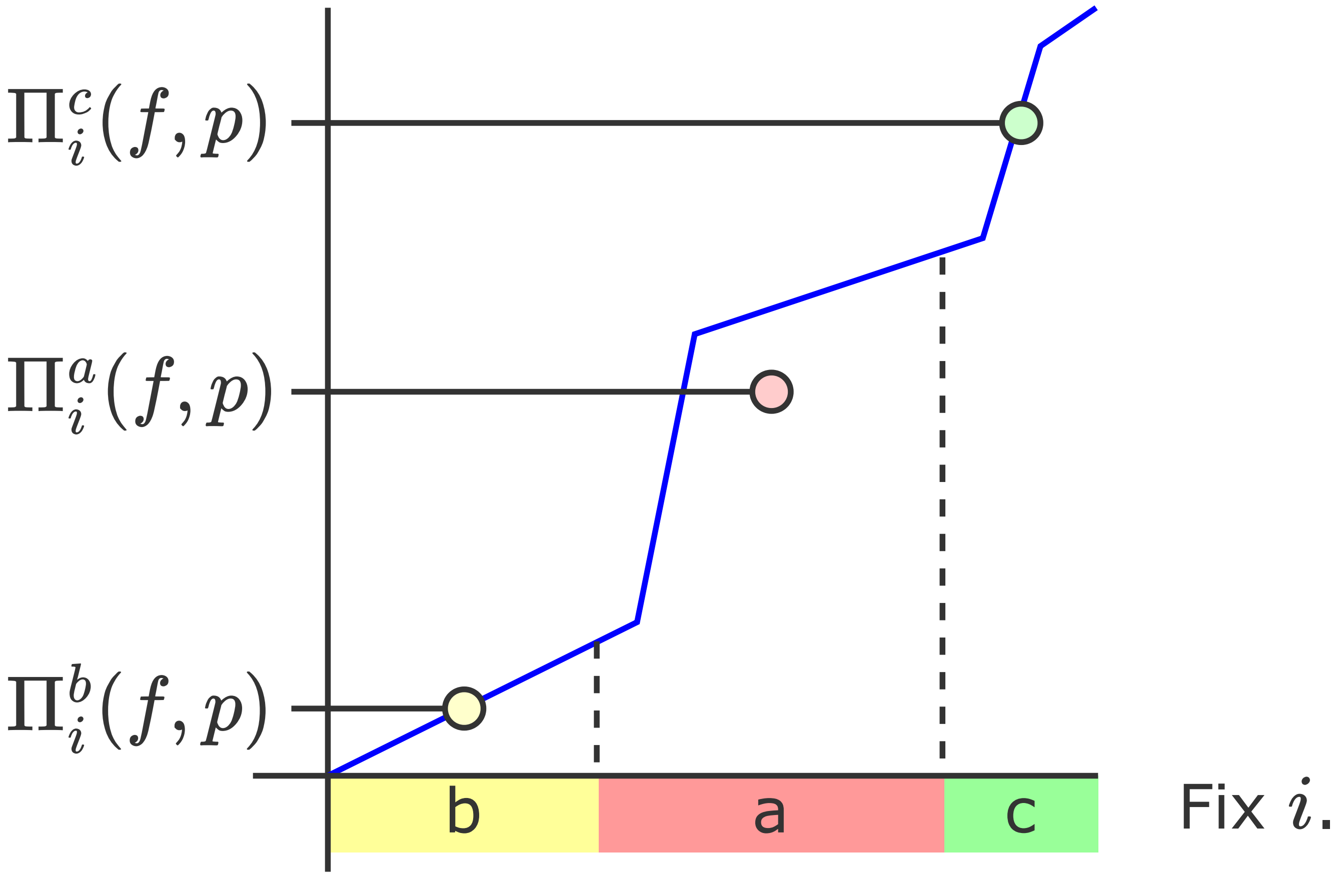

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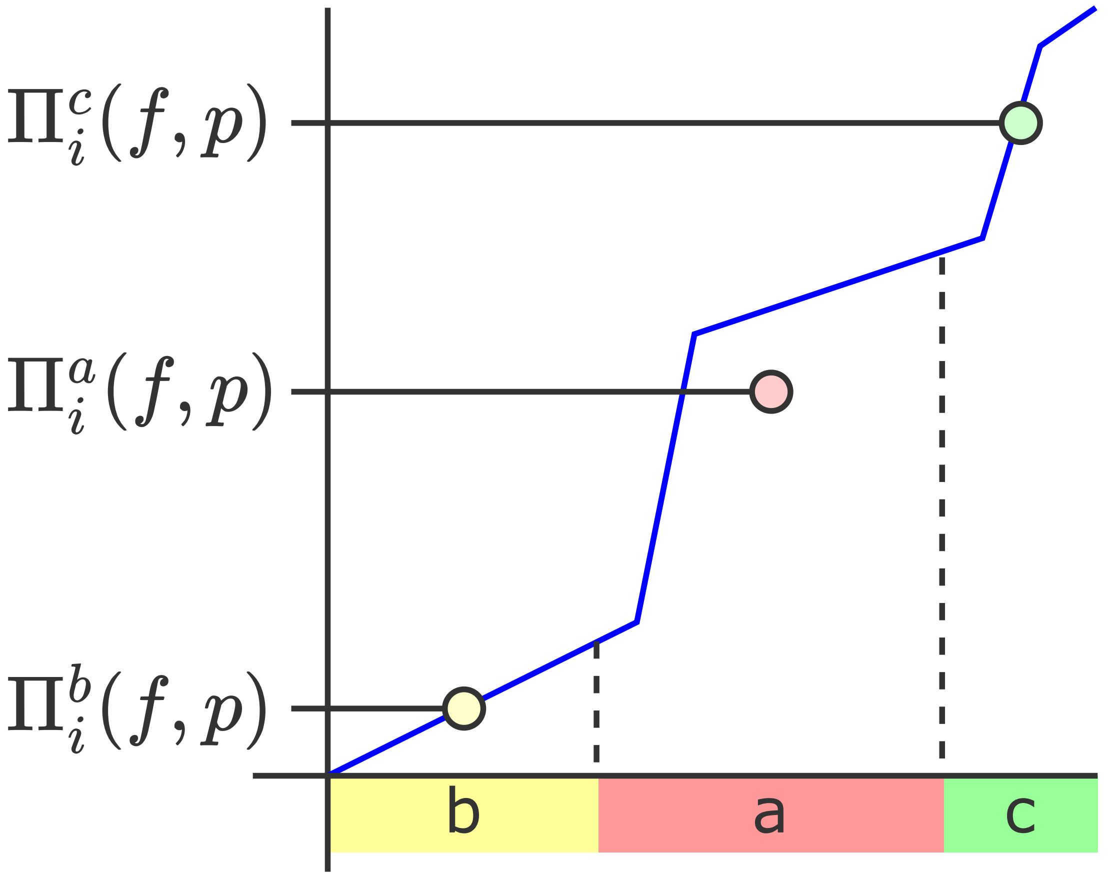

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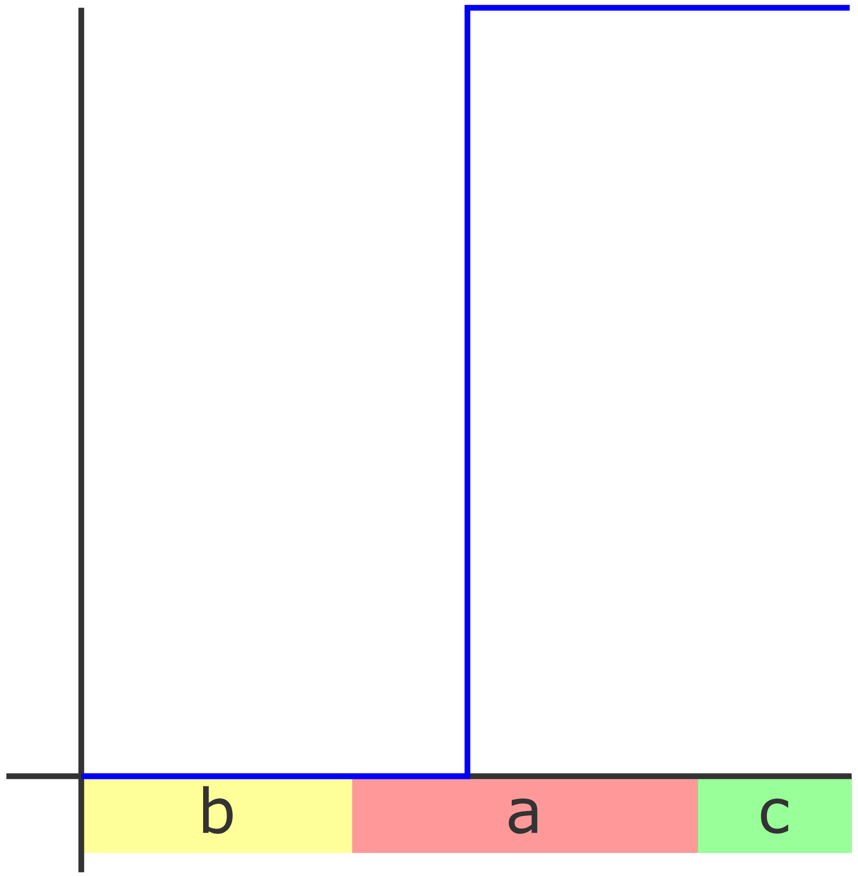

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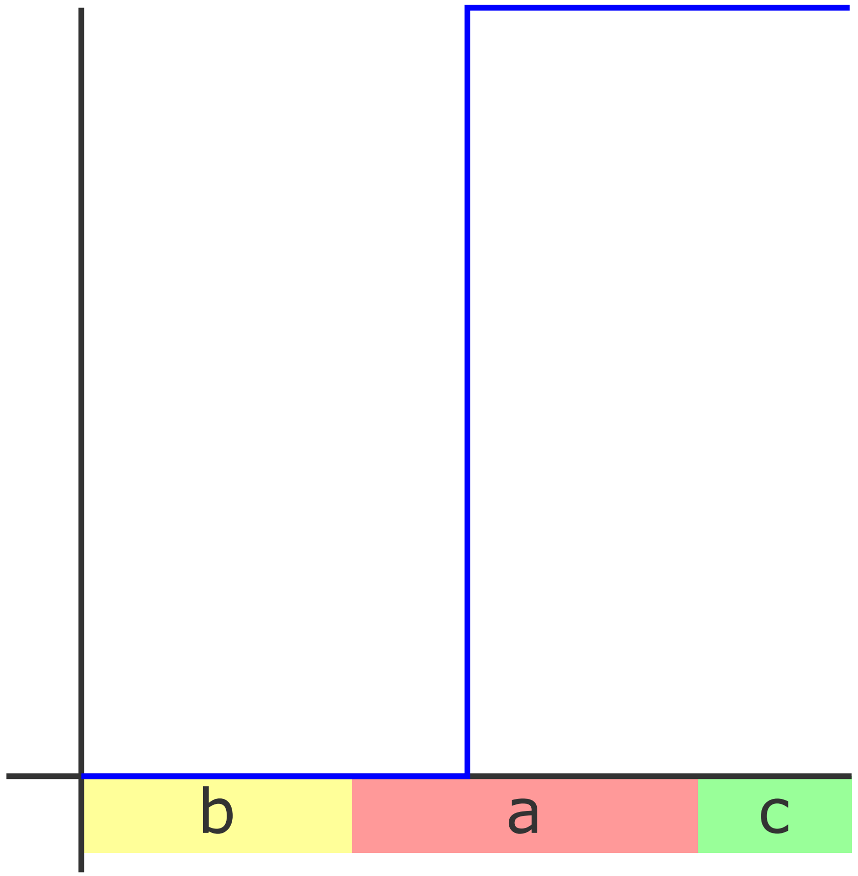

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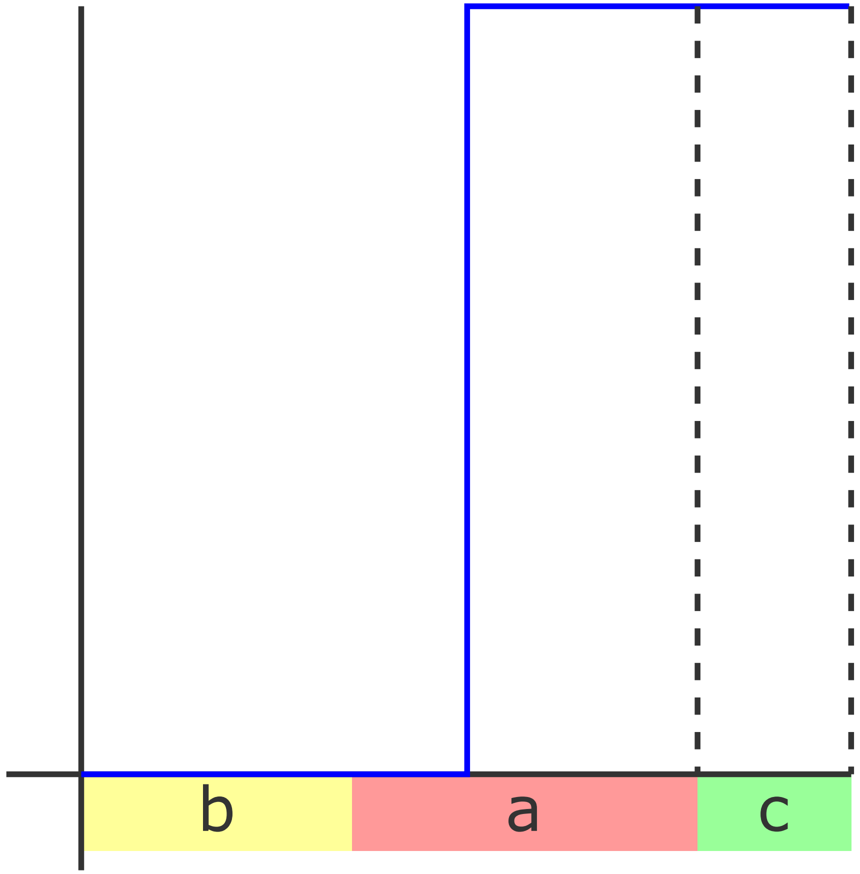

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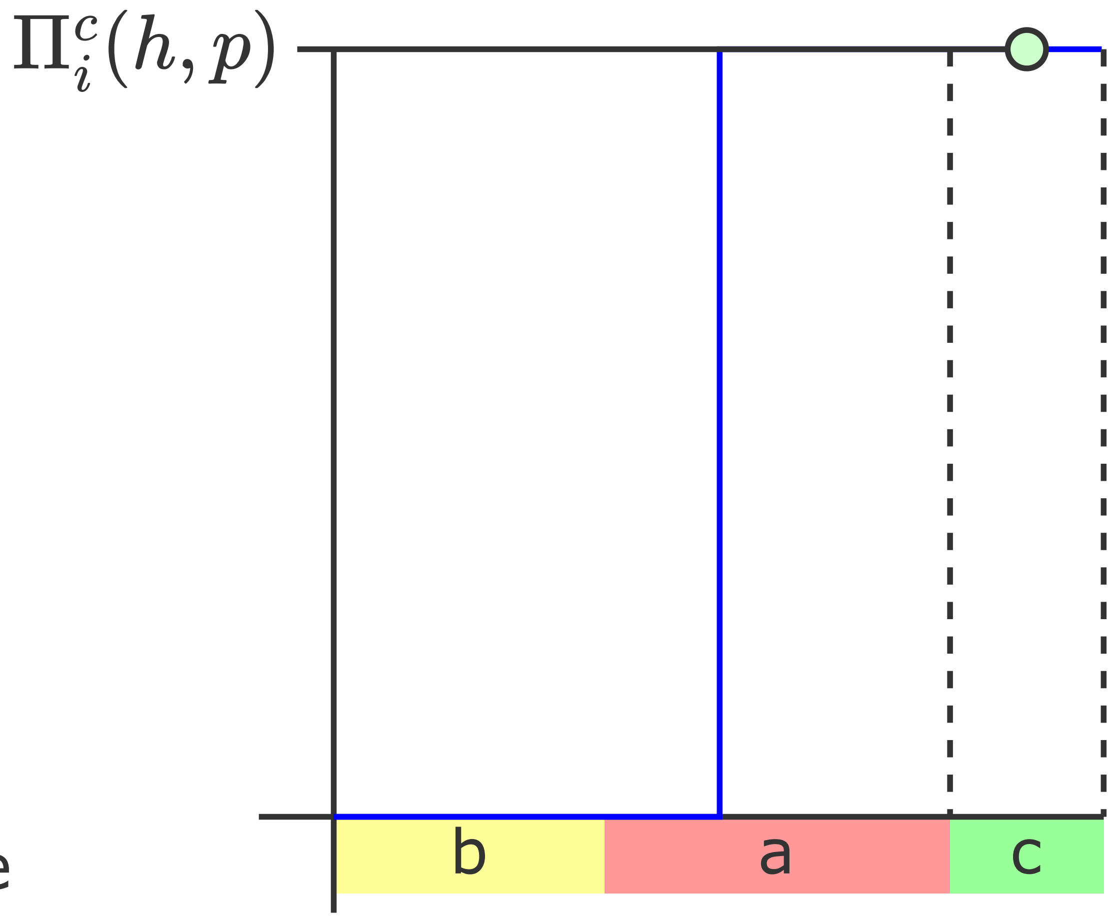

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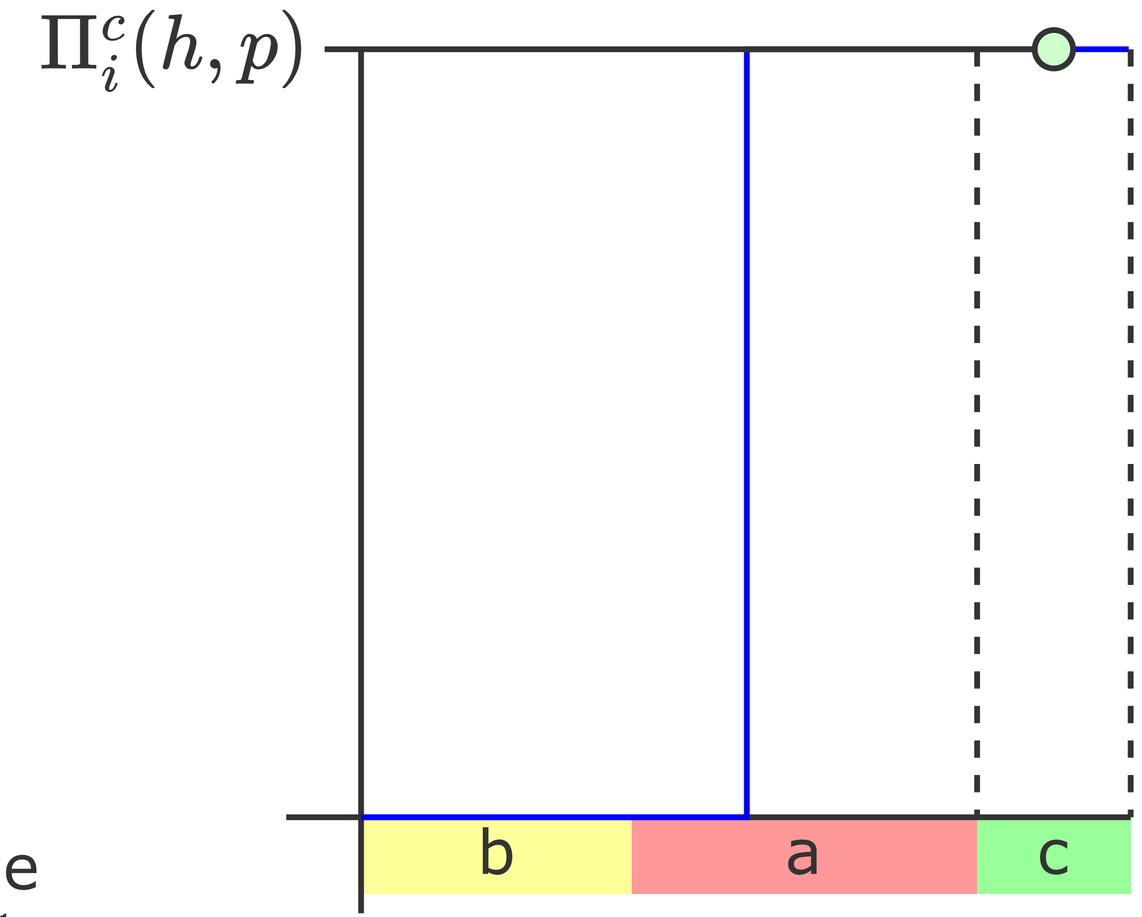

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Since these voters have $\Pi_i^c(h,p) = 1$, the expected score over $i \sim [n]$ is greater than $\frac{1}{2}$, contradicting the Lemma.

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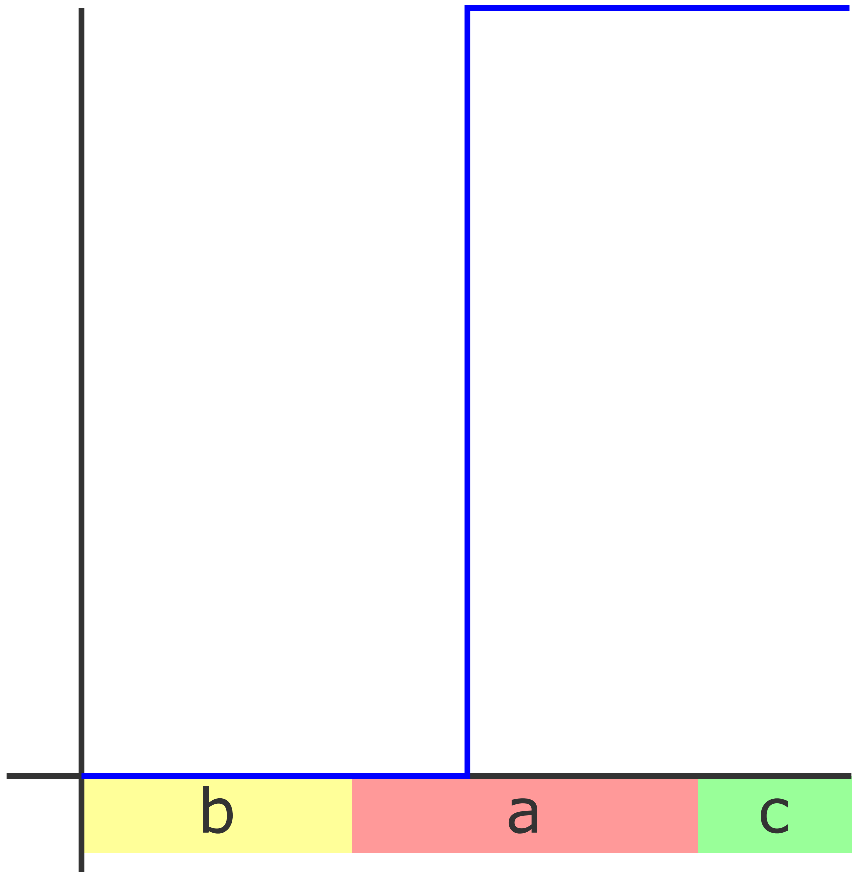

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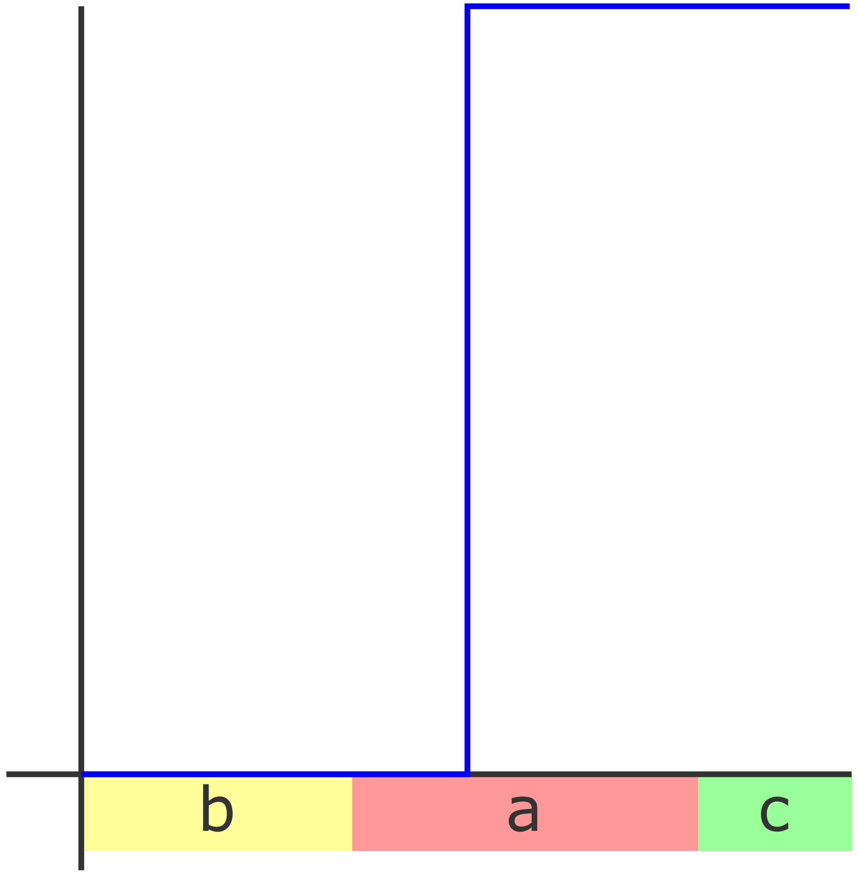

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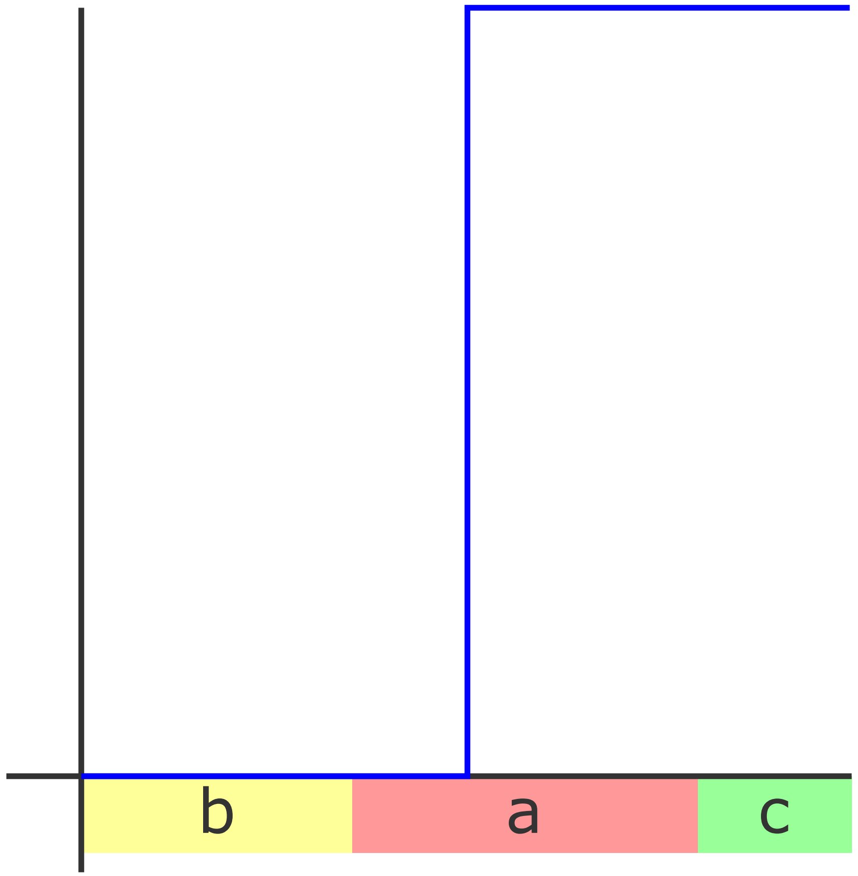

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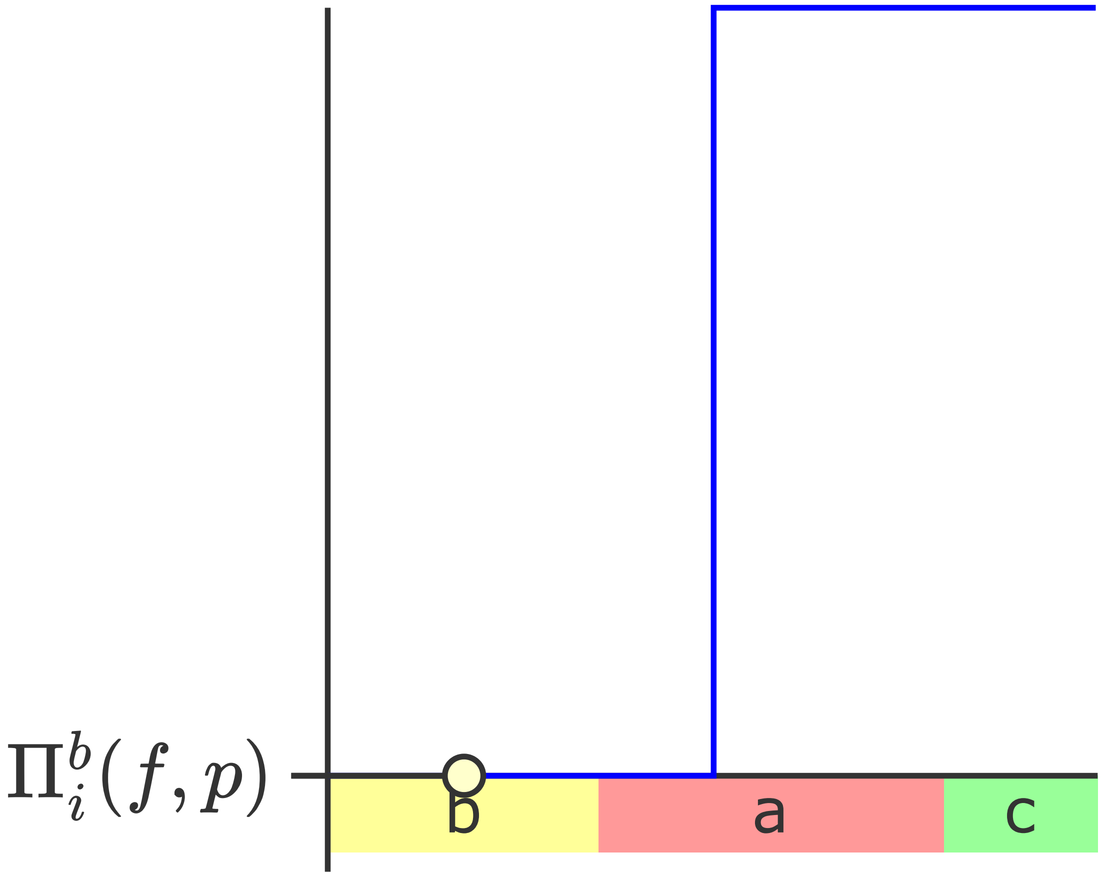

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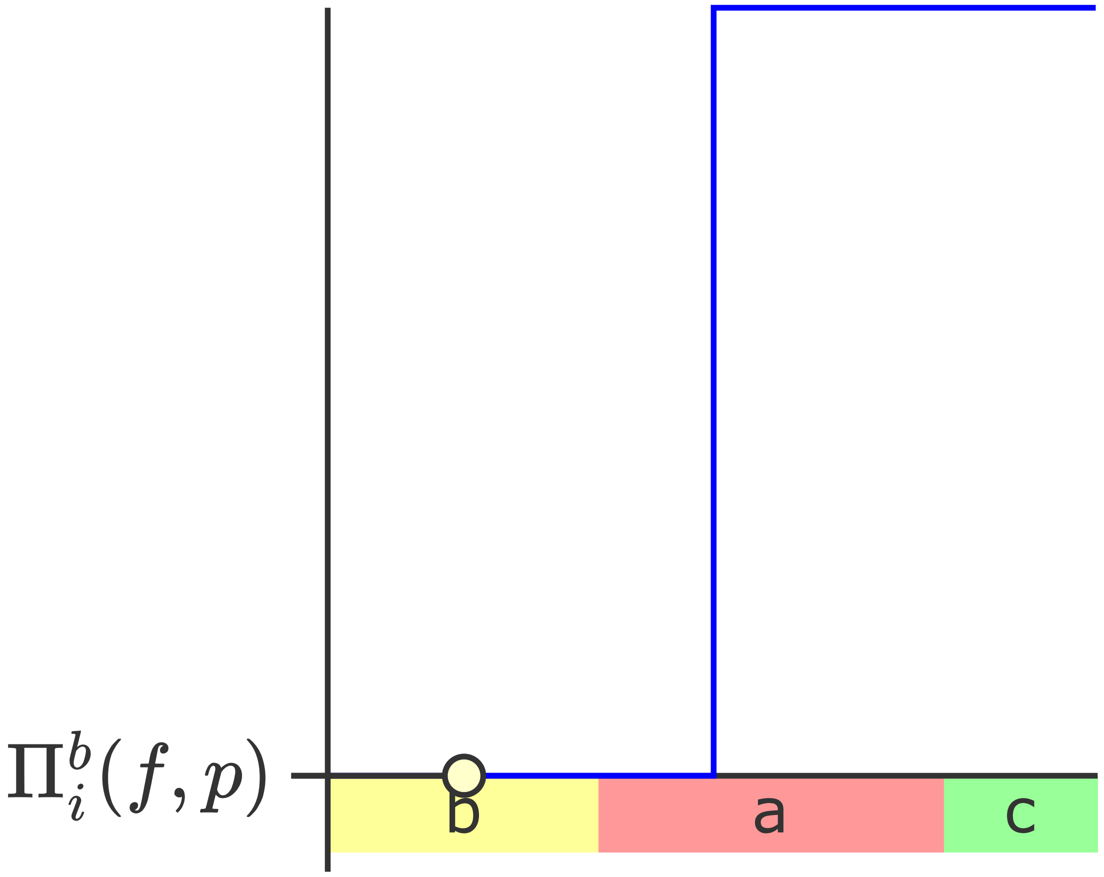

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Application to intransitive dice

k-dice game:

1. Alice makes a set of arbitrarily many dice, with arbitrarily many sides, and arbitrary numbers on the faces
2. Bob picks *k* dice
3. Alice picks any one other die
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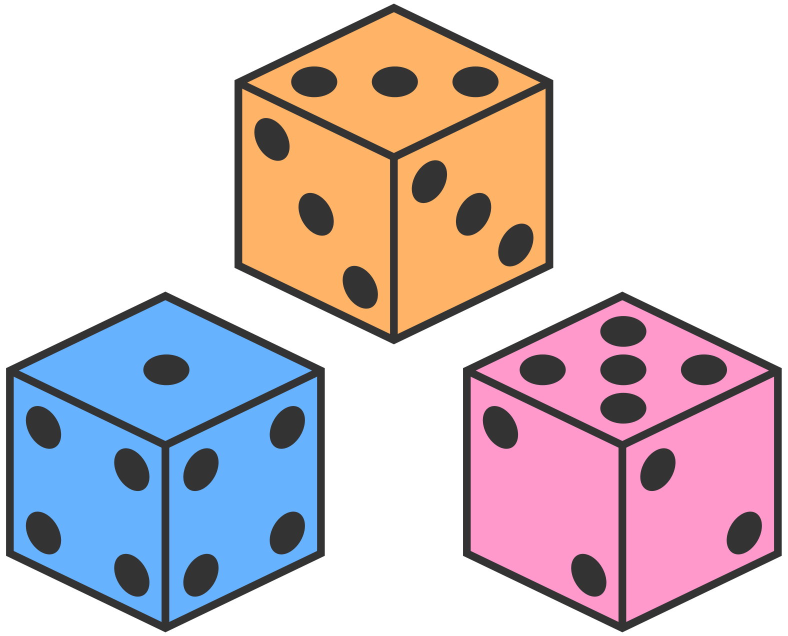

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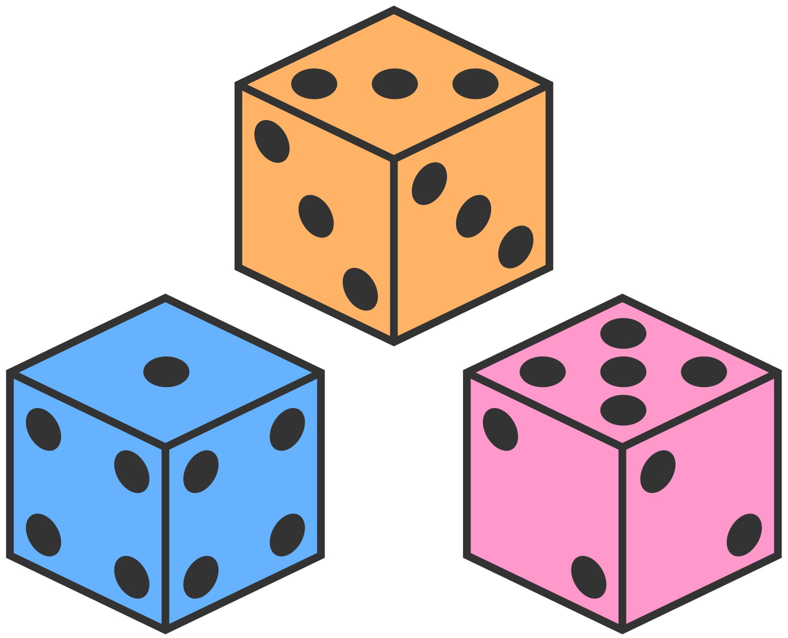

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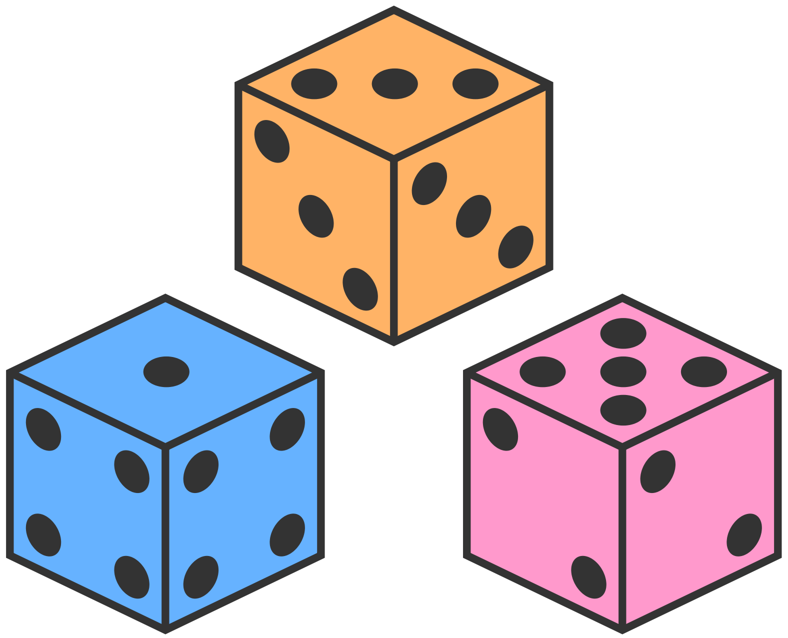

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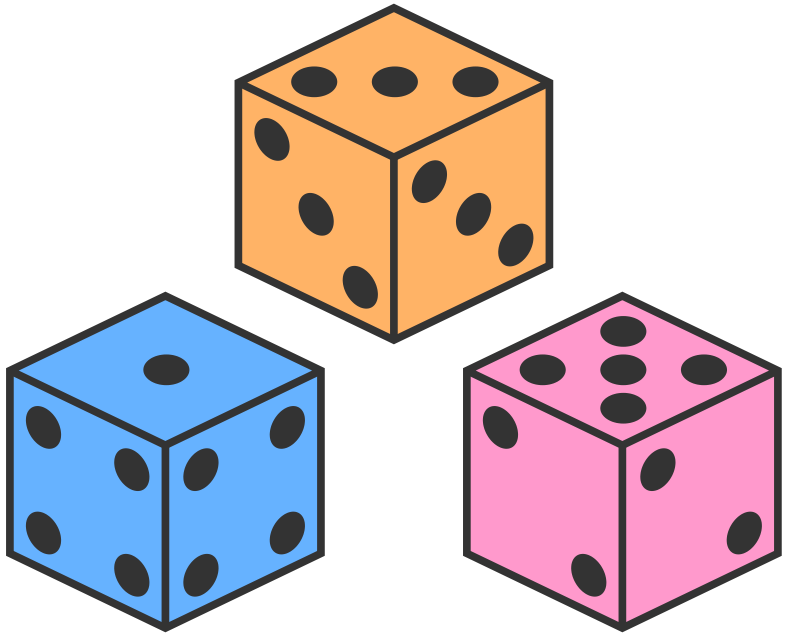

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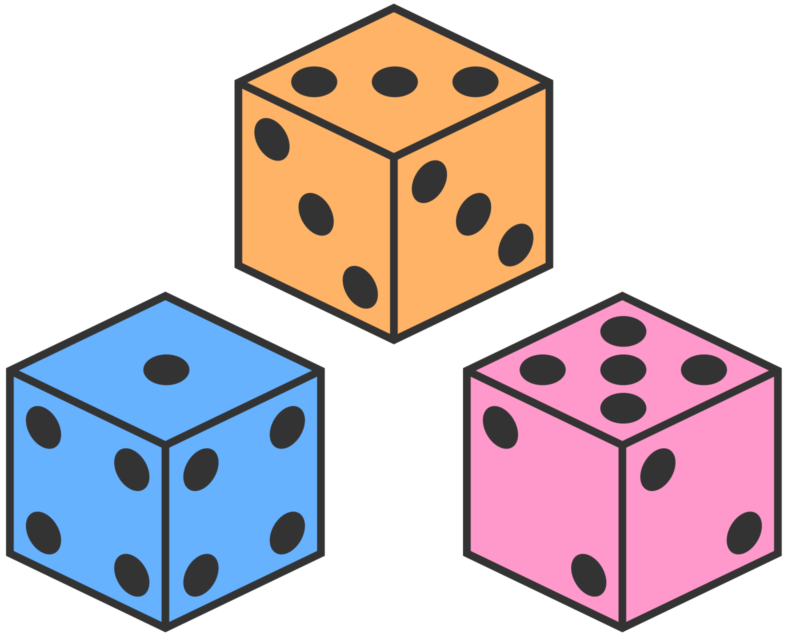

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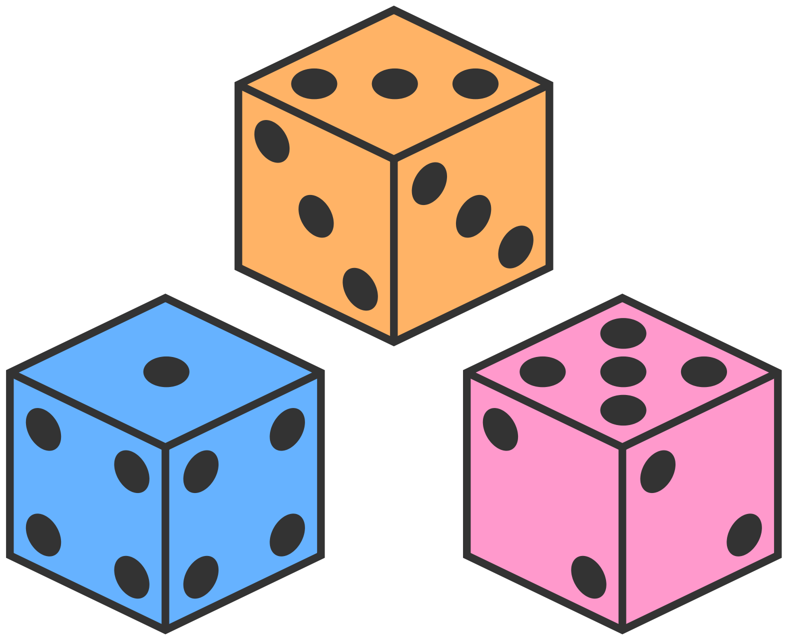

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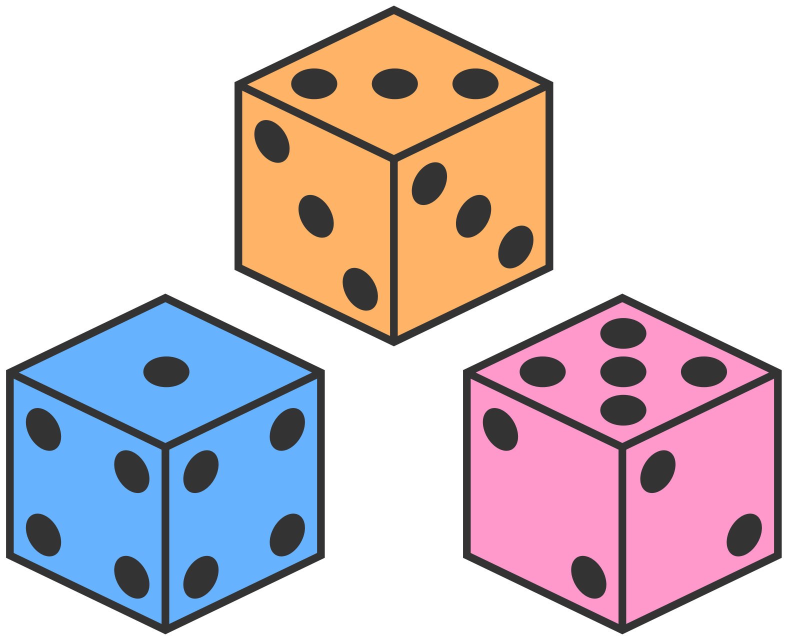

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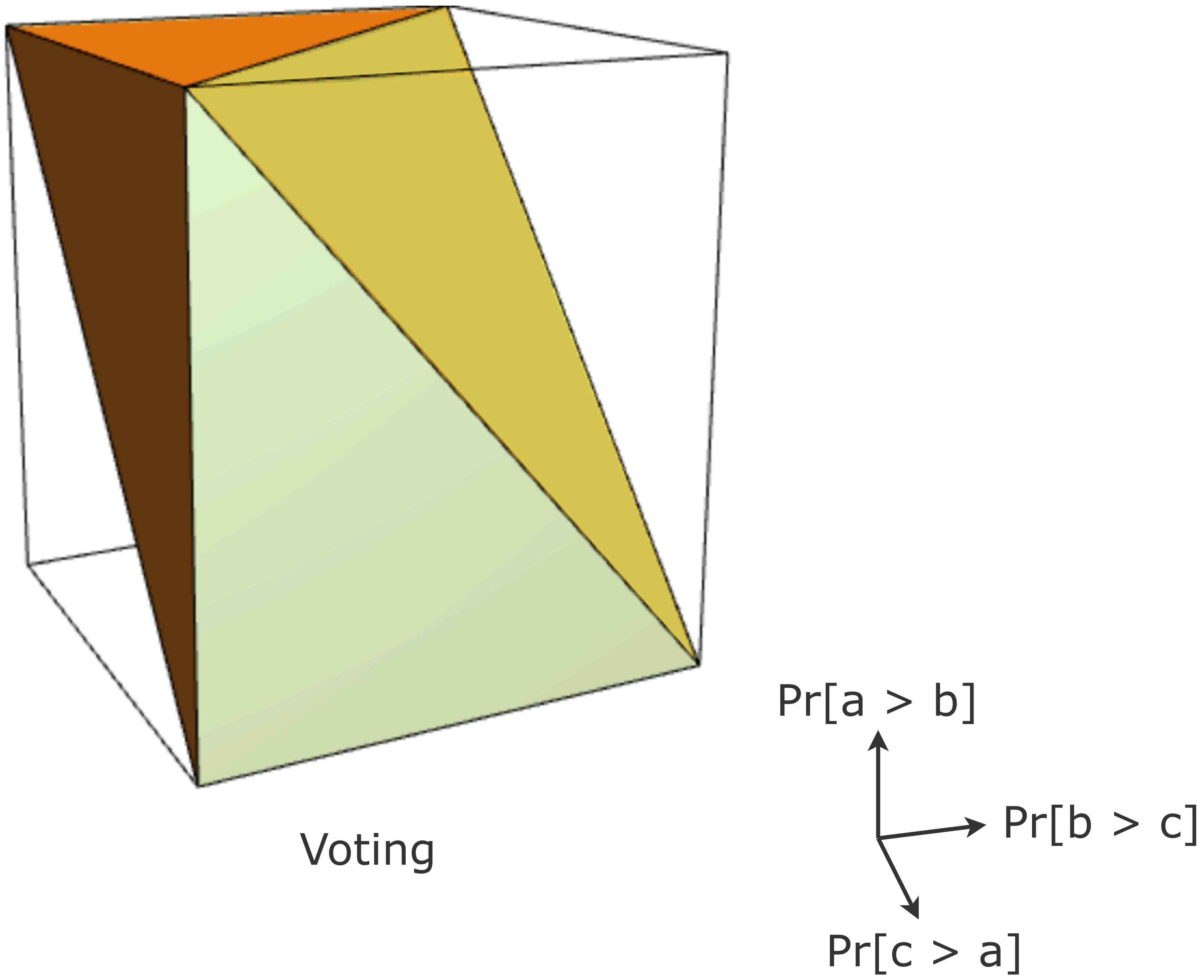
Voting versus independent dice



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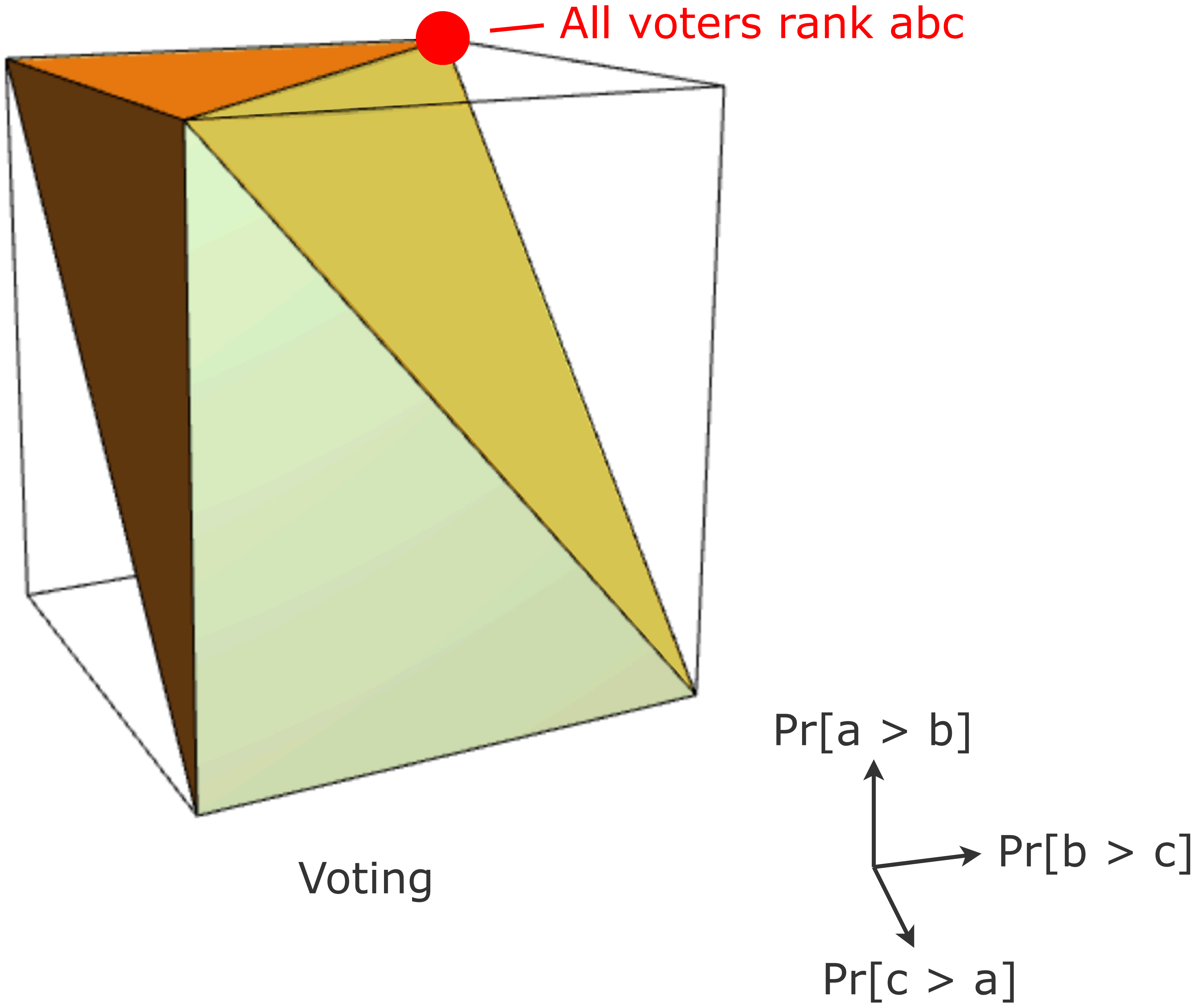

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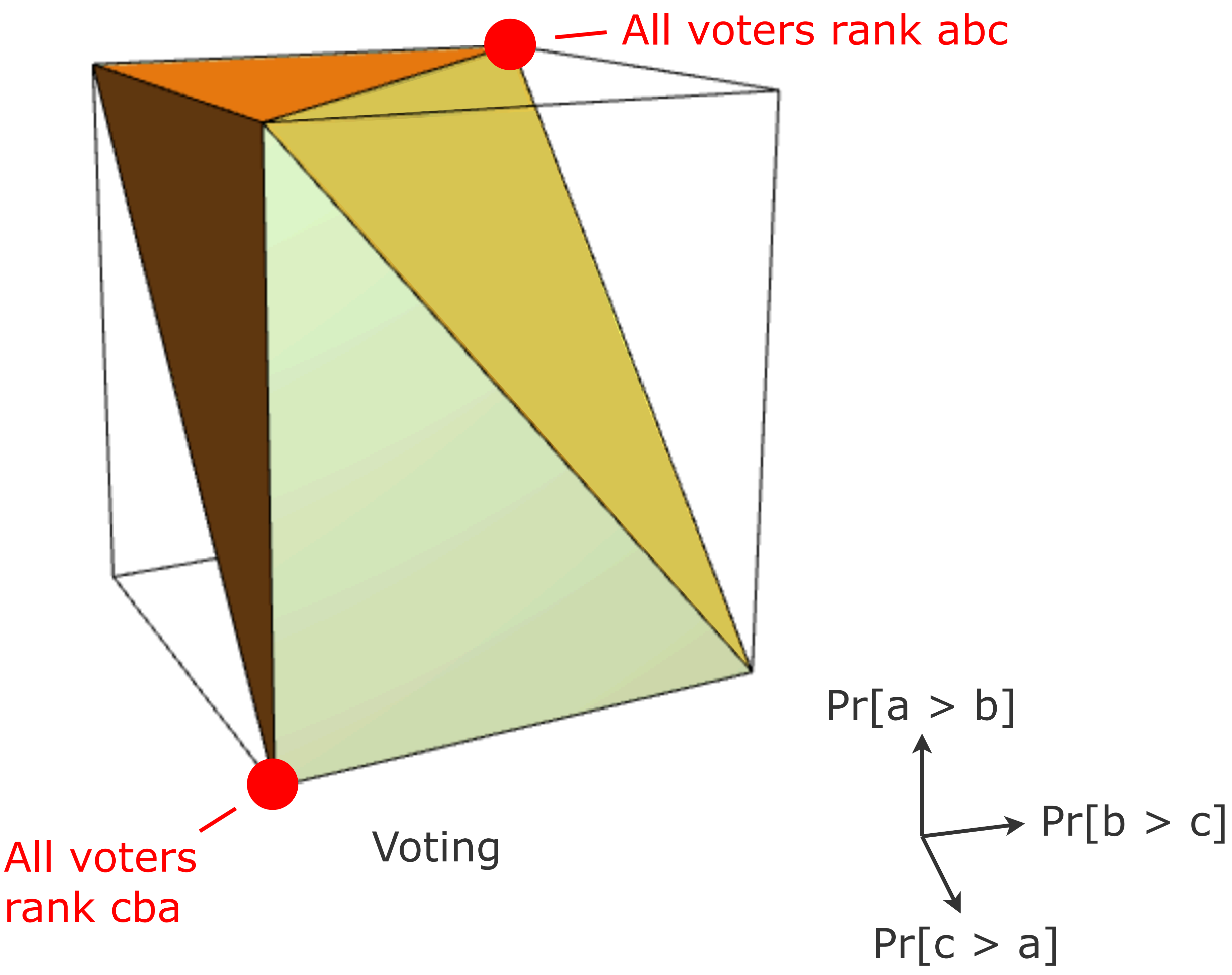

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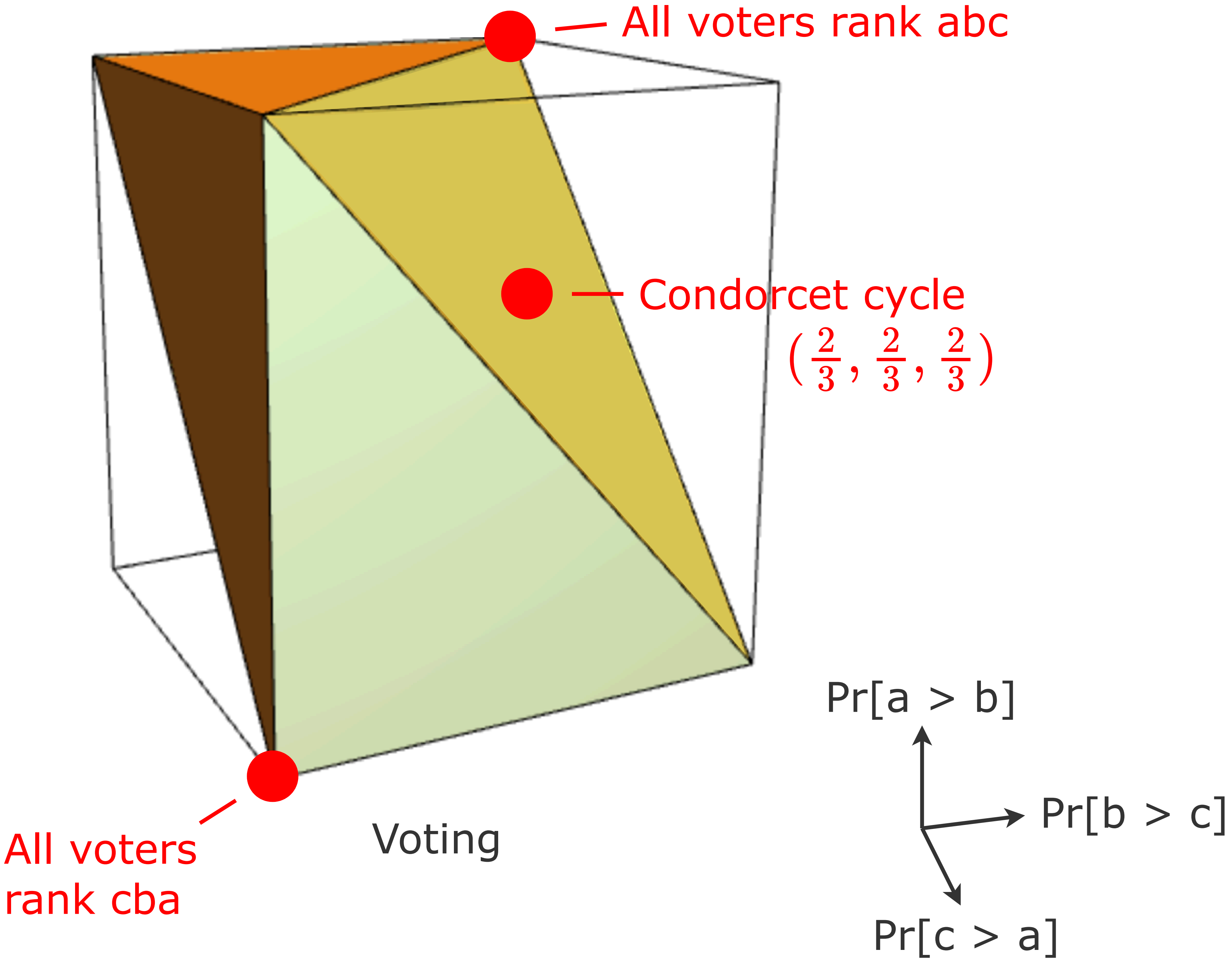

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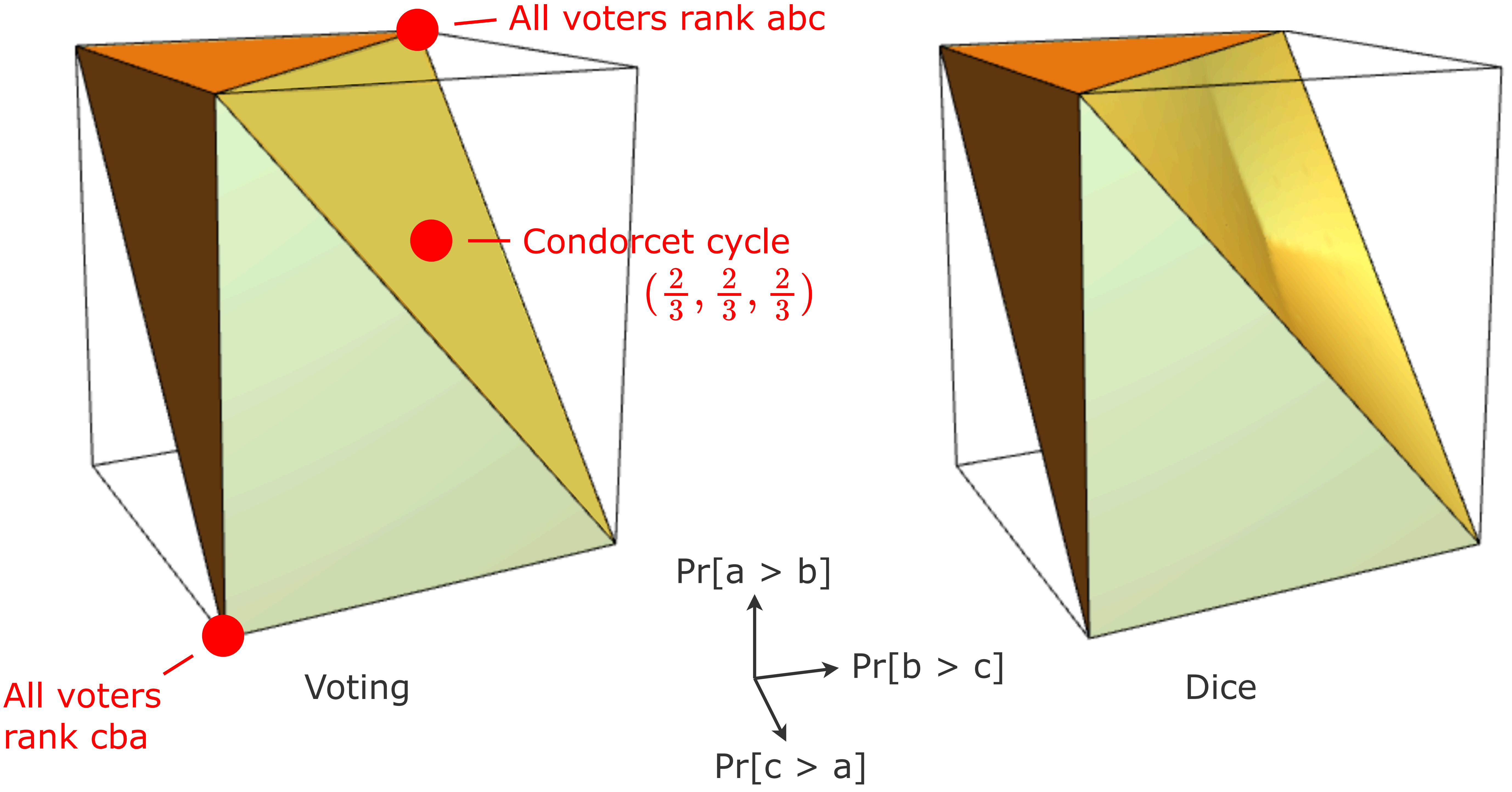

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